Name ________________________ Score ________________

Spring 2005

MATH 582 Homework 2

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. Consider the pde given by

$$au_{xx} + cu_{yy} + du_x + eu_y + fu = g(x, y), \quad \forall\ (x, y) \ \in\ \Omega$$

$$u(x, y) = 0, \forall\ (x, y)\ \in \partial\Omega,$$

where $\Omega = (0, 1) \times (0, 1)$. Assume $a, c < 0$ and $f > 0$ with a, c, d, e, f real-valued constants. Use a discretization of the unit square Ω with Δx and Δy mesh widths in the $x-$ and $y-$ directions, respectively. Apply the standard centered difference operators for each of the differential terms in the pde. Let N_x = number of interior nodes in the $x-$ direction. Similarly, let N_y = number of interior nodes in the $y-$ direction. Choose the lexicographical ordering of the unknowns; that is, define the vector containing our numerical approximation to be

$$\mathbf{U} = [U_{1,1}\ U_{2,1}\ \ldots\ U_{N_x,1}\ U_{1,2}\ U_{2,2}\ \ldots\ U_{N_x,2}\ \ldots\ U_{N_y,N_x}]^T.$$

(a) Write down the explicit form of the linear system to be solved. In particular, give the necessary details of the system matrix structure and right-hand side. (i.e. $A\mathbf{U} = \mathbf{b}$)

(b) Show that if Δx and Δy can be taken to be sufficiently small so that

$$0 < \Delta x < \frac{-2a}{|d|} \quad \text{and} \quad 0 < \Delta y < \frac{-2c}{|e|},$$

then the system matrix A is (strictly) diagonally dominant and hence non-singular.

2. Consider the pde given by

$$u_{xx} + u_{yy} + f(x, y) = 0, \quad (x, y)\ \in\ \Omega$$

$$u(x, 0) = p(x),$$

$$u_x(0, y) = q(y),$$

$$u(x, \sqrt{1 - x^2}) = r(x),$$

where the region Ω is described as the open set bounded by $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$. Here, p, q, r, f are given functions. Use a uniform square grid with $\Delta x = \Delta y = h = 1/3$. Explicitly construct the system of linear equations obtained from approximating the solution u using the standard central difference scheme. Note that there is a Neumann boundary condition imposed along the $y-$axis, and you should introduce two ghost points in order to develop the appropriate equations for the grid points located along that axis. Note that there are six unknowns, $U_1, U_2, \ldots U_6$, to be determined by six equations.

Notation to be used in problem 2:

$$\begin{aligned}
O &= (0,0)\\
A &= (h,0)\\
B &= (2h,0)\\
C &= (0,h)\\
D &= (0,2h)\\
P &= (\sqrt{1-h^2},h)\\
Q &= (\sqrt{1-(2h)^2},2h)\\
R &= (2h,\sqrt{1-(2h)^2})\\
S &= (h,\sqrt{1-h^2})\\
T &= (0,1)
\end{aligned}$$