Name ________________________ Score ____________________

Spring 2005

MATH 582 Homework 3

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. Consider the following boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u(0) = 0, &\qquad u(1) = 0, \end{aligned}$$

   (a) Give the variational formulation of the equation. Be sure to identify the function space that you are using.

   (b) Consider the discretized form of the variational problem. Assume that one uses a non-uniform grid with the partition $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ for the interval $[0,1]$, and denote the width of each interval by $h_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots N+1$. In terms of a general finite dimensional basis, give the form of the stiffness matrix, K, for solving the discrete problem.

   (c) Using the piecewise linear basis functions (the "hat" functions that we have described in class), show that the stiffness matrix, K, has a tridiagonal structure. One can explicitly calculate the actual values that form the tridiagonal. Show that $K_{ii} = \frac{1}{h_i} + \frac{1}{h_{i+1}}$ and $K_{i,i+1} = K_{i+1,i} = -\frac{1}{h_{i+1}}$ for $i = 2, 3, \ldots N$.

   (d) If a uniform grid is used, then give the special form of the system matrix.

2. Using the boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u(0) = 0, &\qquad u(1) = 0, \end{aligned}$$

   consider using a uniform stepsize, $h = \frac{1}{N+1}$, and the usual $O(h^2)$ centered-difference formula to approximate the solution to the boundary value problem. Give the form of the system matrix associated with this approximation. Compare with Number 1(d). For more on this relationship, Section 0.5 of the Brenner & Scott handout.

3. Consider the two-point boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \quad x \in (1,2), \\ u'(1) = 0, &\qquad u(2) = 0. \end{aligned}$$

   (a) Write down the standard weak formulation; indicate any boundary constraints you are imposing on the space.

   (b) Demonstrate that the solution to the classical (differential equation) problem satisfies your weak formulation.

   (c) Assume that the solution to the weak form problem is sufficiently smooth, and show that it also satisfies the classical problem.