Name ______________________ Score ________________

Spring 2005

MATH 582 Homework 6

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. Let V be an inner product space with inner product denoted by $(\cdot, \cdot)$. Verify that the function defined below satisfies all the properties of a norm.

$$\|u\| = \sqrt{(u, u)} \quad \forall\, u \in V$$

2. Consider the boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u'(0) = 0, &\qquad u(1) = 0, \end{aligned}$$

   (a) Give the variational formulation of the equation. Be sure to identify the function space that you are using.

   (b) Consider the discretized form of the variational problem. Assume that one uses a non-uniform grid with the partition $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ for the interval $[0, 1]$, and denote the width of each interval by $h_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots N$. In terms of a general finite dimensional basis $\{\phi_i\}$, give the form of the stiffness matrix, K for solving the discrete problem.

   (c) Using the piecewise linear basis functions (the "hat functions" that we have described in class), show that the stiffness matrix, K has a tridiagonal structure. Explicitly calculate the values that form the matrices. Note: you have already done most of the work for the stiffness matrix, with the exception of a couple of rows, so you may quote those results from previous homeworks.

   (d) Write a Matlab script which implements this problem for an abitrary right-hand side function $f(x)$. To test your code, generate a test problem using $u(x) = \cos(2\pi x) - 1$. Compute the error (just use the discrete 2-norm), and construct a table to show that the absolute error $\to 0$ as $h \to 0$. Also generate a plot of the true soln and the computed approximation.

3. Consider the following boundary value problem where the constants $a > 0, b \geq 0$

$$\begin{aligned} -au''(x) + bu(x) &= f(x), \quad x \in (0,1), \\ u'(0) = 0, &\qquad u'(1) = 0, \end{aligned}$$

   (a) Give the variational formulation of the equation. Be sure to identify the function space that you are using.

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(b) Consider the discretized form of the variational problem. Assume that one uses a non-uniform grid with the partition $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ for the interval $[0,1]$, and denote the width of each interval by $h_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots N$. In terms of a general finite dimensional basis $\{\phi_i\}$, give the form of the stiffness matrix, K, and the mass matrix, M, for solving the discrete problem.

(c) Using the piecewise linear basis functions (the "hat functions" that we have described in class), show that the stiffness matrix, K, and the mass matrix, M, have a tridiagonal structure. Explicitly calculate the values that form the matrices.

(d) Write a Matlab script which implements this problem for an abitrary right-hand side function $f(x)$. To test your code, generate a test problem using $a = 1, b = 1$ and $u(x) = \cos(\pi x)$. Compute the error (just use the discrete 2-norm), and construct a table to show that the absolute error $\to 0$ as $h \to 0$. Also generate a plot of the true soln and the computed approximation.