Name ________________________ Score __________________

MATH 581 - Fall 2004 - Homework 5

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. Use the DFTs as discussed in the Thomas text to show that the Crank-Nicolson method applied to the parabolic equation

$$u_t = \epsilon u_{xx}, \quad 0 < x < 1 \;\; \epsilon > 0$$
$$u(0,t) = u(1,t) = 0, \quad t > 0$$
$$u(x,0) = u^0(x), \quad 0 \le x \le 1$$

   is unconditionally stable.

2. Use the Gerschgorin Circle Theorem (GCT) to show that all strictly diagonally dominant matrices are nonsingular. A couple of the definitions and results stated below may be useful.

   Define the spectrum of a matrix B to be the set of all eigenvalues of B. This is denoted as

$$\Sigma(A) = \{\lambda \colon \exists\, \mathbf{v} \neq \mathbf{0} \text{ with } A\mathbf{v} = \lambda \mathbf{v}\}$$

   The Gerschgorin Circle Theorem can be stated as follows:
   Let B be a $d \times d$ matrix with arbitrary complex entries. Then

$$\Sigma(B) \subset \bigcup_{i=1}^{d} S_i,$$

   where

$$S_i = \{z \in \mathcal{C} \colon |z - b_{ii}| \le \sum_{j=1, j\neq i}^{d} |b_{ij}|\}$$

   Moreover, an eigenvalue $\lambda \in \Sigma(B)$ may lie in ∂S_K for some $K \in \{1, 2, \ldots, d\}$ only if it lies in ∂S_i for all $i = 1, 2, \ldots d$. The sets S_i are known as the Gerschgorin discs.

3. Assume $\epsilon > 0$ for this problem.

$$\begin{aligned} u_t &= \epsilon u_{xx}, \quad x \in (0,2), t > 0 \\ u(x,0) &= \cos\frac{\pi x}{2}, \quad x \in [0,2] \\ u_x(0,t) &= u_x(2,t) = 0, \quad t \ge 0. \end{aligned}$$

   In Homework 4, you developed a Crank-Nicolson scheme using the "ghost point" method for handling the boundary conditions. Write the scheme in the form $Q_1\mathbf{u}^{n+1} = Q\mathbf{u}^n$, and use the GCT to show that this scheme satisfies the necessary condition for stability given in Prop. 3.1.2. (3.1.2 is from the Thomas text. Hint: Follow Example 3.3.2 in the Thomas text to see the details of a very similar problem.)