Name ______________________ Score __________________

Spring 2005

MATH 582 Midterm Exam

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers. Answer 3 of the 4 questions.

1. Let w be a continuous function, and let V consist of continuous functions on the interval $[0,1]$ for which the derivative $v'(x)$ is piecewise continuous and bounded on $[0,1]$ and $v(0) = v(1) = 0$. Show that if

$$\int_0^1 w(x)v(x)dx = 0 \quad \text{for all } v \in V,$$

   then $w = 0$.

2. Let $V = \{v \in C^2[0,1] \colon v(0) = \alpha\}$

$$f(u) = \int_0^1 \left[\frac{1}{2}p(x)(u')^2 + \frac{1}{2}q(x)(u)^2 - r(x)u\right] dx \quad \forall\, u \in V,$$

   where $p \in C^1[0,1]$, $p(x) > 0$ for all $x \in [0,1]$, $q, r \in C[0,1]$ and $q(x) \geq 0$ for all $x \in [0,1]$. Derive the Euler-Lagrange D. E. Be sure to clearly identify the natural as well as the essential boundary conditions. Also clearly identify the linear space of test functions given by $\tilde{V}$.

3. Problem 0.x.6 in Brenner & Scott handout, page 20. Note, the norm indicated is the L^2-norm. That is, the inequality is given by

$$\|u - u_I\|_2 \leq Ch^2\|u''\|_2, \quad \text{where} \|z\|_2 = \left[\int_0^1 [z(x)]^2 dx\right]^{1/2}.$$

   NOTE: Answering the question addressing how small you can make $\tilde{c}$ is a BONUS question. It is worth 5 points.

4. The first-order directional derivative of the functional

$$f(u) = \int_a^b F(x, u, u')dx, \quad u \in V$$

   is given by

$$f^{(1)}(u;\eta) = \int_a^b \frac{\partial F}{\partial u'}\eta' + \frac{\partial F}{\partial u}\eta dx,$$

   for all $\eta \in \tilde{V}$ with $\|\eta\| = 1$. Give the corresponding expression for the second-order directional derivative $f^{(2)}(u;\eta)$. Assume that the function F is sufficiently smooth for all of the partial derivatives that you use.