Additivity of factorization algebras

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Abstract

We prove an additivity result for factorization algebras, which articulates how a factorization algebra on a product of two topological spaces can be codified as a factorization algebra on each factor. In the context of manifolds, we prove this result restricts as an additivity result for locally constant factorization algebras. Specializing this latter result to Euclidean spaces supplies a new proof of Dunn's additivity for E_n-algebras.

Contents

1 Introduction

Factorization algebras were developed by Costello and Gwilliam in [CG16] to understand the algebraic structure of observables in perturbative quantum field theory. They are related to chiral algebras which were developed by Beilinson and Drinfeld in [BD04] to understand vertex algebras in conformal field theory.

A typical source of examples of factorization algebras comes from perturbative σ-models. One such example is the Poisson σ-model. The classical Poisson σ-model aims to understand the mapping space $\mathsf{Map}(\Sigma, X)$ where X is a Poisson manifold and Σ is a compact oriented surface. In general, this mapping space is quite complicated. A first approximation to understanding $\mathsf{Map}(\Sigma, X)$ is to fix a map $\varphi \in \mathsf{Map}(\Sigma, X)$, such as a constant map, and consider an infinitesimal neighborhood of φ. This is known as the perturbative

Poisson σ-model. In [CG16] Costello and Gwilliam laid the foundation for understanding field theories in this way. Using their framework, the classical observables in perturbative σ-models possess the structure of a factorization algebra.

For example, in the classical Poisson σ-model, for X a target Poisson manifold, the space of fields on an open $U \subset \Sigma$ can be taken to be the smooth stack $\mathsf{Map}(U_{dR}, X)$. Here, U_{dR} is the de Rham stack of U, whose global functions is the de Rham complex of U. Notice the embedding $X \xhookrightarrow{\mathsf{const}} \mathsf{Map}(U_{dR}, X)$ of constant maps. For perturbative σ-models, one is interested in the infinitesimal neighborhood of this embedding. For open $U \subset \Sigma$, the fields for this perturbative σ-model can be taken to be the dg Lie algebra $\Omega^*(U) \otimes \mathfrak{g}_X$, where $\mathfrak{g}_X$ is a curved L_∞-algebra defined from the Poisson structure on X. Therefore, the classical observables for this perturbative σ-model can be taken to be $\mathsf{C}_*^{\mathsf{Lie}}(\Omega_{\mathsf{c}}^*(U) \otimes \mathfrak{g}_X)$. This expression is functorial in U

$$\mathsf{Obs}^{\mathsf{cl}} : U \mapsto \mathsf{C}_*^{\mathsf{Lie}}(\Omega_{\mathsf{c}}^*(U) \otimes \mathfrak{g}_X) \ . \tag{1}$$

Further, this functor is a factorization algebra. We refer the reader to [CZ20] for more details.

Similarly, given a based space $* \in Z$ and an n-manifold M, consider the functor

$$\mathsf{Map}_{\mathsf{c}}(-, Z) : \mathsf{open}(M) \to \mathsf{Spaces} \ , \qquad U \mapsto \mathsf{Map}_{\mathsf{c}}(U, Z) \ . \tag{2}$$

If Z is $(n-1)$-connected, non-abelian Poincaré duality (see the main theorem of [AF21] or Corollary 3.6.4 therein, or Corollary 4.8 of [AF20b]) implies the functor $\mathsf{Map}_{\mathsf{c}}(-, Z)$ is given by factorization homology, $\int_{-} A_Z \simeq \mathsf{Map}_{\mathsf{c}}(-, Z)$, for a certain $E_{\mathsf{BO}(n)}$-algebra A_Z that is determined by Z. Proposition 3.14 of [AF20b] implies factorization homology defines a locally constant factorization algebra on M. Together, these results imply the following.

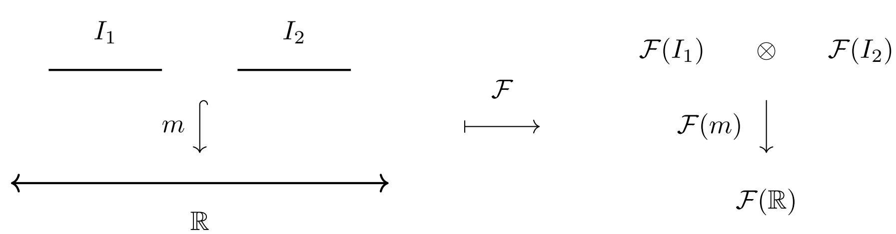
Proposition 1.1. *Let Z be a pointed $(n-1)$-connected space. Let M be an n-manifold. The functor $\mathsf{Map}_{\mathsf{c}}(-, Z)\colon \mathsf{open}(M) \to \mathsf{Spaces}$ has the structure of a factorization algebra on M that is locally constant.*

Heuristically, a factorization algebra on a topological space X valued in a symmetric monoidal category $(\mathcal{V}, \otimes)$ is a functor $\mathcal{F} : \mathsf{open}(X) \to \mathcal{V}$ that possesses a local-to-global property and that takes disjoint unions of open sets to tensor products in $\mathcal{V}$. Here, $\mathsf{open}(X)$ denotes the poset of open subsets of X. Note that disjoint union is only a partially defined operation on $\mathsf{open}(X)$, so we cannot simply require that $\mathcal{F}$ is a symmetric monoidal functor. There are algebraic gadgets called operads that capture this notion of a partial operation. We use this formalism to give a more precise description of factorization algebras later. There is a special class of factorization algebras called locally constant factorization algebras. A factorization algebra $\mathcal{F}$ is called locally constant if each isotopy equivalence $U \hookrightarrow V$ in $\mathsf{open}(X)$ is carried to an equivalence $\mathcal{F}(U) \xrightarrow{\simeq} \mathcal{F}(V)$ in $\mathcal{V}$. For example, the classical observables of the Poisson σ-model as given in equation (1) form a locally constant factorization algebra.

Factorization algebras also provide an approach to understanding (higher) algebraic structures. For instance, we will now show how a locally constant factorization algebra on $\mathbb{R}$ gives rise to an associative algebra. Consider a locally constant factorization algebra on $\mathbb{R}$

$$\mathcal{F} : \mathsf{open}(\mathbb{R}) \to \mathsf{Vect}_{\mathbb{R}}$$

valued in the category of real vector spaces equipped with the symmetric monoidal structure provided by $\otimes_{\mathbb{R}}$. Recall that the open subsets of $\mathbb{R}$ are generated by intervals. Consider the inclusion of two disjoint intervals $I_1 \amalg I_2 \xhookrightarrow{m} \mathbb{R}$. Since $\mathcal{F}$ takes disjoint unions to tensor products in $\mathcal{V}$, the map m gets carried to a linear map of vector spaces as depicted below.



Let A denote the vector space $\mathcal{F}(\mathbb{R})$. Since $\mathcal{F}$ is assumed to be locally constant, given any open interval $I \hookrightarrow \mathbb{R}$, the induced map $\mathcal{F}(I) \to \mathcal{F}(\mathbb{R}) =: A$ is an equivalence. Thus, $\mathcal{F}(I) \cong A$ for each interval $I \in \mathsf{open}(X)$. Therefore, the map m gets carried to a linear map $A \otimes_{\mathbb{R}} A \xrightarrow{\mathcal{F}(m)} A$. One can check that this endows A with the structure of an associative algebra over $\mathbb{R}$.

As shown in [CG16], there is an equivalence of categories

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Vect}_k) \simeq \mathsf{AssocAlg}_k \ ,$$

between the category of locally constant factorization algebras on $\mathbb{R}$ valued in the symmetric monoidal (with respect to $\otimes_k$) category of vector spaces over a field, k, and the category of associative algebras over k. More generally, for a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, in [Lur17] Lurie proves that there is an equivalence of ∞-categories

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^n}(\mathcal{V}^{\otimes}) \simeq \mathsf{Alg}_{\mathsf{E}_n}(\mathcal{V}^{\otimes}) \tag{3}$$

between the ∞-category of locally constant factorization algebras on $\mathbb{R}^n$ valued in $\mathcal{V}^{\otimes}$ and the ∞-category of E_n-algebras in $\mathcal{V}^{\otimes}$.

In [Dun88] Dunn proved a celebrated theorem about the E_n-operads that is referred to as Dunn's additivity. Lurie generalized Dunn's additivity to the setting of ∞-operads as Theorem 5.2.2.2 of [Lur17]. Dunn's additivity asserts that for nonnegative integers $n, m \geq 0$, the E_{n+m}-operad is a tensor product of the E_n-operad with the E_m-operad. In particular, for a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, there is an equivlence of ∞-categories

$$\mathsf{Alg}_{\mathsf{E}_{n+m}}(\mathcal{V}^{\otimes}) \simeq \mathsf{Alg}_{\mathsf{E}_n}(\mathsf{Alg}_{\mathsf{E}_m}(\mathcal{V}^{\otimes})) \tag{4}$$

between the ∞-category of E_{n+m}-algebras in $\mathcal{V}^{\otimes}$ and the ∞-category of E_n-algebras in the ∞-category of E_m-algebras in $\mathcal{V}^{\otimes}$. Note that the ∞-category $\mathsf{Alg}_{\mathsf{E}_m}(\mathcal{V}^{\otimes})$ is a symmetric monoidal ∞-category via pointwise tensor product in $\mathcal{V}^{\otimes}$, thus the right-hand side of equation (4) makes sense.

Using equation (3), we can reformulate the statement of Dunn's additivity as an equivalence of ∞-categories

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^{n+m}}(\mathcal{V}^{\otimes}) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^n}(\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^m}(\mathcal{V}^{\otimes})) \ . \tag{5}$$

There are several natural generalizations of this statement that we contemplate in this paper:

Question 1.2. Is there an analog of equation (5) for factorization algebras that are not necessarily locally constant?

Question 1.3. Is there an analog of equation (5) when one considers factorization algebras over topological spaces other than Euclidean space?

In this paper, we provide solutions to both Question 1.2 and Question 1.3. A novelty of our approach is that we recover Dunn's additivity as a corollary. Lurie provides a highly non-trivial proof of Dunn's additivity (Theorem 5.1.2.2 of [Lur17]). In particular, our methods provide a new proof.

We reformulate factorization algebras within the context of ∞-operads as developed by Lurie in [Lur17]. The poset $\mathsf{open}(X)$ can be regarded as a multicategory, which captures the notion that disjoint union is only a partially defined operation. There is a standard way of regarding a multicategory as an ∞-operad, and we will denote the ∞-operad associated to $\mathsf{open}(X)$ by $\mathsf{open}(X)^{\otimes}$. An element of $\mathsf{open}(X)^{\otimes}$ can be thought of as a pair $(I_+, (U_i))$ consisting of a based finite set I_+ and an I-indexed list (U_i) of open sets in X. Symmetric monoidal ∞-categories are also defined using the framework of ∞-operads. Using this language, a factorization algebra is then a functor of ∞-operads $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$, again satisfying a local-to-global principle and the condition that disjoint unions map to tensor products.

This operadic formulation of factorization algebras provides us with a natural approach to answering Question 1.2 and Question 1.3.

1.1 General additivity

We first provide the following answer to Question 1.2:

Theorem 1.4. *Let X and Y be topological spaces, and let $\mathcal{V}^{\otimes}$ be a $\otimes$-presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathit{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathit{Fact}_X(\mathit{Fact}_Y(\mathcal{V}^{\otimes})) \ .$$

1.1.1 Key ideas

The ∞-category of factorization algebras is an ∞-subcategory

$$\mathsf{Fact}_X(\mathcal{V}^{\otimes}) \hookrightarrow \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})$$

of the functors of ∞-operads between the $\mathsf{open}(X)^{\otimes}$ and $\mathcal{V}^{\otimes}$. Thus, the statement of additivity is making a comparison between an ∞-subcategory of

$$\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}, \mathcal{V}^{\otimes})$$

and an ∞-subcategory of

$$\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})) \ .$$

The category of ∞-operads possesses a tensor product with the property that

$$\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})) \simeq \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}\otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \ .$$

The defining feature of the tensor product of ∞-operads is such that there is an equivalence of ∞-categories

$$\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}\otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ ,$$

where the righthand side is the ∞-category of bifunctors of ∞-operads. A bifunctor is a special type of functor out of $\mathsf{open}(X)^{\otimes}\times \mathsf{open}(Y)^{\otimes}$. There is a natural bifunctor

$$\rho : \mathsf{open}(X)^{\otimes}\times \mathsf{open}(Y)^{\otimes} \to \mathsf{open}(X\times Y)^{\otimes}$$

given by taking the product of open sets in X with open sets in Y to produce an open set in $X\times Y$. Restriction along ρ provides a comparison

$$\rho^* : \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}, \mathcal{V}^{\otimes}) \to \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

There is a left adjoint to ρ^* given by left Kan extension. The strategy for proving Theorem 1.4 is to show that this adjunction restricts to an equivalence between the ∞-subcategories of factorization algebras.

1.2 Locally constant additivity

Next, we provide an affirmative answer to Question 1.3 with the caveat that we now require the topological spaces X and Y to be topological manifolds:

Theorem 1.5. *Let X and Y be topological manifolds and let $\mathcal{V}^{\otimes}$ be a $\otimes$-presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathit{Fact}^{l.c.}_{X\times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathit{Fact}^{l.c.}_X(\mathit{Fact}^{l.c.}_Y(\mathcal{V}^{\otimes})) \ .$$

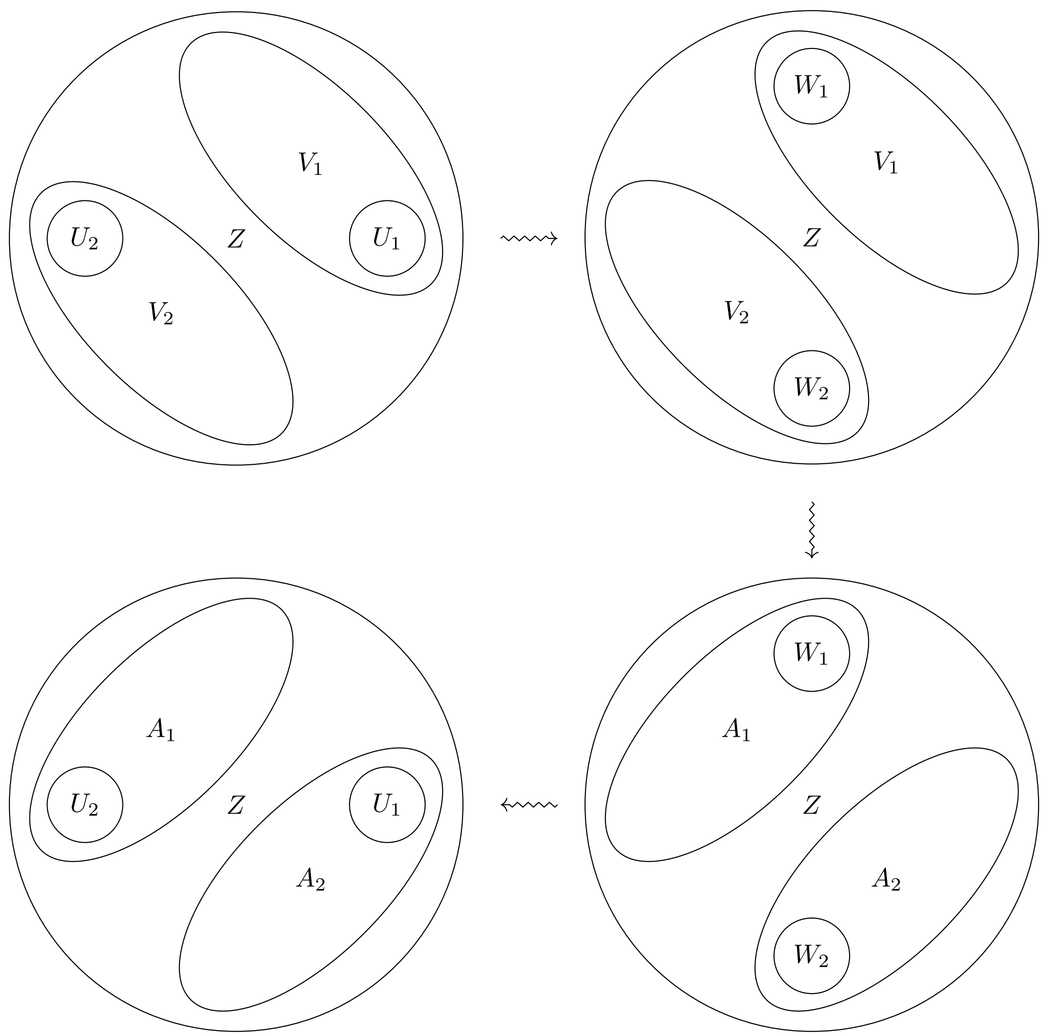
Before discussing the key ideas of the proof, we first mention an immediate implication of this theorem. Let $\mathsf{Ch}_k^{\otimes}$ denote the symmetric monoidal ∞-category of chain complexes over a fixed field. Theorem 1.5 implies that an algebra in $\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})$ does not simply possess two multiplication rules. Rather, there is a space of multiplication rules. We now describe how to see this space has an interesting topology to it. Namely, we can see a nontrivial loop of multiplications. Using equation (3), there is an equivalence

$$\mathsf{Alg}(\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}_{\mathbb{R}}^{\mathsf{l.c.}}(\mathsf{Fact}_{\mathbb{R}}^{\mathsf{l.c.}}(\mathsf{Ch}_k^{\otimes})) \ .$$

Theorem 1.5 further asserts an equivalence

$$\mathsf{Alg}(\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}_{\mathbb{R}}^{\mathsf{l.c.}}(\mathsf{Fact}_{\mathbb{R}}^{\mathsf{l.c.}}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}_{\mathbb{R}^2}^{\mathsf{l.c.}}(\mathsf{Ch}_k^{\otimes}) \ .$$

Now take $\mathcal{F} \in \mathsf{Alg}(\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}_{\mathbb{R}^2}^{\mathsf{l.c.}}(\mathsf{Ch}_k^{\otimes})$. Consider two disjoint disks $U_1 \amalg U_2 \xhookrightarrow{\varphi} Z \subset \mathbb{R}^2$ including into a larger disk, as depicted below. Note that each inclusion of a disk into $\mathbb{R}^2$ is an isotopy equivalence. Since $\mathcal{F}$ is assumed to be locally constant, this implies that $\mathcal{F}(U_1) \simeq \mathcal{F}(U_2) \simeq \mathcal{F}(Z) \simeq \mathcal{F}(\mathbb{R}^2)$. Therefore, $\mathcal{F}$ carries φ to a morphism $\mathcal{F}(\varphi) : \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \to \mathcal{F}(Z)$ in $\mathsf{Ch}_k^{\otimes}$. We now illustrate a zig-zag of inclusions that produces an example of a loop in the space of multiplications.



If we apply $\mathcal{F}$ to the morphisms indicated above, we get the following diagram in $\mathsf{Ch}_k^{\otimes}$

$$\begin{tikzcd}
 & \mathcal{F}(U_1)\otimes\mathcal{F}(U_2) \arrow[dl, "\simeq"'] \arrow[d] \arrow[dr, "\simeq"] & \\
\mathcal{F}(V_1)\otimes\mathcal{F}(V_2) \arrow[r] & \mathcal{F}(Z) & \mathcal{F}(A_1)\otimes\mathcal{F}(A_2) \arrow[l] \\
 & \mathcal{F}(W_1)\otimes\mathcal{F}(W_2) \arrow[ul, "\simeq"] \arrow[u] \arrow[ur, "\simeq"'] &
\end{tikzcd} .$$

1.2.1 Key ideas

Recall that a factorization algebra $\mathcal{F}$ is called locally constant if it carries an isotopy equivalence of open sets in X to an equivalence in $\mathcal{V}$. We now place the locally constant condition within the operadic formulation of factorization algebras. Note that there is an ∞-subcategory $\mathcal{I}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ consiting of the same objects as $\mathsf{open}(X)^{\otimes}$, but only those morphisms $(I_+,(U_i)) \to (J_+,(V_j))$ for which the map of based finite sets $I_+ \xrightarrow{f} J_+$ is a bijection such that for all $i \in I$, the inclusion $U_i \hookrightarrow V_{f(i)}$ is an isotopy equivalence. The condition for a factorization algebra $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ to be locally constant can be phrased as the condition that $\mathcal{F}$ factors

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathsf{loc}"] & \mathcal{V}^{\otimes} \\
\mathsf{open}(X)^{\otimes}[\mathcal{I}(X)^{\otimes -1}] \arrow[ur, dashed] &
\end{tikzcd}$$

through the localization of $\mathsf{open}(X)^{\otimes}$ at the isotopy equivalences $\mathcal{I}(X)^{\otimes}$. The localization $\mathsf{open}(X)^{\otimes}[\mathcal{I}(X)^{\otimes -1}]$ is a little intractable. This is precisely where we employ the additional requirement that X and Y are topological manifolds. As such, we can reduce the situation to analyzing functors out of $\mathsf{disk}(X)^{\otimes}$, the full ∞-sub-operad consisting of those open sets that are homeomorphic to a finite disjoint union of disks. Let $\mathcal{J}(X)^{\otimes}$ denote the full ∞-subcategory of $\mathcal{I}(X)^{\otimes}$ consisting of those objects that lie in $\mathsf{disk}(X)^{\otimes}$. The locally constant condition can then be reduced to analyzing the localization $\mathsf{disk}(X)^{\otimes}[\mathcal{J}(X)^{\otimes -1}]$. By evaluating disks at their centers, we can understand this localization in terms of configuration spaces.

1.3 An application

Consider the moduli space

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right)$$

of $U(1)$-bundles on the space $\mathbb{R} \times (S^1)^{\amalg r}$ that are trivialized outside of a compact set. We can use Theorem 1.5 to identify the algebra of chains on this moduli space. There is a homotopy equivalence

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right) \simeq \mathsf{Map}_{\mathsf{c}}(\mathbb{R} \times (S^1)^{\amalg r}, \mathsf{B}U(1))$$

between the moduli space of $U(1)$-bundles and the space of compactly supported maps into $\mathsf{B}U(1)$. Note that $\mathsf{B}U(1)$ is 1-connected, since $U(1)$ is 0-connected, and $\mathbb{R} \times (S^1)^{\amalg r}$ is 2-dimensional. Consider the functor

$$C_*\left(\mathsf{Map}_{\mathsf{c}}(-, \mathsf{B}U(1))\right) : \mathsf{open}\left(\mathbb{R} \times (S^1)^{\amalg r}\right) \to \mathsf{Ch}_k^{\otimes}$$

where Ch_k denotes the ∞-category of chain complexes over a field k. This defines a locally constant factorization algebra, per the discussion surrounding equation (2). That is,

$$C_*\left(\mathsf{Map}_{\mathsf{c}}(-, \mathsf{B}U(1))\right) \in \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}\times(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes}) . \tag{6}$$

By Theorem 1.5, we know

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}\times(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes}) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Fact}^{\mathsf{l.c.}}_{(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Alg}(\mathsf{Fact}^{\mathsf{l.c.}}_{(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes})) .$$

Therefore, if we evaluate the factorization algebra in equation (6) on the total space $\mathbb{R} \times (S^1)^{\amalg r}$ we then obtain an object in $\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})$. Further, this algebra is

$$C_*\left(\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right)\right) .$$

In this case, we can explicitly identify this algebra by other means. Namely, note that for any space Y and pointed space Z

$$\begin{aligned}
\mathsf{Map}_{\mathsf{c}}(\mathbb{R} \times Y, Z) &:= \mathsf{Map}_*((\mathbb{R} \times Y)^+, Z) \\
&\cong \mathsf{Map}_*(\mathbb{R}^+ \wedge Y^+, Z) \\
&\cong \mathsf{Map}_*(\mathbb{R}^+, \mathsf{Map}_*(Y^+, Z)) \\
&\cong \Omega \mathsf{Map}_{\mathsf{c}}(Y, Z) \\
&\cong \mathsf{Map}_{\mathsf{c}}(Y, \Omega Z) .
\end{aligned}$$

Here, $\mathsf{Map}_*(-,-)$ denotes the space of based maps, Y^+ denotes the one-point compactification, and ΩZ denotes the based loop space. There, if we take $Y = (S^1)^{\amalg r}$ and $Z = \mathsf{B}U(1)$, we have

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right) \simeq \mathsf{Map}_{\mathsf{c}}(\mathbb{R} \times (S^1)^{\amalg r}, \mathsf{B}U(1)) \cong \mathsf{Map}_{\mathsf{c}}((S^1)^{\amalg r}, U(1)) .$$

By the universal property of coproducts,

$$\mathsf{Map}_{\mathsf{c}}((S^1)^{\amalg r}, U(1)) \cong \mathsf{Map}_{\mathsf{c}}(S^1, U(1))^{\times r} .$$

Further, there is a homeomorphism

$$\mathsf{Map}_{\mathsf{c}}(S^1, U(1))^{\times r} \xrightarrow{\cong} \left(\mathsf{Map}_*(S^1, U(1)) \times U(1)\right)^{\times r}$$

given factorwise by

$$(S^1 \xrightarrow{f} U(1)) \mapsto \left(S^1 \xrightarrow{f(1)^{-1} \cdot f} U(1), f(1)\right) .$$

Noting that $\mathsf{Map}_*(S^1, U(1)) \simeq \mathbb{Z}$, we see

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right) \simeq (\mathbb{Z} \times U(1))^{\times r} .$$

All told,

$$C_*\left(\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right); k\right) \simeq k[x^{\pm 1}]^{\otimes r} \otimes (k[\varepsilon]_{/(\varepsilon^2)})^{\otimes} r \simeq \left(k[x^{\pm 1}, \varepsilon]_{/(\varepsilon^2)}\right)^{\otimes r} ,$$

where $\mathsf{deg}(\varepsilon) = 1$. Therefore, we have identified the algebra of global sections of the locally constant factorization algebra given in equation (6) with the algebra $\left(k[x^{\pm 1}, \varepsilon]_{/(\varepsilon^2)}\right)^{\otimes r}$.

1.4 Future contemplations

Recently, the theory of stratified spaces has been given a solid foundation in the context of ∞-categories by Ayala-Francis-Tanaka in [AFT17]. There is a class of factorization algebras on stratified spaces called constructible factorization algebras. A constructible factorization algebra $\mathcal{F}$ on a stratified space $X \to \mathcal{P}$ is a factorization algebra on X such that it is locally constant when restricted to each stratum. We believe that our methods can be used to provide an additivity statement for constructible factorization algebras on stratified spaces. Namely, we conjecture:

Conjecture 1.6. *For nice stratified spaces X and Y, and $\mathcal{V}^{\otimes}$ a $\otimes$-presentable ∞-category, there is an equivalence of ∞-categories*

$$\mathit{Fact}^{\mathit{cbl}}_{X \times Y}(\mathcal{V}^{\otimes}) \to \mathit{Fact}^{\mathit{cbl}}_{X}(\mathit{Fact}^{\mathit{cbl}}_{Y}(\mathcal{V}^{\otimes})) .$$

1.5 Acknowledgements

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2 Background

In this section we establish preliminary conventions, notation, and definitions. Throughout this paper we use the theory of ∞-categories. There are various models for the foundations of ∞-category theory, for example quasicategories as introduced by Joyal and developed by Lurie in [Lur09], and complete Segal spaces as developed by Rezk in [Rez01]. In this paper we work model independently, with the notable exception being our explicit use of complete Segal spaces in the proof of Theorem 4.1.

As indicated in the introduction, a factorization algebra on a topological space X is a functor that assigns data to each open subset $U \subset X$. Furthermore, a factorization algebra satisfies a particular local-to-global property and behaves nicely with respect to disjoint unions of open sets. A good example to keep in mind is the following:

Example 2.1. There is an interesting class of factorization algebras that one can associate to a Lie algebra called the universal enveloping E_n-algebras. Let $n \geq 1$ be an integer and $\mathfrak{g}$ be a Lie algebra over a field k. Consider the functor

$$\mathcal{U}_n\mathfrak{g} : \mathsf{open}(\mathbb{R}^n) \to \mathsf{Ch}_k$$

from the poset of open sets in $\mathbb{R}^n$ to the category of chain complexes over a field k given by sending $U \in \mathsf{open}(\mathbb{R}^n)$ to

$$U \mapsto \mathsf{C}_*^{\mathsf{Lie}}(\Omega_{\mathsf{c}}^*(U) \otimes \mathfrak{g}) ,$$

the Lie algebra chains on the dgla of compactly supported de Rham forms on U with values in $\mathfrak{g}$. This defines a locally constant factorization algebra on $\mathbb{R}^n$.

We formulate factorization algebras within the framework of ∞-operads, and freely use this theory. The data of an ∞-operad is an ∞-category $\mathcal{O}^\otimes$ and a functor $\mathcal{O}^\otimes \to \mathsf{Fin}_*$ to the ∞-category of based finite sets. As ∞-operads form the base of this paper, we have recalled the basic definitions in the appendix. The appendix also contains other foundational definitions and results that we do our best to cite as we use them.

2.1 Conventions

Here we compile a list of basic notation and conventions that we use throughout this chapter. Note that many of these items are discussed in more detail in the appendix, so we recommend looking there if more information is desired.

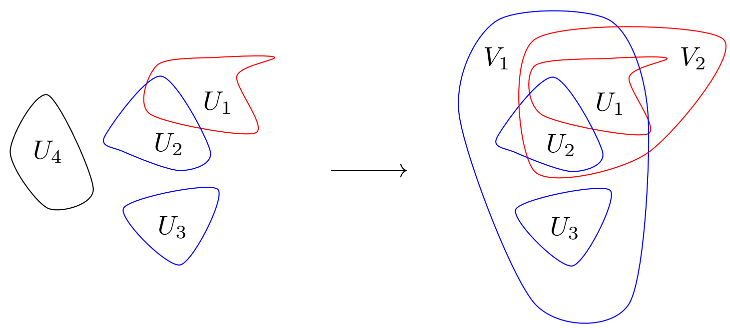
- Fin_* denotes the category of based finite sets and based maps between them.
- $[p]$ dentotes the poset $\{0 < 1 < \cdots < p\}$.
- For $\mathcal{C} \xrightarrow{F} \mathcal{D}$ a functor between categories and $d \in \mathcal{D}$, we let
  - $\mathcal{C}_{/d}$ denote the overcategory consisting of objects $c \in \mathcal{C}$ equipped with a morphism $F(c) \to d$ in $\mathcal{D}$.
  - $\mathcal{C}_{|d}$ denote the fiber of $\mathcal{C}$ over d. This consists of objects $c \in \mathcal{C}$ for which $F(c) = d$.

2.2 Preliminary definitions

Throughout the remainder of this chapter, unless otherwise specified, we will let $\mathcal{V}^{\otimes}$ denote a $\otimes$-presentable symmetric monoidal ∞-category, as in Definition A.10. Note that this is not too restrictive of a condition though, and encompasses the prototypical codomains of factorization algebras. In particular, the ∞-category of chain complexes over a fixed ring is $\otimes$-presentable.

Definition 2.2. For X a topological space, let $\mathsf{open}(X)$ denote the poset of open sets in X with partial order given by inclusion.

The poset $\mathsf{open}(X)$ gives rise to an ∞-operad in a standard way. We denote the resulting ∞-operad by $\mathsf{open}(X)^{\otimes}$. An object in $\mathsf{open}(X)^{\otimes}$ is a pair $(I_+, (U_i))$ consisting of a based finite set I_+ and an I-indexed list of open sets in X. A morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ in $\mathsf{open}(X)^{\otimes}$ is a map of based finite sets $f : I_+ \to J_+$ such that for each $j \in J$, the set $\{U_i \mid i \in f^{-1}(j)\}$ is a collection of pairwise disjoint open subsets of V_j. Below is an example of a morphism in $\mathsf{open}(\mathbb{R}^2)^{\otimes}$ given by the map of based finite sets $\{1, 2, 3, 4\}_+ \to \{1, 2\}_+$ that sends $2, 3 \mapsto 1$, $1 \mapsto 2$, and $4 \mapsto +$.



Remark 2.3. We emphasize the fact that $\mathsf{open}(X)^{\otimes}$ is an ordinary category. Additionally, so is the full ∞-sub-operad $\mathsf{disk}(X)^{\otimes}$ consisting of open sets that are homeomorphic to a disjoint union of open disks, as defined in Defininition 4.2. Throughout the remainder of this section, we will use a number of variations on $\mathsf{open}(X)^{\otimes}$ and $\mathsf{disk}(X)^{\otimes}$. Note that these are also ordinary categories. This is an important fact that enables us to do explicit constructions.

We now make precise the idea that factorization algebras behave nicely with respect to disjoint unions. First, note that disjoint union is only a partially defined operation on $\mathsf{open}(X)$. Indeed, if $U, V \in \mathsf{open}(X)$ such that $U \cap V \neq \emptyset$, then $U \amalg V \notin \mathsf{open}(X)$. The next observation characterizes disjoint unions as a particular class of morphisms in $\mathsf{open}(X)^{\otimes}$.

Observation 2.4. Note that coCartesian morphisms in $\mathsf{open}(X)^{\otimes}$ are of the form

$$(I_+, (U_i)) \xrightarrow{f} \left(J_+, \left(\coprod_{i \in f^{-1}(j)} U_i \right) \right) .$$

Further, these coCartesian morphisms exist precisely when for each $j \in J$, the collection $(U_i)_{i \in f^{-1}(j)}$ is a pairwise disjoint collection of open subsets.

Definition 2.5. We say that a functor of ∞-operads $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ is *multiplicative* if $\mathcal{F}$ carries all coCartesian morphisms in $\mathsf{open}(X)^{\otimes}$ to coCartesian morphisms in $\mathcal{V}^{\otimes}$. Define the ∞-category $\mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}) \subset \mathsf{Fun}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})$ to be the full ∞-subcategory consisting of the multiplicative functors of ∞-operads.

Observation 2.6. Recall that $\mathcal{V}^{\otimes}$ is a symmetric monoidal ∞-category. In particular, this means that $\mathcal{V}^{\otimes} \to \mathsf{Fin}_*$ is a coCartesian fibration. Analogous to Observation 2.4, a coCartesian morphism in $\mathcal{V}^{\otimes}$ is of the form

$$(I_+, (V_i)) \xrightarrow{f} \left(J_+, \left(\bigotimes_{i \in f^{-1}(j)} V_i\right)\right) .$$

In other words, coCartesian morphisms in $\mathcal{V}^{\otimes}$ are given by tensor products.

In light of Observations 2.4 and 2.6, the condition for a functor of ∞-operads

$$\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$$

to be multiplicative is an articulation of the idea that disjoint unions of open sets get carried to tensor products in $\mathcal{V}$.

Next, we describe the type of local-to-global condition that factorization algebras satisfy. There is the standard Grothendieck topology on $\mathsf{open}(X)$ where a cover corresponds to an ordinary open cover. However, this is not the correct form of descent for factorization algebras. As discussed in the introduction (Proposition 1.1), a prototypical example of a factorization algebra on a locally compact topological space X looks like the functor

$$\mathsf{Map_c}(-, Z) : \mathsf{open}(X) \to \mathsf{Spaces} , \qquad U \mapsto \mathsf{Map_c}(U, Z)$$

that sends an open set U to the space of compactly supported maps valued in a pointed $(n-1)$-connected space Z. We use this prototypical example to motivate what local-to-globacl condition factorization algebras satisfy. First observe that this functor $\mathsf{Map_c}(-, Z)$ is *not* a cosheaf with respect to the standard topology on $\mathsf{open}(X)$. Indeed, for $U, V \subset X$ two non-empty disjoint open subsets, consider the canonical maps

$$\mathsf{Map_c}(U, Z) \amalg \mathsf{Map_c}(V, Z) \longrightarrow \mathsf{Map_c}(U \cup V, Z) \longrightarrow \mathsf{Map_c}(U, Z) \times \mathsf{Map_c}(V, Z) .$$

Though the second map is an equivalence, the first map is never an equivalence and therefore $\mathsf{Map_c}(-, Z)$ is *not* a cosheaf with respect to the standard Grothendieck topology on $\mathsf{open}(X)$. Let us outline, nonetheless, what local-to-global condition one might expect $\mathsf{Map_c}(-, Z)$ to satisfy. For starters, fix a compactly supported map $X \xrightarrow{f} Z$. Suppose X is a compact smooth n-manifold, and choose a triangulation of X. Using that Z is $(n-1)$-connected, it is possible to choose a null-homotopy of the restriction $f_{|\mathsf{sk}_{n-1}(X)}$ to the $(n-1)$-skeleton of X. Using this, and using that the inclusion $\mathsf{sk}_{n-1}(X) \hookrightarrow X$ is a cofibration, replace f upto compactly supported homotopy to assume the restriction of f to a neighborhood of $\mathsf{sk}_{n-1}(X) \subset X$ is constant at the base point of Z. Under this assumption, the support of f is contained in a finite disjoint union of Euclidean (open) disks in X, one about the center of each n-simplex. This implies the canonical map

$$\coprod_{D \in \mathsf{disk}(X)} \mathsf{Map_c}(D, Z) \longrightarrow \mathsf{Map_c}(X, Z)$$

is surjective on π_0. In fact, replacing this coproduct by a (homotopy) colimit supplies a map

$$\underset{D \in \mathsf{disk}(X)}{\mathsf{hocolim}} \ \mathsf{Map_c}(D, Z) \longrightarrow \mathsf{Map_c}(X, Z)$$

which is a (weak) homotopy equivalence. (This is non-abelian Poincaré duality (see the main theorem of [AF21] or Corollary 3.6.4 therein, or Corollary 4.8 of [AF20b]), together with a technical result in [AF15] (Proposition 2.19).) As a mapping space can naturally be regarded as a (homotopy) limit, to describe it as a (homotopy) colimit is surprising. Let us further expand on this theme. As X is an n-manifold, $\mathsf{disk}(X) \subset \mathsf{open}(X)$ has the property that, for any finite subset $S \subset X$, there is an element $D \in \mathsf{disk}(X)$ such that $S \subset D$. More generally, for $\mathcal{U} \subset \mathsf{open}(X)$ a collection of opens with this property, then

$$\underset{U \in \mathcal{U}}{\mathsf{hocolim}} \ \mathsf{Map_c}(U, Z) \longrightarrow \mathsf{Map_c}(X, Z) .$$

is a (weak) homotopy equivalence (this follows from Proposition 3.14 of [AF20b]). This is a (homotopy) cosheaf condition with respect to the following non-standard topology on $\mathsf{open}(X)$.

Definition 2.7. Let $U \subset X$ be an open subset of a manifold. We declare a subset $\mathcal{U} \subset \mathsf{open}(X)_{/U} = \mathsf{open}(U)$ to be a *naive* J_∞*-cover* of U if for all finite subsets $S \subset U$, there exists some $U_S \in \mathcal{U}$ such that $S \subset U_S$. A naive J_∞-cover $\mathcal{U}$ is a J_∞*-cover* if for any finite subset $\{U_1, \ldots, U_n\} \subset \mathcal{U}$, the subset $\mathcal{U}_{/U_1 \cap \cdots \cap U_n} \subset \mathsf{open}(U_1 \cap \cdots \cap U_n)$ is a naive J_∞-cover of $U_1 \cap \cdots \cap U_n$. We call the induced topology on $\mathsf{open}(X)$ the J_∞ *(or Weiss) topology.*

Remark 2.8. A naive J_∞-cover determines a J_∞-cover. Indeed, for $\mathcal{U}$ a naive J_∞-cover, the subset consisting of all finite fold intersections of members of $\mathcal{U}$ is a J_∞-cover.

Example 2.9. For M a d-manifold, consider $\mathsf{disk_c}(M) \subset \mathsf{open}(M)$ the subposet consisting of those open subsets $U \subset M$ for which $U \cong \mathbb{R}^d$. While $\mathsf{disk_c}(M)$ is an ordinary cover of M, it is *not* a J_∞-cover of M. In fact, it is not even a naive J_∞-cover.

Recall that the *right cone* of an ∞-category $\mathcal{U}$ is defined by

$$\mathcal{U}^{\triangleright} := \mathcal{U} \times \{0,1\} \coprod_{\mathcal{U}\times\{1\}} * \ .$$

Thus, the objects of $\mathcal{U}^{\triangleright}$ consist of the same objects as $\mathcal{U}$ together with an additional object $*$ that recieves a unique morphism from every other object in $\mathcal{U}$.

Definition 2.10. We call a functor $F : \mathsf{open}(X) \to \mathcal{V}$ a J_∞*-cosheaf* if for all $O \in \mathsf{open}(X)$ and J_∞-covers $\mathcal{U}$ of O, the composite functor

$$\mathcal{U}^{\triangleright} \to \mathsf{open}(X)_{/O} \xrightarrow{\mathsf{fgt}} \mathsf{open}(X) \xrightarrow{F} \mathcal{V}$$

is a colimit diagram. We let $\mathsf{Fun}^{J_\infty}(\mathsf{open}(X), \mathcal{V}) \hookrightarrow \mathsf{Fun}(\mathsf{open}(X), \mathcal{V})$ denote the full ∞-subcategory consisting of those functors that are J_∞-cosheaves.

Definition 2.11. We let $\mathsf{Fun}^{J_\infty,\mathsf{opd}}(\mathsf{open}(X)^\otimes, \mathcal{V}^\otimes) \hookrightarrow \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^\otimes, \mathcal{V}^\otimes)$ denote the full ∞-subcategory consisting of those functors of ∞-operads $\mathcal{F} : \mathsf{open}(X)^\otimes \to \mathcal{V}^\otimes$ for which the restriction $\mathcal{F}_{|_{1_+}} : \mathsf{open}(X)^\otimes_{|_{1_+}} \to \mathcal{V}^\otimes_{|_{1_+}}$ is a J_∞-cosheaf.

We now define the ∞-category of factorization algebras on a topological space X.

Definition 2.12. The ∞-category of *factorization algebras on* X is defined as the pullback

$$\begin{tikzcd}
\mathsf{Fact}_X(\mathcal{V}^\otimes) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}^{\mathsf{m},\mathsf{opd}}(\mathsf{open}(X)^\otimes, \mathcal{V}^\otimes) \arrow[d] \\
\mathsf{Fun}^{J_\infty}(\mathsf{open}(X), \mathcal{V}) \arrow[r, hook] & \mathsf{Fun}(\mathsf{open}(X), \mathcal{V})
\end{tikzcd} .$$

That is, a factorization algebra on X is a functor of ∞-operads

$$\mathcal{F} : \mathsf{open}(X)^\otimes \to \mathcal{V}^\otimes$$

that restricts to a J_∞-cosheaf and that takes coCartesian morphisms in $\mathsf{open}(X)^\otimes$ to coCartesian morphisms in $\mathcal{V}^\otimes$.

There is a special class of factorization algebras that will be of interest in the second half of this paper. These are the locally constant factorization algebras.

Definition 2.13. Let $\mathcal{F} : \mathsf{open}(X)^\otimes \to \mathcal{V}^\otimes$ be a factorization algebra. We say that $\mathcal{F}$ is *locally constant* if the restriction

$$\mathcal{F}_{|_{1_+}} : \mathsf{open}(X) \simeq \mathsf{open}(X)^\otimes_{|_{1_+}} \to \mathcal{V}^\otimes_{|_{1_+}} \simeq \mathcal{V}$$

carries isotopy equivalences of open sets to equivalences in $\mathcal{V}$. We let $\mathsf{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^\otimes) \hookrightarrow \mathsf{Fact}_X(\mathcal{V}^\otimes)$ denote the full ∞-subcategory consisting of the locally constant factorization algebras.

There is another useful way of thinking about the locally constant condition.

Definition 2.14. Consider the wide subcategory $\mathcal{I}(X) \subset \mathsf{open}(X)$ consisting of the same objects as $\mathsf{open}(X)$, but only those morphisms that are isotopy equivalences. Additionally, define the ∞-subcategory $\mathcal{I}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ over Fin_* that consists of the same objects as $\mathsf{open}(X)^{\otimes}$, but only those morphisms $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ that are a bijection of based sets such that for all $i \in I$ the inclusion $U_i \hookrightarrow V_{f(i)}$ is an isotopy equivalence.

In Lemma 2.16 below, we give alternate characterizations of what it means for a factorization algebra to be locally constant. These use the idea of localization of ∞-categories. To prove Lemma 2.16, we make use of the following observation.

Observation 2.15. The inclusion $\mathcal{I}(X) \hookrightarrow \mathcal{I}(X)^{\otimes}$ witnesses $\mathcal{I}(X)^{\otimes}$ as the free ∞-operad on $\mathcal{I}(X)$.

Lemma 2.16. *Let $\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$. The following are equivalent:*

1. *$\mathcal{F}$ is locally constant.*
2. *the induced functor between underlying ∞-categories*

$$\begin{tikzcd} \mathsf{open}(X)^{\otimes}_{|1_+} \arrow[r, "\mathcal{F}_{|1_+}"] \arrow[d] & \mathcal{V}^{\otimes}_{|1_+} \\ \mathsf{open}(X)^{\otimes}_{|1_+}[\mathcal{I}(X)^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

uniquely factors through the localization on isotopy equivalences, $\mathcal{I}(X)$.

3. *$\mathcal{F}$ uniquely factors*

$$\begin{tikzcd} \mathsf{open}(X)^{\otimes} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{V}^{\otimes} \\ \mathsf{open}(X)^{\otimes}[\mathcal{I}(X)^{\otimes -1}] \arrow[ur, dashed] & \end{tikzcd}$$

through the localization about $\mathcal{I}(X)^{\otimes}$.

Proof. The equivalence of conditions 1 and 2 follows immediately from the definition of localization. Condition 3 immediately implies condition 2. We now show that condition 2 implies condition 3. From the definition of localization, the dotted arrow in condition 2 is equivalent to a unique filler

$$\begin{tikzcd} \mathcal{I}(X) \arrow[r] \arrow[d] & \mathsf{open}(X)^{\otimes}_{|1_+} \arrow[r, "\mathcal{F}_{|1_+}"] & \mathcal{V}^{\otimes}_{|1_+} \\ \mathsf{B}\mathcal{I}(X) \arrow[urr, dashed, bend right] & & \end{tikzcd} \tag{7}$$

Now, there is a forgetful functor $(-)_{|1_+} : \mathsf{Op}_\infty \to \mathsf{Cat}_\infty$ from the ∞-category of ∞-operads to the ∞-category of ∞-categories. This functor sends an ∞-operad $\mathcal{O}^{\otimes}$ to its underlying ∞-category $\mathcal{O}^{\otimes}_{|1_+}$. This functor is a right adjoint with left adjoint given by $(-)^{\amalg}$, as in Construction 2.4.3.1 in [Lur17]. Thus we see that

$$\mathsf{open}(X)^{\otimes}_{|1_+} \xrightarrow{\mathcal{F}_{|1_+}} \mathcal{V}^{\otimes}_{|1_+}$$

is the right adjoint $(-)_{|1_+}$ applied to the functor of ∞-operads

$$\mathsf{open}(X)^{\otimes} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes} .$$

As such, a functor $\mathcal{I}(X) \to \mathsf{open}(X)^{\otimes}_{|1_+}$ is equivalent to a functor of ∞-operads $\mathcal{I}(X)^{\amalg} \to \mathsf{open}(X)^{\otimes}$. Similarly, a functor $\mathsf{B}\mathcal{I}(X) \to \mathcal{V}^{\otimes}_{|1_+}$ is equivalent to a functor of ∞-operads $\mathsf{B}\mathcal{I}(X)^{\amalg} \to \mathcal{V}^{\otimes}$. Recall the classifying space B is defined as a left adjoint (Definition A.19). Since left adjoints commute, we can unambiguously write $\mathsf{B}\mathcal{I}(X)^{\amalg}$. Therefore, the diagram in equation (7) is equivalent to the following

$$\begin{tikzcd} \mathcal{I}(X)^{\otimes} \arrow[r, "\simeq", "\text{Observation 2.15}"'] \arrow[d] & \mathcal{I}(X)^{\amalg} \arrow[r] \arrow[d] & \mathsf{open}(X)^{\otimes} \arrow[r, "\mathcal{F}"] & \mathcal{V}^{\otimes} \\ \mathsf{B}\mathcal{I}(X)^{\otimes} \arrow[r, "\simeq", "\text{Observation 2.15}"'] & \mathsf{B}\mathcal{I}(X)^{\amalg} \arrow[urr, dashed] & & \end{tikzcd} . \tag{8}$$

Again, by the definition of localization, this is precisely condition 3 in the statement of this lemma. $\square$

Convention 2.17. For the duration of this paper we will further assume that the unit $\mathbb{1} \in \mathcal{V}$ is initial.

Example 2.18. Let $\mathcal{V}^{\otimes}$ be a symmetric monoidal ∞-category. Denote its symmetric monoidal unit as $\mathbb{1} \in \mathcal{V}$. The ∞-undercategory $\mathcal{V}^{\mathbb{1}})$ canonically inherits a symmetric monoidal structure, $(\mathcal{V}^{\mathbb{1}/})^{\otimes}$, with respect to which the forgetful functor $\mathcal{V}^{\mathbb{1}/} \to \mathcal{V}$ is is canonically symmetric monoidal: $(\mathcal{V}^{\otimes})^{\mathbb{1}/} \to \mathcal{V}^{\otimes}$. This symmetric monoidal ∞-category $(\mathcal{V}^{\otimes})^{\mathbb{1}/}$ abides by Convention 2.17.

In many cases of interest, a native symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ does not abide by Convention 2.17. In such cases, we replace $\mathcal{V}^{\otimes}$ by $(\mathcal{V}^{\otimes})^{\mathbb{1}/}$. This maneuver is validated through the following.

Proposition 2.19 ([Lur17] Proposition 2.3.1.11)**.** *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and let $\mathcal{V}^{\otimes}$ be a symmetric monoidal ∞-category. The forgetful functor $(\mathcal{V}^{\otimes})^{\mathbb{1}/} \to \mathcal{V}^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathit{Fun}^{\mathit{opd}}(\mathcal{O}^{\otimes}, (\mathcal{V}^{\otimes})^{\mathbb{1}/}) \xrightarrow{\simeq} \mathit{Fun}^{\mathit{opd}}(\mathcal{O}^{\otimes}, \mathcal{V}^{\otimes}) \ .$$

Note that both $\mathsf{open}(X)^{\otimes}$ and $\mathsf{disk}(X)^{\otimes}$ (see Definition 4.2 below) are unital, with the empty set $\emptyset$ as the unit. The above proposition then justifies our assumption that the unit $\mathbb{1} \in \mathcal{V}^{\otimes}$ is initial. Below we will also need to work with the ∞-category of bifunctors. We note that a similar statement also holds for bifunctors:

Proposition 2.20. *Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be unital ∞-operads and let $\mathcal{V}^{\otimes}$ be a symmetric monoidal ∞-category. The forgetful functor $(\mathcal{V}^{\otimes})^{\mathbb{1}/} \to \mathcal{V}^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathit{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; (\mathcal{V}^{\otimes})^{\mathbb{1}/}) \xrightarrow{\simeq} \mathit{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

Proof. This follows by the same logic used in the proof of Proposition 2.19 in [Lur17]. In particular, using Lemma 2.3.1.12 therein. $\square$

3 Additivity of factorization algebras

In this section we prove the following additivity statement for factorization algebras. Its proof can be found after Lemma 3.17 below.

Theorem 3.1. *Let X and Y be topological spaces and let $\mathcal{V}^{\otimes}$ be a $\otimes$-presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathit{Fact}_{X \times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathit{Fact}_X(\mathit{Fact}_Y(\mathcal{V}^{\otimes})) \ . \tag{9}$$

Before delving into proving this theorem, we first give a brief outline of the logic involved. The ∞-category $\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes}))$ is the more complicated object in the statement of Theorem 3.1. To understand this ∞-category, we are inspired by the tensor-hom adjunction. A classical version of this adjunction is the following. Let R be a commutative ring. Recall that given two R-modules O and P, we can form a new

R-module $O \otimes_R P$. Now let V be a third R-module. In this setting, the tensor-hom adjunction asserts there is an isomorphism

$$\mathsf{Hom}_R(O \otimes_R P, V) \cong \mathsf{Hom}_R(O, \mathsf{Hom}_R(P, V)) \ .$$

As shown in §2.2.5 of [Lur17], the ∞-category of ∞-operads possess a tensor product. Similarly, for ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$, and a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, there is an equivalence

$$\mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes}, \mathsf{Fun}^{\mathsf{opd}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes})) \ .$$

Recall that an R-linear map $O \otimes_R P \to V$ is the same as an R-bilinear map $O \times P \to V$. We again have an analogous statement for ∞-operads, where the role of bilinear maps is played by bifunctors of ∞-operads. (We refer the unfamiliar reader to Definition A.44 and the surrounding discussion in the appendix.) The statement in this setting is that there is an equivalence of ∞-categories

$$\mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

Thus, we use the theory of bifunctors of ∞-operads to make sense of the ∞-category $\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes}))$.

The proof of Theorem 3.1 now consists of two major components. First, we identify $\mathsf{Fact}_{X\times Y}(\mathcal{V}^{\otimes})$ with $\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$. This is the statement of Proposition 3.16. Then we identify $\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$ with $\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes}))$. This is the statement of Lemma 3.17. Lemma 3.17 involves simply verifying that the aforementioned tensor-hom adjunction respects the two additional conditions of being a factorization algebra: multiplicativity and J_∞-cosheaf. The main content of this section lies in establishing Proposition 3.16.

The beginning of this section mostly addresses more technical issues. In particular, Lemma 3.9, Lemma 3.10, and Corollary 3.11 allow us to reduce our consideration of functors out of the category of open sets to functors out of a simpler category consisting of open sets that have finitely many connected components. Then, Corollary 3.12 provides an explicit formula that enables us to prove the logical crux of Proposition 3.14. We now proceed towards the proof of Theorem 3.1.

Consider the natural bifunctor

$$\rho : \mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes} \to \mathsf{open}(X \times Y)^{\otimes} \ , \quad (I_+, (U_i)), (J_+, (V_j)) \mapsto (I_+ \wedge J_+, (U_i \times V_j)) \ .$$

Restriction along ρ has a left adjoint given formally by left Kan extension

$$\rho_! : \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X \times Y)^{\otimes}, \mathcal{V}^{\otimes}) : \rho^* \ . \tag{10}$$

We provide a formula for how this left Kan extension evaluates in Proposition 3.12 below.

Definition 3.2. Let $\mathcal{F} \in \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$ be a bifunctor.

- We say $\mathcal{F}$ is a *multiplicative bifunctor* if $\mathcal{F}$ takes all pairs of coCartesian morphisms in $\mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}$ to coCartesian morphisms in $\mathcal{V}^{\otimes}$. Let

$$\mathsf{BiFun}^{\mathsf{m}}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$$

  denote the full sub ∞-category consisting of the multiplicative bifunctors.

- We say $\mathcal{F}$ is a *J_∞-bifunctor* if the restriction $\mathcal{F}_{|1_+} \to \mathcal{V}^{\otimes}_{|1_+}$ is a J_∞-cosheaf separately in each variable. Let

$$\mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$$

  denote the full ∞-subcategory consisting of the J_∞-bifunctors.

Proposition 3.16 below establishes that the adjunction in equation (10) restricts as an equivalence

$$\rho_! : \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \rightleftarrows \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{open}(X \times Y)^\otimes, \mathcal{V}^\otimes) : \rho^* \ .$$

Key to the proof of Proposition 3.16 is the colimit expression for left Kan extension along a bifunctor that we provide in Corollary 3.12. To establish this, we use an alternate characterization of multiplicative factorization algebras that we now describe.

Definition 3.3. Let $\mathsf{open}(X)^\otimes_{\mathsf{fin}} \hookrightarrow \mathsf{open}(X)^\otimes$ denote the full ∞-sub-operad consisting of those objects $(I_+, (U_i))$ for which each U_i has finitely many connected components.

Definition 3.4. Let $\mathsf{open}(X)^\otimes_{\mathsf{c}} \hookrightarrow \mathsf{open}(X)^\otimes_{\mathsf{fin}}$ denote the full ∞-sub-operad consisting of those objects $(I_+, (U_i))$ for which each U_i has a single connected component.

Observation 3.5. Note that for connected $U \in \mathsf{open}(X)$, the composite functor

$$\mathsf{open}(X)^\otimes_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(1_+,U)}} \hookrightarrow \mathsf{open}(X)^\otimes_{\mathsf{fin}}{}_{/^{\mathsf{act}}_{(1_+,U)}} \xrightarrow{(-)_!} \mathsf{open}(X)_{\mathsf{fin}/U} \simeq \mathsf{open}(U)_{\mathsf{fin}}$$

is an equivalence. Here, $(-)_!$ denotes the coCartesian monodromy functor which exists in light of Observation 2.4.

Definition 3.6. We define $\mathsf{Fun}^{J_\infty,\mathsf{opd}}(\mathsf{open}(X)^\otimes_{\mathsf{c}}, \mathcal{V}^\otimes) \hookrightarrow \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^\otimes_{\mathsf{c}}, \mathcal{V}^\otimes)$ to be the full ∞-subcategory consisting of those morphisms of operads $\mathcal{F}$ for which for each $U \in \mathsf{open}(X)_{\mathsf{fin}}$, the composite

$$\mathsf{open}(X)_{\mathsf{fin}/U} \xrightarrow{\simeq} \mathsf{open}(X)^\otimes_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(1_+,U)}} \xrightarrow{\mathcal{F}} \mathcal{V}^\otimes_{/1_+} \xrightarrow{\otimes} \mathcal{V} \ ,$$

is a J_∞-cosheaf. Here, the functor

$$\mathcal{V}^\otimes_{/1_+} \xrightarrow{\otimes} \mathcal{V} \ ,$$

takes an object of $\mathcal{V}^\otimes_{/1_+}$ and tensors the preimage of 1 to produce an object of $\mathcal{V}$.

Observation 3.7. Note that $\mathsf{open}(X)_{\mathsf{fin}} \to \mathsf{open}(X)$ is a basis for the J_∞ Grothendieck topology on $\mathsf{open}(X)$. Since we are only interested in J_∞-cosheaves, this observation justifies our restriction to $\mathsf{open}(X)^\otimes_{\mathsf{fin}}$.

There is a functor $\mathsf{expand} : \mathsf{open}(X)^\otimes_{\mathsf{fin}} \to \mathsf{open}(X)^\otimes_{\mathsf{c}}$ given by

$$(I_+, (U_i)) \mapsto \left(\left(\coprod_{i \in I} \pi_0(U_i) \right)_+ , (U_{i_\alpha})_{\alpha \in \pi_0(U_i)} \right) \ ,$$

which expands an I-indexed list of open sets. This functor is right adjoint to the inclusion

$$\iota_{\mathsf{c}} : \mathsf{open}(X)^\otimes_{\mathsf{c}} \rightleftarrows \mathsf{open}(X)^\otimes_{\mathsf{fin}} : \mathsf{expand} \ . \tag{11}$$

Observation 3.8. A morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ in $\mathsf{open}(X)^\otimes_{\mathsf{fin}}$ is coCartesian if and only if $\mathsf{expand}(f)$ is coCartesian in $\mathsf{open}(X)^\otimes_{\mathsf{c}}$. Further, note that the only morphisms in $\mathsf{open}(X)^\otimes_{\mathsf{c}}$ are the identity morphisms.

Lemma 3.9. *Given a functor of ∞-operads $\mathcal{F} : \mathsf{open}(X)^\otimes_{\mathsf{c}} \to \mathcal{V}^\otimes$, there exists a unique multiplicative filler to the following diagram*

$$\begin{tikzcd} \mathsf{open}(X)^\otimes_{c} \arrow[r, "\mathcal{F}"] \arrow[d, "\iota_c"'] & \mathcal{V}^\otimes \\ \mathsf{open}(X)^\otimes_{\mathsf{fin}} \arrow[ur, dashed] & \end{tikzcd} \ .$$

Proof. The counit of the adjunction in equation (11) defines a functor

$$\varepsilon : \mathsf{open}(X)^{\otimes}_{\mathsf{fin}} \to \mathsf{Ar}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}) \ .$$

Define the composite

$$\widetilde{\mathcal{F}} : \mathsf{open}(X)^{\otimes}_{\mathsf{fin}} \xrightarrow{\varepsilon} \mathsf{Ar}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}) \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times_{\mathsf{Fin}_*} \mathsf{Ar}(\mathsf{Fin}_*) \xrightarrow{\mathcal{F}\times \mathsf{id}} \mathcal{V}^{\otimes} \times_{\mathsf{Fin}_*} \mathsf{Ar}(\mathsf{Fin}_*) \xrightarrow{(-)_!} \mathcal{V}^{\otimes} \ .$$

Here, the fiber product $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times_{\mathsf{Fin}_*} \mathsf{Ar}(\mathsf{Fin}_*)$ is taken with respect to the functor $\mathsf{Ar}(\mathsf{Fin}_*) \xrightarrow{\langle s\rangle} \mathsf{Fin}_*$. Using Observation 3.8, one can show that $\widetilde{\mathcal{F}}$ is indeed the unique multiplicative filler to the diagram. □

Lemma 3.10. *There is an equivalence of ∞-categories*

$$\mathit{Fun}^{m,J_\infty,opd}(\mathit{open}(X)^{\otimes}_{\mathit{fin}}, \mathcal{V}^{\otimes}) \xrightarrow[\iota_c^*]{\simeq} \mathit{Fun}^{opd}(\mathit{open}(X)^{\otimes}_{c}, \mathcal{V}^{\otimes}) \ .$$

Furthermore, this equivalence restricts to an equivalence between the J_∞ subcategories

$$\mathit{Fun}^{m,J_\infty,opd}(\mathit{open}(X)^{\otimes}_{\mathit{fin}}, \mathcal{V}^{\otimes}) \xrightarrow[\iota_c^*]{\simeq} \mathit{Fun}^{J_\infty,opd}(\mathit{open}(X)^{\otimes}_{c}, \mathcal{V}^{\otimes}) \ .$$

Proof. Lemma 3.9 defines a functor $\mathsf{Fun}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes}) \to \mathsf{Fun}^{\mathsf{m}}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}, \mathcal{V}^{\otimes})$ which is inverse to the restriction ι_{c}^*. The J_∞ statement is readily verified since $\mathsf{open}(X)^{\otimes}_{\mathsf{c}}$ is a J_∞-basis for $\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}$. □

By applying Lemma 3.10 in each factor, we obtain the following corollary.

Corollary 3.11. *There is an equivalence of ∞-categories*

$$\mathit{BiFun}^{m}(\mathit{open}(X)^{\otimes}_{\mathit{fin}}, \mathit{open}(Y)^{\otimes}_{\mathit{fin}}; \mathcal{V}^{\otimes}) \xrightarrow[\iota_c^*]{\simeq} \mathit{BiFun}(\mathit{open}(X)^{\otimes}_{c}, \mathit{open}(Y)^{\otimes}_{c}; \mathcal{V}^{\otimes}) \ .$$

Furthermore, this equivalence restricts as an equivalence between the J_∞ subcategories

$$\mathit{BiFun}^{m,J_\infty}(\mathit{open}(X)^{\otimes}_{\mathit{fin}}, \mathit{open}(Y)^{\otimes}_{\mathit{fin}}; \mathcal{V}^{\otimes}) \xrightarrow[\iota_c^*]{\simeq} \mathit{BiFun}^{J_\infty}(\mathit{open}(X)^{\otimes}_{c}, \mathit{open}(Y)^{\otimes}_{c}; \mathcal{V}^{\otimes}) \ .$$

In light of Observation 3.7, Lemma 3.10, and Corollary 3.11 we restrict attention to the bifunctor

$$\rho_{\mathsf{c}} : \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}} \to \mathsf{open}(X\times Y)^{\otimes}_{\mathsf{c}}$$

defined in the same way as ρ. Using Proposition A.50, we obtain the following.

Corollary 3.12. *For $\mathcal{F} \in \mathit{BiFun}(\mathit{open}(X)^{\otimes}_{c}, \mathit{open}(Y)^{\otimes}_{c}; \mathcal{V}^{\otimes})$, the left adjoint $(\rho_c)_!\mathcal{F}$ evaluates on $(I_+,(U_i)) \in \mathit{open}(X\times Y)^{\otimes}$ as the colimit*

$$\operatorname{colim}\left(\mathit{open}(X)^{\otimes}_{c} \times \mathit{open}(Y)^{\otimes}_{c}{}_{/(I_+,(U_i))} \to \mathit{open}(X)^{\otimes}_{c} \times \mathit{open}(Y)^{\otimes}_{c}{}_{/I_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/I_+} \xrightarrow{(-)_!} \mathcal{V}^{\otimes}_{|I_+} \right) \ .$$

Further, if $\mathcal{F}$ is J_∞, then $(\rho_c)_!\mathcal{F}$ is J_∞.

Proof. Proposition A.50 applied to

$$\begin{tikzcd} \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}} \arrow[r, "\mathcal{F}"] \arrow[d, "\rho_{\mathsf{c}}"] & \mathcal{V}^{\otimes} \arrow[d] \\ \mathsf{open}(X\times Y)^{\otimes}_{\mathsf{c}} \arrow[r, "\wedge"] & \mathsf{Fin}_*. \end{tikzcd}$$

tells us the colimit expression defines a functor over Fin_*. It remains to check that $(\rho_{\mathsf{c}})_!$ carries inert-coCartesian morphisms to inert-coCartesian morphisms, and that $(\rho_{\mathsf{c}})_!\mathcal{F}$ is J_∞ if $\mathcal{F}$ is a J_∞ bifunctor.

First we will verify that $(\rho_{\mathsf{c}})_!$ carries inert-coCartesian morphisms to inert-coCartesian morphisms. Let $(I_+,(U_i)) \xrightarrow{f} (J_+,(U_j))$ be an inert-coCartesian morphism in $\mathsf{open}(X\times Y)^{\otimes}_{\mathsf{c}}$. Since f is inert and $\mathcal{V}^{\otimes}$ is a coCartesian fibration, the monodromy functor $\mathcal{V}^{\otimes}_{|I_+} \to \mathcal{V}^{\otimes}_{|J_+}$ is given by projection. As such, the monodromy functor preserves colimits. This implies that

$$\begin{aligned}(\rho_{\mathsf{c}})_!\mathcal{F}((I_+,(U_i))) &\simeq \operatorname{colim}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathcal{V}^{\otimes}_{|I_+})\\ &\simeq \operatorname{colim}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathcal{V}^{\otimes}_{|I_+} \xrightarrow{\mathsf{pr}} \mathcal{V}^{\otimes}_{|J_+}) .\end{aligned}$$

Now, observe the commutative diagram

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \arrow[r] \arrow[d] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/I_+} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{V}^{\otimes}_{|I_+} \arrow[d, "\mathsf{pr}"] \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))} \arrow[r] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/J_+} \arrow[r, "\mathcal{F}"] & \mathcal{V}^{\otimes}_{|J_+}
\end{tikzcd} .$$

This implies $(\rho_{\mathsf{c}})_!\mathcal{F}((I_+,(U_i)))$ is equivalent to

$$\operatorname{colim}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))} \to \mathcal{V}^{\otimes}_{|J_+}) .$$

Finally, we use Quillen's Theorem A (Theorem A.61) to show the functor

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))}$$

is final. The hypothesis of Theorem A.61 requires us to verify that for any object $\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+ \xrightarrow{\alpha} J_+\big)$ in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))}$, the classifying space of the undercategory

$$\Big(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)^{\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+ \xrightarrow{\alpha} J_+\big)/} \tag{12}$$

is contractible. To see this, note the assumption that $f: I_+ \to J_+$ is inert allows us to define a map $K_+\wedge L_+ \xrightarrow{\beta} I_+$ via $(k,\ell)\mapsto f^{-1}(\alpha(k,\ell))$. The object $\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+ \xrightarrow{\beta} I_+\big)$ is then seen to be initial in the undercategory of equation (12). Hence its classifying space is contractible by Observation A.21. This completes the proof that $(\rho_{\mathsf{c}})_!\mathcal{F}$ is a functor of ∞-operads.

Now, for $\mathcal{F} \in \mathsf{BiFun}^{\mathcal{J}_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}},\mathsf{open}(Y)^{\otimes}_{\mathsf{c}};\mathcal{V}^{\otimes})$, we will show that $(\rho_{\mathsf{c}})_!\mathcal{F} \in \mathsf{Fun}^{\mathcal{J}_\infty,\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}_{\mathsf{c}},\mathcal{V}^{\otimes})$. Note that products of open sets form a basis for $\mathsf{open}(X\times Y)$. Thus, we only need to check this statement holds for products. This is precisely what is shown in Proposition 3.14 below. $\square$

In Proposition 3.14 below, we verify that $(\rho_{\mathsf{c}})_!\mathcal{F}(1_+,U\times V) \simeq \mathcal{F}((1_+,U),(1_+,V))$. Corollary 3.12 tells us that we can compute $(\rho_{\mathsf{c}})_!\mathcal{F}(1_+,U\times V)$ as the colimit of the following functor

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+,U\times V)} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/1_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/!_+} \xrightarrow{(-)_!} \mathcal{V} . \tag{13}$$

The general strategy of the proof of Proposition 3.14 exploits some additional functoriality coming from the unitality of $\mathcal{V}^{\otimes}$. We do this by extending the domain of the functor in equation (13) to a larger category. Using this extension, we then compute $(\rho_c)_!\mathcal{F}$ as an iterated left Kan extension. First, we lay out some necessary definitions.

Define the category $\mathsf{Fin}_* \times \mathsf{Fin}_{*/^{\mathsf{min}}_{1_+}}$ to consist of the same objects as $\mathsf{Fin}_* \times \mathsf{Fin}_{*/1_+}$ but with a morphism $(I_+,J_+,I_+\wedge J_+ \xrightarrow{f} 1_+) \to (K_+,L_+,K_+\wedge L_+ \xrightarrow{g} 1_+)$ given by a map of based finite sets $I_+\wedge J_+ \xrightarrow{\alpha} K_+\wedge L_+$ such that the following conditions hold:

1. the diagram

$$\begin{tikzcd} I_+ \wedge J_+ \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & K_+ \wedge L_+ \arrow[dl, "g"] \\ & 1_+ & \end{tikzcd}$$

commutes;

2. the resulting containment $f^{-1}(1) \subseteq \alpha^{-1}(K \times L)$ is entire: $f^{-1}(1) \subseteq \alpha^{-1}(K \times L)$;

3. for all $(i, j) \in \alpha^{-1}(K \times L)$, the projection $\mathsf{pr}_K(\alpha(i, j))$ is independent of j, and the projection $\mathsf{pr}_L(\alpha(i, j))$ is independent of i.

For notational purposes, let us now define

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{1_+}} := \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}} \times_{\mathsf{Fin}_*} \mathsf{Fin}_* \times \mathsf{Fin}_*{}_{/^{\mathsf{min}}_{1_+}} .$$

Definition 3.13. We define the category $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U \times V)}}$ to consist of the same objects as $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+, U \times V)}$ but with a morphism

$$\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \to \big((K_+, (D_k)), (L_+, (E_\ell)), K_+ \wedge L_+ \xrightarrow{g} 1_+\big)$$

given by a morphism $\alpha \in \mathsf{Fin}_* \times \mathsf{Fin}_*{}_{/^{\mathsf{min}}_{1_+}}$ satisfying the following conditions for all $(k, \ell) \in K \times L$:

1. the diagram

$$\begin{tikzcd} I_+ \wedge J_+ \arrow[rr, "\alpha"] \arrow[dr, "f"] & & K_+ \wedge L_+ \arrow[dl, "g"] \\ & 1_+ & \end{tikzcd}$$

commutes;

2. $\alpha^{-1}(K \times L) = f^{-1}(1)$;

3. for all $(i, j) \in \alpha^{-1}(K \times L)$, the projection $\mathsf{pr}_K(\alpha(i, j))$ is independent of j, and the projection $\mathsf{pr}_L(\alpha(i, j))$ is independent of i.

4. the collection of the sets A_i indexed over all $i \in I$ for which there exists $j \in J$ such that $\alpha(i, j) = (k, \ell)$ form a pairwise disjoint collection of open subsets of D_k;

5. the collection of the sets B_j indexed over all $j \in J$ for which there exists $i \in I$ such that $\alpha(i, j) = (k, \ell)$ form a pairwise disjoint collection of open subsets of E_ℓ;

The idea of the category $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U \times V)}}$ is that it selects out 'minimal' morphisms between partial grids in the open set $U \times V$. We give some visual intuition for what we mean by this below. Namely, condition 2 eliminates flexibility of the underlying morphisms of based finite sets by forcing the morphism of sets to collapse everything possible to the basepoint. This is illustrated in Figure 2. Condition 3 ensures that grids do not get split up, as illustrated in Figure 3. Conditions 4 and 5 ensure that we can only include grids, as illustrated in Figure 4.

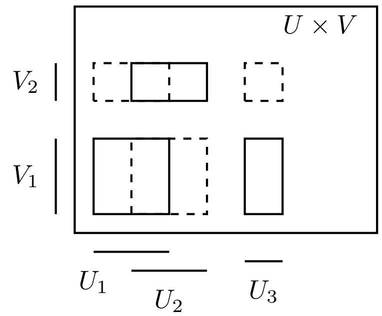


Figure 1: A typical object in $\mathsf{open}(\mathbb{R})_{\mathsf{c}}^{\otimes} \times \mathsf{open}(\mathbb{R})_{\mathsf{c}}^{\otimes}{}_{/(1_+, U\times V)}$. We have indicated the map $\{1,2,3\}_+ \wedge \{1,2\}_+ \to 1_+$ by the style of line used. Namely, those boxes that are sent to 1 are styled with solid boundary, and those that are sent to $+$ are styled with dashed boundary.

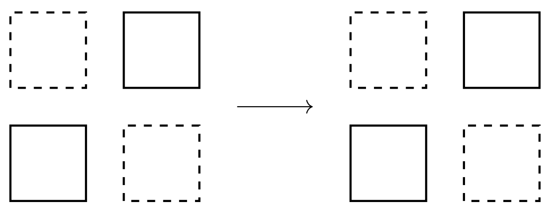


Figure 2: Condition 2) of Definition 3.13 disallows, for example, the identity morphism on underlying finite sets such as the one indicated.

Proposition 3.14. *For $\mathcal{F} \in \mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)_{\mathsf{c}}^{\otimes}, \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}; \mathcal{V}^{\otimes})$ and connected open subsets $U \in \mathsf{open}(X)_{\mathsf{c}}$ and $V \in \mathsf{open}(Y)_{\mathsf{c}}$, there is an equivalence in $\mathcal{V}$*

$$\mathcal{F}((1_+, U), (1_+, V)) \xrightarrow{\simeq} (\rho_c)_! \mathcal{F}(1_+, U \times V)\ .$$

Proof. Observe the evident functor

$$\Phi : \mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/(1_+, U\times V)} \to \mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}}\ .$$

To define the aforementioned extension, observe the solid commutative diagram

$$\begin{tikzcd}
\mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/(1_+, U\times V)} \arrow[r] \arrow[d, "\Phi"] & \mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/1_+} \arrow[r, "\mathcal{F}_{/1_+}"] \arrow[d] & \mathcal{V}^{\otimes}_{/1_+} \arrow[r, "(-)_!"] & \mathcal{V} \\
\mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}} \arrow[r] & \mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/^{\mathsf{min}}_{1_+}} \arrow[ur, dashed, "\mathcal{F}^{\mathsf{min}}_{/1_+}"'] & &
\end{tikzcd}\ .$$

We now define the dashed arrow, $\mathcal{F}^{\mathsf{min}}_{/1_+}$. On objects, $\mathcal{F}^{\mathsf{min}}_{/1_+}$ evaluates the same as $\mathcal{F}_{/1_+}$. Consider a morphism

$$\big((I_+, (A_i)), (J_+, (B_j)), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \xrightarrow{\alpha} \big((K_+, (D_k)), (L_+, (E_\ell)), K_+ \wedge L_+ \xrightarrow{g} 1_+\big)$$

in $\mathsf{open}(X)_{\mathsf{c}}^{\otimes} \times \mathsf{open}(Y)_{\mathsf{c}}^{\otimes}{}_{/^{\mathsf{min}}_{1_+}}$. We define a morphism in $\mathcal{V}^{\otimes}_{/1_+}$ between their images as the following composite

$$(\mathcal{F}(A_i, B_j))_{I\times J} \xrightarrow{\mathsf{inert}} (\mathcal{F}(A_i, B_j))_{f^{-1}(1)} \xrightarrow{(-)_!} \left(\bigotimes_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j)\right)_{K\times L} \to (\mathcal{F}(D_k, E_\ell))_{K\times L}\ . \qquad (14)$$

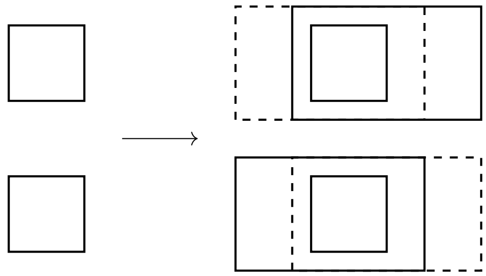


Figure 3: Condition 3) of Definition 3.13 disallows, for example, a morphism that splits up gridded elements such as the one indicated.

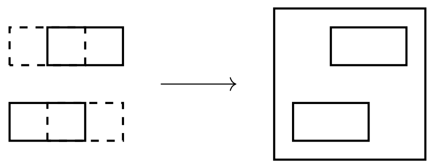


Figure 4: Condition 4) of Definition 3.13 disallows, for example, an inclusion of overlapping elements in either axis, such as the one indicated.

The middle morphism is the coCartesian monodromy functor applied to the morphism of based finite sets $f^{-1}(1) \xrightarrow{\alpha_|} K \times L$. We now describe the third morphism in equation (14). This is a morphism over the identity $K \times L \to K \times L$, so we describe this morphism for each $(k, \ell) \in K \times L$. Fix $(k, \ell) \in K \times L$. If $\alpha^{-1}(k, \ell) = \emptyset$, then we take the empty tensor product

$$\bigotimes_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j)$$

to be the unit $\mathbb{1} \in \mathcal{V}$. By the assumed initiality of the unit (Convention 2.17), there is a unique morphism $\mathbb{1} \to \mathcal{F}(D_k, E_\ell)$. If $\alpha^{-1}(k, \ell) \neq \emptyset$ then we claim there is a morphism in $\mathcal{V}$

$$\bigotimes_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j) \to \mathcal{F}(D_k, E_\ell) \ .$$

This follows precisely from the conditions that we placed on morphisms in the overcategory $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{1_+}}$.

Now, $(\rho_{\mathsf{c}})_!\mathcal{F}(1_+, U \times V)$ is given by the colimit of the top composite arrow in the following diagram

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+, U\times V)} \arrow[r] \arrow[d, "\Phi"] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/1_+} \arrow[r, "\mathcal{F}_{/1_+}"] & \mathcal{V}^{\otimes}_{/1_+} \arrow[r, "(-)_!"] & \mathcal{V} \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}} \arrow[d, "!"] & & & \\
* \, . & & &
\end{tikzcd}$$

We will henceforth refer to this top composite functor as $\mathcal{F}_{/1_+}$. Note that the colimit of $\mathcal{F}_{/1_+}$ is equivalent to the left Kan extension of $\mathcal{F}_{/1_+}$ along the unique functor to $*$. By Proposition A.48, this is equivalent to the colimit of the left Kan extension $\Phi_!\mathcal{F}_{/1_+}$. That is,

$$(\rho_{\mathsf{c}})_!\mathcal{F}(1_+, U \times V) \simeq \operatorname{colim} \Phi_!\mathcal{F}_{/1_+} \ .$$

Next, we will identify our constructed extension $\mathcal{F}^{\mathsf{min}}_{/1_+}$ with $\Phi_!\mathcal{F}_{/1_+}$. To do this, we construct a section $\mathcal{F}^{\mathsf{min}}_{/1_+} \xrightarrow{\Psi} \Phi_!\mathcal{F}_{/1_+}$ under $\mathcal{F}_{/1_+}$. By Proposition A.34, the coCartesian monodromy functor $\mathcal{V}^{\otimes}_{/1_+} \to \mathcal{V}$ is a left adjoint and thus preserves colimits. Thus, we will work in $\mathcal{V}^{\otimes}_{/1_+}$. On objects $\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big)$, the definition of the section is clear since $\mathcal{F}^{\mathsf{min}}_{/1_+}$ evaluates the same as $\mathcal{F}_{/1_+}$. There is a natural morphism in $\mathcal{V}^{\otimes}_{/1_+}$

$$\mathcal{F}_{/1_+}\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \to \Phi_!\mathcal{F}_{/1_+}\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big)$$

since left Kan extensions are defined as initial extensions. Now, consider a morphism

$$\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \xrightarrow{\alpha} \big((K_+, (D_k)), (L_+, (E_\ell)), K_+ \wedge L_+ \xrightarrow{g} 1_+\big)$$

in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}}$. Observe the factorization in $\mathcal{V}^{\otimes}_{/1_+}$ from equation (14) also allows us the following factorization

$$\begin{tikzcd}
\big(\mathcal{F}(A_i, B_j)\big)_{I\times J} \arrow[r, "\mathsf{inert}"] \arrow[d, "\Psi"] & \big(\mathcal{F}(A_i, B_j)\big)_{f^{-1}(1)} \arrow[r, "(-)_!"] \arrow[d] & \left(\bigotimes\limits_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j)\right)_{K\times L} \arrow[r] \arrow[d, "\mathsf{canon}"] & \big(\mathcal{F}(D_k, E_\ell)\big)_{K\times L} \arrow[d, "\Psi"] \\
\big(\Phi_!\mathcal{F}_{/1_+}(A_i, B_j)\big)_{I\times J} \arrow[r, "\mathsf{inert}"] & \big(\Phi_!\mathcal{F}_{/1_+}(A_i, B_j)\big)_{f^{-1}(1)} \arrow[r, "(-)_!"] & \left(\bigotimes\limits_{(i,j)\in\alpha^{-1}(k,\ell)} \Phi_!\mathcal{F}_{/1_+}(A_i, B_j)\right)_{K\times L} \arrow[r] & \big(\Phi_!\mathcal{F}_{/1_+}(D_k, E_\ell)\big)_{K\times L} .
\end{tikzcd}$$

Note the bottom left object is equivalent to

$$\Phi_!\mathcal{F}_{/1_+}((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+) \ .$$

Similarly, the bottom right object is equivalent to

$$\Phi_!\mathcal{F}_{/1_+}((K_+, (D_k)), (L_+, (E_\ell)), K_+ \wedge L_+ \xrightarrow{g} 1_+) \ .$$

The two left-most squares commute because coCartesian monodromy is a functor. We now explain why the right hand square commutes. Note that both right-most horizontal arrows come from morphisms in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+, U\times V)}$. Further, the vertical arrow labeled canon is from the definition of left Kan extension as the initial extension of $\mathcal{F}_{/1_+}$. Thus, the right hand square commutes since it is the evaluation of the canonical natural transformation from the definition of left Kan extension. This completes the construction of the natural transformation

$$\Psi : \mathcal{F}^{\mathsf{min}}_{/1_+} \to \Phi_!\mathcal{F}_{/1_+} \ .$$

Note that this evidently lies under $\mathcal{F}_{/1_+}$, and clearly defines a section of $\Phi_!\mathcal{F}_{/1_+}$. By initiality of the left Kan extension, this implies the equivalence $\Phi_!\mathcal{F}_{/1_+} \simeq \mathcal{F}^{\mathsf{min}}_{/1_+}$. Therefore, we have shown

$$(\rho_{\mathsf{c}})_!\mathcal{F}(1_+, U \times V) \simeq \operatorname{colim} \Phi_!\mathcal{F}_{/1_+} \simeq \operatorname{colim} \mathcal{F}^{\mathsf{min}}_{/1_+} \ . \tag{15}$$

We will now show that $\mathcal{F}((1_+,U),(1_+,V)) \simeq \operatorname{colim} \mathcal{F}^{\mathsf{min}}_{/1_+}$, which will complete the proof. Consider the object $\big((1_+,U),(1_+,V),\mathsf{id}\big) \in \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}$. Observe the forgetful functor

$$\nabla : \left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}} .$$

We will show the counit $\nabla_!\nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+} \to \mathcal{F}^{\mathsf{min}}_{/1_+}$ is an equivalence. Before doing so, we explain why this implies $\mathcal{F}((1_+,U),(1_+,V)) \simeq (\rho_{\mathsf{c}})_!\mathcal{F}(1_+,U\times V)$. Note that the category

$$\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})}$$

has a final object $((1_+,U),(1_+,V),\mathsf{id})$. This can be verified using Quillen's Theorem A (Theorem A.61). Thus,

$$\operatorname{colim} \nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+} \simeq \mathcal{F}^{\mathsf{min}}_{/1_+}((1_+,U),(1_+,V),\mathsf{id}) \simeq \mathcal{F}((1_+,U),(1_+,V)) .$$

Further, since left Kan extensions compose by Proposition A.48, we have

$$\operatorname{colim} \nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+} \simeq \operatorname{colim} \nabla_!\nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+} .$$

Using equation (15), upon taking the colimit of the counit, we have

$$\mathcal{F}((1_+,U),(1_+,V)) \simeq \operatorname{colim} \nabla_!\nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+} \to \operatorname{colim} \mathcal{F}^{\mathsf{min}}_{/1_+} \simeq (\rho_{\mathsf{c}})_!\mathcal{F}(1_+,U\times V) .$$

Finally, we now complete the proof by showing the counit $\nabla_!\nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+} \to \mathcal{F}^{\mathsf{min}}_{/1_+}$ is an equivalence. To do this, we will use the assumption that $\mathcal{F}$ is a J_∞-cosheaf in each factor. We show this using Lemma 3.15 below.

First, note that for

$$((I_+,(A_i)),(J_+,(B_j)),I_+\wedge J_+ \xrightarrow{f} 1_+) \in \mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}_{/^{\mathsf{min}}_{(1_+,U\times V)}} ,$$

the functor $\nabla_!\nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+}$ evaluates as the colimit of the composite

$$\begin{array}{c}
\left(\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})}\right)_{/((I_+,(A_i)),(J_+,(B_j)),I_+\wedge J_+\xrightarrow{f}1_+)} \\
\downarrow \\
\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})} \\
\downarrow{\scriptstyle\nabla} \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}} \\
\downarrow{\scriptstyle\mathcal{F}^{\mathsf{min}}_{/1_+}} \\
\mathcal{V} .
\end{array}$$

Further, since $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}$ is a subposet of $\mathsf{open}(U) \times \mathsf{open}(V)$, so is the domain of this composite. Now, we verify that

$$\left(\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})}\right)_{/((I_+,(A_i)),(J_+,(B_j)),I_+\wedge J_+\xrightarrow{f}1_+)} \tag{16}$$

satisfies the hypotheses of Lemma 3.15. First, note that the projection onto X of the overcategory in equation (16) is simply identified as $\mathsf{open}_{\mathsf{fin}}(\amalg A_i)$. Next, consider the full subposet $\mathsf{open}_{\mathsf{surj}}(\amalg A_i) \subset \mathsf{open}_{\mathsf{fin}}(\amalg A_i)$ consisting of those open sets that are surjective on connected components. This subposet of $\mathsf{open}(\amalg A_i)$ is a J_∞-cover. Using Theorem A.61, this full subposet is checked to be final. Indeed, for $(U \subseteq \amalg A_i) \in \mathsf{open}_{\mathsf{fin}}(\amalg A_i)$, consider the subset $I_U := \{i' \in I \mid A_{i'} \cap U = \emptyset\} \subseteq I$. Then $(U \cup \coprod_{i' \in I_U} A_{i'}) \in \mathsf{open}_{\mathsf{surj}}(\amalg A_i)^{U/}$ is a final object. Next, let $D \in \mathsf{open}_{\mathsf{surj}}(\amalg A_i)$. Consider the F_D of equation (16) over D. Projection onto the Y-factor determines a functor $F_D \to \mathsf{open}(\amalg B_j)$ between posets. Let $S \subset \amalg B_j$ be a finite subset. If the induced map $S \to \pi_0(\amalg B_j)$ is not surjective, choose a finite superset $S \subset S' \subset \amalg B_j$ such that the induced map $S' \to \pi_0(\amalg B_j)$ is a surjection. For each $y \in S'$, choose an open disk $E_y \subset \amalg B_j$ such that $y \in E_y$. Using that S' is finite, shrink each E_y such that $y \neq y'$ implies $E_y \cap E_{y'} = \emptyset$. Denote the S'-indexed disjoint union $E := \amalg E_y$. So there is an inclusion $S' \subset \amalg E_y$ such that the induced map on components $S' \to \pi_0(E)$ is a bijection. Using that $D \hookrightarrow \amalg A_i$ is surjective on components, there exists a minimal subset of $\pi_0(D) \times \pi_0(E)$. Such a minimal subset corresponds, in particular, to a selection of components of $D \times E$ whose projection to E is entire. Furthermore, such a minimal subset defines an object in F_D that is carried by the functor $F_D \to \mathsf{open}(\amalg B_j)$ to E. Therefore, the functor $F_D \to \mathsf{open}(\amalg B_j)$ is a J_∞-cover. Finally, observe that the functor from the poset (16) to the poset $\mathsf{open}_{\mathsf{fin}}(\amalg A_i)$ given by projection onto the X-factor is a coCartesian fibration (as in Definition A.32). Therefore, the base-change along $\mathsf{open}_{\mathsf{surj}}(\amalg A_i) \hookrightarrow \mathsf{open}_{\mathsf{fin}}(\amalg A_i)$ remains a coCartesian fibration. So Lemma 3.15 applies. □

Lemma 3.15. *Let $U \in \mathsf{open}(X)_c$ and $V \in \mathsf{open}(Y)_c$. Assume $\mathcal{F} : \mathsf{open}(U) \times \mathsf{open}(V) \to \mathcal{V}$ is a J_∞-cosheaf in each factor. For $\mathcal{U} \subset \mathsf{open}(U) \times \mathsf{open}(V)$ a full subposet, let $\mathcal{U}_U := \mathsf{pr}_U \mathcal{U} \subset \mathsf{open}(U)$ denote the full subposet given by projection onto U. If*

1. *$\mathcal{U}_U$ is a J_∞ cover,*
2. *there exists a final full subcategory $\mathcal{U}_U^0 \subset \mathcal{U}_U$ such that for each $D \in \mathcal{U}_U^0$, the fiber $\mathcal{U}_{|D} \subset \mathsf{open}(V)$ is a J_∞-cover, and*
3. *the functor $\mathcal{U}_{|\mathcal{U}_U^0} \to \mathcal{U}_U^0$ is a coCartesian fibration,*

then

$$\operatorname{colim}\Big(\mathcal{U} \hookrightarrow \mathsf{open}(U) \times \mathsf{open}(V) \xrightarrow{\mathcal{F}} \mathcal{V}\Big) \simeq \mathcal{F}(U, V) \ .$$

Proof. Note the functor $\mathcal{U} \xrightarrow{\mathsf{pr}} \mathcal{U}_U$. By Proposition A.48 and the assumption that $\mathcal{U}_U^0$ is final, we have

$$\operatorname{colim}\Big(\mathcal{U} \hookrightarrow \mathsf{open}(U) \times \mathsf{open}(V) \xrightarrow{\mathcal{F}} \mathcal{V}\Big) \simeq \operatorname{colim}(\mathsf{pr}_! \mathcal{F}) \simeq \operatorname{colim}\Big(\mathcal{U}_U^0 \hookrightarrow \mathcal{U}_U \xrightarrow{\mathsf{pr}_! \mathcal{F}} \mathcal{V}\Big) \ .$$

By assumption, $\mathcal{U}_{|\mathcal{U}_U^0} \to \mathcal{U}_U^0$ is a coCartesian fibration. Therefore, by Proposition A.49, the left Kan extension can be computed as a fiberwise colimit. That is, the functor

$$\mathcal{U}_U^0 \hookrightarrow \mathcal{U}_U \xrightarrow{\mathsf{pr}_! \mathcal{F}} \mathcal{V}$$

evaluates on $D \in \mathcal{U}_U^0$ as $D \mapsto \operatorname{colim}\Big(\mathcal{U}_{|D} \xrightarrow{\mathcal{F}} \mathcal{V}\Big) \simeq \mathcal{F}(D, V)$ The equivalence is the assumption that $\mathcal{F}$ is a J_∞-cosheaf in the second factor. Thus, since $\mathcal{F}$ is assumed to be a J_∞-cosheaf in the first factor, taking the colimit of the fiberwise evaluations is equivalent to $\mathcal{F}(U, V)$, as desired. □

Proposition 3.16. *Let $\mathcal{V}^\otimes$ be a $\otimes$-presentable ∞-category. The adjunction in equation (10) restricts to an equivalence of ∞-categories*

$$\rho_! : \mathsf{BiFun}^{m, J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathsf{Fact}_{X \times Y}(\mathcal{V}^\otimes) : \rho^* \ .$$

Proof. Observe the commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}_{\mathsf{fin}},\mathcal{V}^{\otimes}) \arrow[r] \arrow[d, "\rho^*"] & \mathsf{Fun}^{J_\infty,\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}_{\mathsf{c}},\mathcal{V}^{\otimes}) \arrow[d, "\rho_{\mathsf{c}}^*"] \\
\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}\times\mathsf{open}(Y)^{\otimes}_{\mathsf{fin}},\mathcal{V}^{\otimes}) \arrow[r] & \mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{open}(Y)^{\otimes}_{\mathsf{c}},\mathcal{V}^{\otimes})
\end{tikzcd} . \tag{17}$$

Lemma 3.10 established that the top horizontal functor is an equivalence. Corollary 3.11 established that the bottom horizontal functor is an equivalence. Thus, it suffices to show that ρ_{c}^* is an equivalence. To prove this, we show that the unit and counit of the adjunction evaluate as equivalences.

First, we show the counit $(\rho_{\mathsf{c}})_!\rho_{\mathsf{c}}^*\mathcal{F}\to\mathcal{F}$ evaluates as an equivalence. Since $\mathsf{pr}:\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ preserves colimits, we have the equivalence $(\rho_{\mathsf{c}})_!\rho_{\mathsf{c}}^*\mathcal{F}(I_+,(U_i))\simeq(I_+,((\rho_{\mathsf{c}})_!\rho_{\mathsf{c}}^*\mathcal{F}(1_+,U_i)))$. Further, since $(\rho_{\mathsf{c}})_!\rho_{\mathsf{c}}^*\mathcal{F}$ is a J_∞-cosheaf, and products of open sets form a basis for the J_∞-topology on $\mathsf{open}(X\times Y)$, its values are determined by its values on products of opens. Proposition 3.14 establishes this.

Similarly, for $\mathcal{G}\in\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}},\mathsf{open}(Y)^{\otimes}_{\mathsf{c}};\mathcal{V}^{\otimes})$, we can use the fact that $\rho_{\mathsf{c}}^*(\rho_{\mathsf{c}})_!\mathcal{G}$ is J_∞ in each factor to reduce the computation to an evaluation on unary objects. Again, since products of opens form a basis, we invoke Proposition 3.14 to show the unit evaluates as equivalences. $\square$

There is a natural functor

$$\mathsf{ev}:\mathsf{BiFun}(\mathsf{open}(X)^{\otimes},\mathsf{open}(Y)^{\otimes};\mathcal{V}^{\otimes})\to\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes},\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes},\mathcal{V}^{\otimes})^{\otimes})\ , \tag{18}$$

given by evaluation. As developed in §2.2.5 of [Lur17], this functor is an equivalence. More precisely, by definition a functor $\mathcal{K}\to\mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes},\mathcal{P}^{\otimes})$ is a dashed arrow filling the diagram

$$\begin{tikzcd}
\mathcal{K}\times\mathcal{O}^{\otimes} \arrow[r, dashed] \arrow[d, "\mathsf{pr}"] & \mathcal{P}^{\otimes} \arrow[dd] \\
\mathcal{O}^{\otimes} \arrow[d] & \\
\mathsf{Fin}_* \arrow[r, "\mathsf{id}"] & \mathsf{Fin}_*
\end{tikzcd} .$$

For $\mathcal{K}\to\mathsf{Fin}_*$ and $\mathcal{V}^{\otimes}$ a symmetric monoidal ∞-category, a functor $\mathcal{K}\to\mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes},\mathcal{V}^{\otimes})^{\otimes}$ over Fin_* is a dashed arrow filling

$$\begin{tikzcd}
\mathcal{K}\times\mathcal{O}^{\otimes} \arrow[r, dashed] \arrow[d] & \mathcal{V}^{\otimes} \arrow[dd] \\
\mathsf{Fin}_*\times\mathsf{Fin}_* \arrow[d, "\wedge"] & \\
\mathsf{Fin}_* \arrow[r] & \mathsf{Fin}_*
\end{tikzcd} .$$

Thus, a functor

$$\mathsf{BiFun}(\mathsf{open}(X)^{\otimes},\mathsf{open}(Y)^{\otimes};\mathcal{V}^{\otimes})\to\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes},\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes},\mathcal{V}^{\otimes})^{\otimes})$$

is a filler

$$\begin{tikzcd}
\big(\mathsf{BiFun}(\mathsf{open}(X)^{\otimes},\mathsf{open}(Y)^{\otimes};\mathcal{V}^{\otimes})\times\mathsf{open}(X)^{\otimes}\big)\times\mathsf{open}(Y)^{\otimes} \arrow[r, dashed] \arrow[d, "\mathsf{pr}"] & \mathcal{V}^{\otimes} \arrow[dd] \\
\mathsf{open}(X)^{\otimes}\times\mathsf{open}(Y)^{\otimes} \arrow[d] & \\
\mathsf{Fin}_*\times\mathsf{Fin}_* \arrow[r, "\wedge"] & \mathsf{Fin}_*
\end{tikzcd} .$$

The natural such filler is what we are denoting by ev.

Lemma 3.17. *The equivalence in equation (18) restricts as an equivalence*

$$\mathit{ev} : \mathit{BiFun}^{\mathsf{m},J_\infty}(\mathit{open}(X)^\otimes, \mathit{open}(Y)^\otimes; \mathcal{V}^\otimes) \to \mathit{Fact}_X(\mathit{Fact}_Y(\mathcal{V}^\otimes)) \ .$$

Proof. Since equation (18) is an equivalence, it suffices to show

$$\mathsf{ev}^{-1}(\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))) \simeq \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \ .$$

So consider

$$\mathcal{G} \in \mathsf{ev}^{-1}(\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))) \subset \mathsf{BiFun}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \ .$$

Since $\mathsf{ev}\mathcal{G} \in \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))$ is a factorization algebra in each factor, it folllows directly that $\mathcal{G}$ takes pairs of coCartesian morphisms to coCartesian morphisms. Also, it follows that $\mathcal{G}$ restricts as a J_∞-cosheaf in each factor. In other words, $\mathcal{G} \in \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$.

It remains to show that if $\mathcal{G} \in \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$, then $\mathsf{ev}\mathcal{G} \in \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))$. Note that $\mathcal{G}$ being multiplicative is equivalent to saying that it takes pairs of morphisms of the form (id, α) and (α, id) to coCartesian morphisms, for α a coCartesian morphism. This implies that $\mathsf{ev}\mathcal{G} \in \mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(X)^\otimes, \mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(Y)^\otimes, \mathcal{V}^\otimes))$. The J_∞ condition on the Y variable follows immediately. Thus, we have $\mathsf{ev}\mathcal{G} \in \mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(X)^\otimes, \mathsf{Fact}_Y(\mathcal{V}^\otimes))$. The J_∞ condition on the X variable is a little subtle. This is because it requires taking a colimit in the ∞-category $\mathsf{Fact}_Y(\mathcal{V}^\otimes)$. It's not clear colimits in this category look like, or if they even exist. To bypass this difficulty, note that

$$\mathsf{Fact}_Y(\mathcal{V}^\otimes) \hookrightarrow \mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(Y)^\otimes, \mathcal{V}^\otimes) \tag{19}$$

is a full ∞-subcategory. Therefore, we can compute the colimit in $\mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(Y)^\otimes, \mathcal{V}^\otimes)$ and then check that this colimit satisfies the J_∞ condition. Since the functor in equation (19) is full, this implies that the colimit in $\mathsf{Fact}_Y(\mathcal{V}^\otimes)$ does exist, and is equivalent to the colimit as computed in $\mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{open}(Y)^\otimes, \mathcal{V}^\otimes)$. □

We now use the results established in this section to prove Theorem 3.1 stated at the beginning of this section.

Proof of Theorem 3.1. Proposition 3.16 asserts that restriction along ρ is an equivalence of ∞-categories

$$\rho^* : \mathsf{Fact}_{X\times Y}(\mathcal{V}^\otimes) \xrightarrow{\simeq} \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \ .$$

Lemma 3.17 then establishes that the evaluation functor is an equivalence of ∞-categories

$$\mathsf{ev} : \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \xrightarrow{\simeq} \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes)) \ .$$

The composition of these two results establishes the desired equivalence

$$\mathsf{Fact}_{X\times Y}(\mathcal{V}^\otimes) \xrightarrow{\simeq} \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes)) \ .$$

□

4 Additivity of locally constant factorization algebras

In this section, we prove the following:

Theorem 4.1. *There is an equivalence of ∞-categories*

$$\mathit{Fact}^{\mathit{l.c.}}_{X\times Y}(\mathcal{V}) \simeq \mathit{Fact}^{\mathit{l.c.}}_X(\mathit{Fact}^{\mathit{l.c.}}_Y(\mathcal{V}^\otimes)).$$

Proof. We organize the structure of this proof into establishing the following commutative diagram:

Proving the statement amounts to showing that the top dashed arrow is an equivalence. To deduce this, we show that the bottom dashed arrow is an equivalence. Theorem 3.1 shows the top adjunction is an equivalence. Lemma 4.6 establishes that the front left vertical adjunction is an equivalence. Lemma 4.7

$$\begin{tikzcd}
\mathsf{Fact}^{\mathsf{l.c.}}_{X\times Y}(\mathcal{V}^{\otimes}) \arrow[rr, dashed] \arrow[dr, hook] \arrow[dd, shift right, leftarrow] \arrow[dd, shift left] & & \mathsf{BiFun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \arrow[dr, hook] \arrow[dd, shift right, leftarrow] \arrow[dd, shift left] & \\
& \mathsf{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \arrow[rr, shift left] \arrow[dd, shift right, leftarrow] \arrow[dd, shift left] & & \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \arrow[ll, shift left] \arrow[dd, shift right, leftarrow] \arrow[dd, shift left] \\
\mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X\times Y)^{\otimes}, \mathcal{V}^{\otimes}) \arrow[rr, dashed] \arrow[dr, hook] & & \mathsf{BiFun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \arrow[dr, hook] & \\
& \mathsf{Fun}^{\mathsf{m},J_\infty}(\mathsf{disk}(X\times Y)^{\otimes}, \mathcal{V}^{\otimes}) \arrow[rr, shift left] & & \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \arrow[ll, shift left]
\end{tikzcd}$$

establishes that the back left vertical adjunction is an equivalence. Lemma 4.10 establishes that the front right vertical adjunction is an equivalence. Lemma 4.11 establishes the back right vertical adjunction is an equivalence. It follows, in particular, that the bottom adjunction is an equivalence. Next, we show that this bottom adjunction restricts to an adjunction between the locally constant ∞-subcategories. This shows the bottom adjunction restricts, and thus is manifestly an equivalence. This statement is Corollary 4.15. This completes the proof. $\square$

We now turn our attention to proving the lemmas mentioned in the above proof.

Definition 4.2. For X a topological manifold of dimension n, there is a full ∞-sub-operad $\mathsf{disk}(X)^{\otimes} \xhookrightarrow{\iota} \mathsf{open}(X)^{\otimes}$ consisting of those $(I_+, (U_i))$ for which each open $U_i \cong \coprod \mathbb{R}^n$ is homeomorphic to a finite disjoint union of Euclidean space. Further, we let $\mathcal{J}(X) \subset \mathcal{I}(X)$ denote the full subcategory consisting of those opens that are homeomorphic to a finite disjoint union of disks.

The functor $\iota : \mathsf{disk}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ induces an adjunction of ∞-categories

$$\iota_! : \mathsf{Fun}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fun}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}) : \iota^* \tag{20}$$

with left adjoint given by operadic left Kan extension. In Proposition A.51 we provide a colimit expression for computing the values of $\iota_!$.

Definition 4.3. For notational purposes, we define

$$\mathsf{Fun}^{J_\infty}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) := \iota_!^{-1}(\mathsf{Fun}^{J_\infty}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}))\ .$$

Definition 4.4. Let $\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \hookrightarrow \mathsf{disk}(X)^{\otimes}$ denote the full ∞-suboperad consisting of those $(I_+, (U_i))$ such that, for each $i \in I$, the space U_i is connected.

Recall Definition 2.5 of *multiplicative* operad morphisms and the notation $\mathsf{Fun}^{\mathsf{m,opd}}$.

Lemma 4.5. *There is an equivalence of ∞-categories*

$$\mathsf{Fun}^{\mathsf{m,opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \xrightarrow[\mathsf{rest}]{\simeq} \mathsf{Fun}^{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes})\ .$$

Proof. This follows the proof of Lemma 3.10. $\square$

Lemma 4.6. *The adjunction in equation (20) restricts to an equivalence*

$$\iota_! : \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fact}_X(\mathcal{V}^{\otimes}) : \iota^*\ .$$

Proof. Note that $\mathsf{disk}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ takes coCartesian morphisms to coCartesian morphisms. So for $\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$, we see that $\iota^*\mathcal{F}$ is multiplicative. We now show $\iota^*\mathcal{F} \in \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes})$. By Definition 4.3, this is to show $\iota_!\iota^*\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$. Because $\mathsf{disk}(X) \subset \mathsf{open}(X)^{\otimes}$ is fully faithful, for $U \in \mathsf{disk}(X)$, the canonical morphism $\iota_!\iota^*\mathcal{F}(U) \simeq\xrightarrow{\simeq} \mathcal{F}(U)$ is an equivalence. Now, as discussed in Example A.58, $\mathsf{disk}(X)$ is a basis for the J_∞ Grothendieck topology on $\mathsf{open}(X)$. Therefore, we can check that $\iota_!\iota^*\mathcal{F}$ satisfies the J_∞-cosheaf condition by checking that it is a J_∞-cosheaf with respect to covers in $\mathsf{disk}(X)$. This follows from the fact that $\mathcal{F}$ is a J_∞-cosheaf. Therefore, the functor ι^* restricts as a functor

$$\mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \longleftarrow \mathsf{Fact}_X(\mathcal{V}^{\otimes}) \colon \iota^* ~,$$

as desired.

We next verify that the functor $\iota_!$ restricts as a functor

$$\iota_! \colon \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \longrightarrow \mathsf{Fact}_X(\mathcal{V}^{\otimes}) ~.$$

Let $\mathcal{G} \in \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes})$. We must show $\iota_!\mathcal{G}$ belongs to the full ∞-subcategory $\mathsf{Fact}_X(\mathcal{V}^{\otimes}) \subset \mathsf{Fun}\left(\mathsf{open}(X), \mathcal{V}^{\otimes}\right)$. By Definition 4.3, the restriction of $\iota_!\mathcal{G}$ to fibers over $1_+ \in \mathsf{Fin}_*$ is a J_∞-cosheaf. By Definition 2.12 of this ∞-subcategory $\mathsf{Fact}_X(\mathcal{V}^{\otimes})$, it remains is to show $\iota_!\mathcal{G}$ is multiplicative.

Denote the functor $c \colon \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \hookrightarrow \mathsf{disk}(X)^{\otimes}$ which, via (11), is a fully faithful right adjoint. By the proof of Lemma 4.5, $\mathcal{G}$ is the unique multiplicative extension of its restriction $c^*\mathcal{G}$. Furthermore, in how the proof of Lemma 4.5 uses the adjunction (11), the canonical morphism $c_!c^*\mathcal{G} \xrightarrow{\simeq} \mathcal{G}$ is an equivalence. Denote $\rho := \iota \circ c$, and denote $\mathcal{H} := c^*\mathcal{G}$. Because left Kan extensions compose, there results are equivalences

$$\rho_!\mathcal{H} ~\simeq~ \rho_! c^*\mathcal{G} ~\simeq~ (\iota \circ c)_! c^*\mathcal{G} ~\simeq~ \iota_! c_! c^*\mathcal{G} ~\simeq~ \iota_!\mathcal{G} ~.$$

Therefore, to $\iota_!\mathcal{G}$ is multiplicative if and only if $\rho_!\mathcal{H}$ is multiplicative. Note that $\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}$ is a unital ∞-operad. The advantage of this maneuver of replacing $\iota_!\mathcal{G}$ with $\rho_!\mathcal{H}$ is that the former involves a multiplicative functor ($\mathcal{G}$) between ∞-operads, while the latter is free from such a condition, so the values of $\rho_!\mathcal{H}$ are given by the formula of Proposition A.51 which are generally manageable to work with: for $(I_+, (U_i)) \in \mathsf{open}(X)^{\otimes}$,

$$\rho_!\mathcal{H}((I_+,(U_i))) \simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(I_+,(U_i))}} \xrightarrow{\mathrm{fgt}} \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{I_+}} \xrightarrow{\mathcal{H}_{/I_+}} \mathcal{V}^{\otimes}_{/^{\mathsf{act}}_{I_+}} \xrightarrow{\otimes} \mathcal{V}^{\otimes}_{|I_+}\right) ~\in~ \mathcal{V}^{\otimes}_{|I_+} ~.$$

We now show $\rho_!\mathcal{H}$ is multiplicative. We do this through the formula for operadic left Kan extension, of Proposition A.51. An active coCartesian morphism in $\mathsf{open}(X)^{\otimes}$,

$$(I_+,(U_i)) \xrightarrow{f} \left(J_+, \left(\coprod_{i \in f^{-1}(j)} U_i\right)\right) ~,$$

induces a functor

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(I_+,(U_i))}} \to \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(\coprod U_i))}}$$

given by postcomposing with f. We claim this functor is final. We verify this through Quillen's Theorem A (Theorem A.61). To use this theorem we must verify that for

$$\left((K_+,(V_k)), K_+ \xrightarrow{\alpha} J_+\right) \in \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(\coprod U_i))}} ~,$$

the classifying space of the undercategory

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(I_+,(U_i))}}^{\left((K_+,(V_k)),K_+ \xrightarrow{\alpha} J_+\right)/}$$

is contractible. To show contractibility, we note there is an initial object in the undercategory. The existence of an initial object comes from the fact that each V_k is connected. This implies that for $k \in \alpha^{-1}(J)$, the connected open set V_k is a subset of a unique U_i. This allows us to define a map $K_+ \xrightarrow{\beta} I_+$. The object $\big((K_+,(V_k)), K_+ \xrightarrow{\beta} I_+\big)$ is initial in the undercategory. Thus,

$$\begin{aligned}
\rho_!\mathcal{H}(f_!(I_+,(U_i))) &\underset{\text{Prop A.51}}{\simeq} \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(\coprod U_i))}} \to \mathcal{V}^{\otimes}_{|J_+}\right)\\
&\simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(I_+,(U_i))}} \to \mathcal{V}^{\otimes}_{|I_+} \xrightarrow{f_!} \mathcal{V}^{\otimes}_{|J_+}\right)\ .
\end{aligned}$$

The canonical projection maps $\mathcal{V}^{\otimes}_{|J_+} \simeq \mathcal{V}^{J} \xrightarrow{\mathsf{pr}_j} \mathcal{V} \simeq \mathcal{V}^{\otimes}_{|\{j\}_+}$ preserve colimits. Therefore, we can compute the colimit separately in each factor of $\mathcal{V}^{\otimes}_{|J_+} \simeq \mathcal{V}^{\times J}$. So without loss of generality, let us consider the case of an active coCartesian morphism $(I_+,(U_i)) \to (1_+, \coprod U_i)$. This general case of an active coCartesian morphism will follow completely analogously to the case $(2_+,(U_1,U_2)) \xrightarrow{f} (1_+, U_1 \amalg U_2)$, and this latter case will ease the notational burden tremendously. Note the canonical functor

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(2_+,(U_1,U_2))}} \xrightarrow{\simeq} \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \times \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}}$$

is an equivalence. Further, the functor

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \times \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}} \xrightarrow{\mathsf{pr}_1} \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}}$$

given by the projection onto the first factor is a coCartesian fibration. Proposition A.49 says that left Kan extension along a coCartesian fibration evaluates as a fiberwise colimit. Therefore,

$$\begin{aligned}
&\rho_!\mathcal{H}(f_!(2_+,(U_1,U_2)))\\
&\qquad\simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(2_+,(U_1,U_2))}} \xrightarrow{\mathcal{H}} \mathcal{V}^{\otimes}_{|2_+} \xrightarrow{\otimes} \mathcal{V}^{\otimes}_{|1_+} \simeq \mathcal{V}\right)\\
&\qquad\simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \times \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}} \xrightarrow{\mathcal{H}} \mathcal{V}^{\otimes}_{|2_+} \xrightarrow{\otimes} \mathcal{V}\right)\\
&\qquad\simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \xrightarrow{\mathsf{pr}_!\mathcal{H}} \mathcal{V}\right)\\
&\qquad\simeq \underset{(K_+,(D_k))\in\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}}}{\operatorname{colim}}\left(\operatorname{colim}\left(\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}}\right)_{|(K_+,(D_k))} \xrightarrow{\mathcal{H}} \mathcal{V}\right)\right)\\
&\qquad\simeq \underset{(K_+,(D_k))\in\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}}}{\operatorname{colim}}\ \underset{(L_+,(E_\ell))\in\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}}}{\operatorname{colim}} \mathcal{H}((K_+,(D_k)))\otimes\mathcal{H}((L_+,(E_\ell)))\\
&\qquad\simeq \rho_!\mathcal{H}(U_1)\otimes\rho_!\mathcal{H}(U_2)\ ,
\end{aligned}$$

as desired. Note that the last equivalence invokes the assumption that $\mathcal{V}^{\otimes}$ is $\otimes$-presentable. In particular, the $\otimes$-presentability of $\mathcal{V}^{\otimes}$ means that for any $V \in \mathcal{V}$, the functor $V \otimes - : \mathcal{V} \to \mathcal{V}$ preserves colimits. This concludes our proof that $\rho_!\mathcal{H}$ is multiplicative.

In summary, we have shown the adjunction in equation (20) restricts as an adjunction

$$\iota_! : \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{opd}}(\mathsf{disk}(X)^{\otimes},\mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fact}_X(\mathcal{V}^{\otimes}) : \iota^*\ .$$

It remains to verify this adjunction is an equivalence. We do this by checking that the unit and counit evaluate as equivalences. Because ι is fully faithful, the unit $\mathsf{id} \to \iota^*\iota_!$ is an equivalence. Next, take $\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$ and consider the counit $\iota_!\iota^*\mathcal{F} \to \mathcal{F}$. Since both functors are J_∞ cosheaves, it suffices to check the counit is an equivalence evaluated on unary elements. Further, since $\mathsf{disk}(X)$ is a J_∞ basis for $\mathsf{open}(X)$, as just mentioned above, it suffices to check this equivalence on elements of $\mathsf{disk}(X)$ which follows immediately. $\square$

Lemma 4.7. *The adjunction in Lemma 4.6 restricts to an adjunction between the respective locally constant subcategories*

$$\iota_! : \mathit{Fun}^{\mathsf{m},J_\infty,\mathsf{l.c.},\mathsf{opd}}(\mathit{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \rightleftarrows \mathit{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^{\otimes}) : \iota^* \quad .$$

In particular, this adjunction is an equivalence.

Proof. First, for $\mathcal{G} \in \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{l.c.},\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes})$, we show $\iota_!\mathcal{G} \in \mathsf{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^{\otimes})$. That is, for $U \overset{\varphi}{\hookrightarrow} V$ in $\mathcal{I}(X)$, we show the canonical morphism induced by φ

$$\iota_!\mathcal{G}(U) \to \iota_!\mathcal{G}(V)$$

is an equivalence. Observe the commutative diagram

$$\begin{tikzcd}
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}} \arrow[r] \arrow[d, "\mathrm{loc}"] & \mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}} \arrow[r, "\mathrm{fgt}"] \arrow[d, "\mathrm{loc}"] & \mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "\mathcal{G}"] & \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "\otimes"] & \mathcal{V} \, . \\
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}}[(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}})^{-1}] \arrow[r] & \mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}}[(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}})^{-1}] \arrow[urrr, dashed, bend right] & & &
\end{tikzcd}$$

The colimit of the top horizontal line is the definition of $\iota_!\mathcal{G}(U)$ and the existence of the dashed arrow follows from $\mathcal{G}$ being locally constant. There are canonical identifications $\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}} \simeq \mathcal{J}(U)$ and $\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}} \simeq \mathsf{disk}(U)$, where $\mathsf{disk}(U)$ is the poset. By Proposition A.25 localizations are final, so

$$\iota_!\mathcal{G}(U) \simeq \operatorname{colim}\left(\mathsf{disk}(U)[\mathcal{J}(U)^{-1}] \to \mathsf{disk}(X)[\mathcal{J}(X)^{-1}] \to \mathcal{V}\right) .$$

By a similar analysis, we see

$$\iota_!\mathcal{G}(V) \simeq \operatorname{colim}\left(\mathsf{disk}(V)[\mathcal{J}(V)^{-1}] \to \mathsf{disk}(X)[\mathcal{J}(X)^{-1}] \to \mathcal{V}\right) .$$

By Proposition 2.19 in [AF15], we see $\mathsf{disk}(U)[\mathcal{J}(U)^{-1}] \simeq \mathsf{Disk}(U)$, and likewise for V. Here, by $\mathsf{Disk}(X)$ we mean the topological category of embedded disks in X, which we regard as an ∞-category via the coherent/simplicial nerve (see [Cor82] for an original reference, or Definition 1.1.5.5 of [Lur17] for a later treatment). (We choose to not dwell on $\mathsf{Disk}(X)$ because we make no further use of it beyond this paragraph. We refer the interested reader to [AF15] for further details.) Finally, we claim that the isotopy equivalence φ induces an equivalence $\mathsf{Disk}(U) \xrightarrow{\simeq} \mathsf{Disk}(V)$. To see this, choose an isotopy inverse to φ, say ψ, together with isotopies $\mathsf{id}_U \overset{\eta}{\simeq} \psi \circ \varphi$ and $\varphi \circ \psi \overset{\epsilon}{\simeq} \mathsf{id}_V$. These data determine an adjunction $\mathsf{Disk}(U) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathsf{Disk}(V)$: the unit is given on $D \in \mathsf{Disk}(U)$ by $D \xrightarrow{\eta(D)} \psi(\varphi(D))$, which is an equivalence in $\mathsf{Disk}(U)$; the counit is given on $E \in \mathsf{Disk}(V)$ by $\varphi(\psi(E)) \xrightarrow{\epsilon(E)} E$, which is an equivalence in $\mathsf{Disk}(V)$. Because the unit and counit of this adjunction are given by equivalences, φ is an equivalence between ∞-categories.

It remains to show that $\iota^*\mathcal{F}$ is locally constant, for $\mathcal{F} \in \mathsf{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^{\otimes})$. This follows from the following diagram

$$\begin{tikzcd}
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}} \arrow[r, "\iota"] \arrow[dd, "\mathrm{loc}"] & \mathsf{open}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}} \arrow[r] \arrow[d, "(-)_!"] & \mathsf{open}(X)^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "\mathcal{F}"] & \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "(-)_!"] & \mathcal{V} \\
& \mathsf{open}(X) \arrow[d, "\mathrm{loc}"] \arrow[urrr, bend right, "\mathcal{F}"] & & & \\
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}}[(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}})^{-1}] \arrow[r, dashed] & \mathsf{open}(X)[\mathcal{I}(X)^{-1}] \arrow[uurrr, bend right] & & &
\end{tikzcd}$$

The bottom right most arrow is the condition of $\mathcal{F}$ being multiplicative, and the existence of the dashed arrow is given by the fact that $((-)_! \circ \iota)(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}}) \subset \mathcal{I}(X)$. $\square$

Lemma 4.8. *The functor of ∞-operads $\mathsf{disk}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ induces an adjunction*

$$\iota_! : \mathsf{BiFun}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes};, \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) : \iota^* \tag{21}$$

in which the left adjoint evaluates on $((I_+, (U_i)), (J_+, (V_j))) \in \mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}$ as the following colimit

$$\operatorname{colim}\left(\mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/((I_+,(U_i)),(J_+,(V_j)))} \to \mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/I_+ \wedge J_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/I_+ \wedge J_+} \xrightarrow{(-)_!} \mathcal{V}^{\otimes}_{|I_+ \wedge J_+}\right) .$$

Proof. Proposition A.50 establishes the formula for the left adjoint, so it remains to verify that $\iota_! \mathcal{F}$ is a bifunctor. To see this, take an inert coCartesian morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (U_j))$. There is a commutative square

$$\begin{tikzcd}
\mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/((I_+,(U_i)),(K_+,(V_k)))} \arrow[r] \arrow[d] & \mathcal{V}^{\otimes}_{|I_+ \wedge K_+} \arrow[d] \\
\mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/((J_+,(U_j)),(K_+,(V_k)))} \arrow[r] & \mathcal{V}^{\otimes}_{|J_+ \wedge K_+}
\end{tikzcd} .$$

Note the left vertical functor is final, as verified using Quillen's Theorem A (Theorem A.61). One can use the fact that $I_+ \xrightarrow{f} J_+$ is inert to show the relevant undercategory in the statement of Theorem A.61 has an initial object. The result then follows from the fact that the projection $\mathcal{V}^{\otimes}_{|I_+ \wedge K_+} \to \mathcal{V}^{\otimes}_{|J_+ \wedge K_+}$ preserves colimits. Note that an analogous argument holds in the second variable. $\square$

To prove Lemma 4.10, we need an analogue of Lemma 4.5 for bifunctors:

Lemma 4.9. *There is an equivalence of ∞-categories*

$$\mathsf{BiFun}^{\mathsf{m}}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \xrightarrow[\mathsf{rest}]{\simeq} \mathsf{BiFun}(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}, \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}; \mathcal{V}^{\otimes}) .$$

Proof. This directly follows the proof of Lemma 3.10. $\square$

Lemma 4.10. *The adjunction in equation (21) restricts to an equivalence*

$$\iota_! : \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}) : \iota^* .$$

Proof. This is proved in the same manner as Lemma 4.6. $\square$

Lemma 4.11. *The adjunction in Lemma 4.10 restricts to an adjunction between the respective locally constant subcategories*

$$\iota_! : \mathit{BiFun}^{m,J_\infty,l.c.}(\mathit{disk}(X)^\otimes, \mathit{disk}(Y)^\otimes; \mathcal{V}^\otimes) \rightleftarrows \mathit{BiFun}^{m,J_\infty,l.c.}(\mathit{open}(X)^\otimes, \mathit{open}(Y)^\otimes; \mathcal{V}) : \iota^* \ .$$

In particular, this adjunction is an equivalence.

Proof. The proof of this follows analogous to Lemma 4.7. □

To prove Proposition 4.14 below, we employ a result from [MG19] that enables us to identify localizations of ∞-categories via complete Segal spaces. This result is recorded as Theorem A.26. For the reader unfamiliar with complete Segal spaces, we devote a section in the appendix to the basic definitions and ideas that we use.

Definition 4.12. We let $\mathcal{J}(X)^\otimes_{\mathsf{c}}$ denote the full ∞-subcategory of $\mathcal{J}(X)^\otimes$ over Fin_* consisting of those $(I_+, (U_i))$ for which U_i is connected.

Lemma 4.13. *Let $(I_+, (U_i)) \in \mathit{disk}(X \times Y)^\otimes_{\mathsf{c}}$. The simplicial space*

$$\mathit{BFun}^{\mathcal{J}(X)^\otimes_c \times \mathcal{J}(Y)^\otimes_c{}_{/(I_+,(U_i))}}\left([\bullet], \mathit{disk}(X)^\otimes_c \times \mathit{disk}(Y)^\otimes_c{}_{/(I_+,(U_i))}\right)$$

is a complete Segal space.

Proof. Let us adopt the notational conventions

$$\mathcal{C} := \mathsf{disk}(X)^\otimes_{\mathsf{c}} \times \mathsf{disk}(Y)^\otimes_{\mathsf{c}}{}_{/(I_+,(U_i))} \ , \qquad \mathcal{W} := \mathcal{J}(X)^\otimes_{\mathsf{c}} \times \mathcal{J}(Y)^\otimes_{\mathsf{c}}{}_{/(I_+,(U_i))} \ . \tag{22}$$

First, we establish the Segal condition. That is, for all $p \geq 0$, we show the diagram of classifying spaces

$$\begin{tikzcd} \mathsf{BFun}^{\mathcal{W}}([p], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{BFun}^{\mathcal{W}}(\{0<1\}, \mathcal{C}) \arrow[d] \\ \mathsf{BFun}^{\mathcal{W}}(\{1 < \cdots < p\}, \mathcal{C}) \arrow[r] & \mathsf{BFun}^{\mathcal{W}}(\{1\}, \mathcal{C}) \end{tikzcd} \tag{23}$$

is a pullback. To show the diagram in equation (23) is a pullback, we make use of Proposition A.8 and show an equivalence of vertical fibers. The defining Segal condition for $(\infty,1)$-categories stipulates that the diagram in Δ, regarded as a diagram among $(\infty,1)$-categories,

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \{0<1\} \arrow[d] \\ \{1 < \cdots < p\} \arrow[r] & {,} \end{tikzcd}$$

is a pushout. This implies the diagram prior to taking classifying spaces

$$\begin{tikzcd} \mathsf{Fun}^{\mathcal{W}}([p], \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}^{\mathcal{W}}(\{0<1\}, \mathcal{C}) \arrow[d] \\ \mathsf{Fun}^{\mathcal{W}}(\{1 < \cdots < p\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}^{\mathcal{W}}(\{1\}, \mathcal{C}) \end{tikzcd} \tag{24}$$

is a pullback. Thus by Proposition A.7, the vertical fibers of the diagram in equation (24) are equivalent, hence their classifying spaces are equivalent. Therefore, we must show that the classifying space of the vertical fibers in the diagram in equation (24) are the vertical fibers of the diagram in equation (23). Quillen's Theorem B (Theorem A.62) provides a method to verify this. By Lemma A.63, we only need to show this for the rightmost vertical functor, so we restrict our attention there. All told, proving the Segal condition reduces to verifying the hypothesis of Quillen's Theorem B for the functor

$$\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C}) \to \mathsf{Fun}^{\mathcal{W}}(\{1\},\mathcal{C}) \ .$$

By Observation A.27, $\mathsf{Fun}^{\mathcal{W}}(\{1\},\mathcal{C}) \simeq \mathcal{W}$, so to invoke Quillen's Theorem B, we need to show that for each $c \xrightarrow{f} c'$ in $\mathcal{W}$, the induced functor

$$\mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{/_c} \xrightarrow{\mathsf{B}(f\circ -)} \mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{/_{c'}} \tag{25}$$

is an equivalence. The vertical functors in the diagram in equation (24) are coCartesian fibrations, so by Proposition A.34 the canonical functor

$$\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|_c} \hookrightarrow \mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{/_c}$$

is a right adjoint, with left adjoint given by the coCartesian monodromy functor. Since, an adjunction induces an equivalence between classifying spaces by Proposition A.22, showing that equation (25) is an equivalence is equivalent to showing that

$$\mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})|_c \xrightarrow{\mathsf{B}(f\circ -)} \mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})|_{c'} \tag{26}$$

is an equivalence. Evaluation at $0 \in \{0<1\}$ defines a functor $\mathsf{Fun}(\{0<1\},\mathcal{C})_{|c} \xrightarrow{\mathsf{ev}_0} \mathcal{C}$. The definition of ∞-overcategories is such that there is an identification $\mathsf{Fun}(\{0<1\},\mathcal{C})_{|c} \simeq \mathcal{C}_{/c}$ between ∞-categories over $\mathcal{C}$. Observe an identification $\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|c} \simeq (\mathsf{Fun}(\{0<1\},\mathcal{C})_{|c})_{|\mathcal{W}}$ between ∞-subcategories of $\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|c}$, in which the latter is the base-change along the inclusion $\mathcal{W} \hookrightarrow \mathcal{C}$. Denote the base-change $(\mathcal{C}_{/c})^{\mathcal{W}} \simeq (\mathcal{C}_{/c})_{|\mathcal{W}}$ along the inclusion $\mathcal{W} \hookrightarrow \mathcal{C}$. So, we have an identification $\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|c} \simeq (\mathcal{C}_{/c})^{\mathcal{W}}$ between ∞-categories over $\mathcal{W}$. Taking classifying spaces results in an equivalence between spaces:

$$\mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|_c} \simeq \mathsf{B}(\mathcal{C}_{/_c})^{\mathcal{W}}.$$

This implies that showing equation (26) is an equivalence is equivalent to showing that the functor

$$\mathsf{B}(\mathcal{C}_{/_c})^{\mathcal{W}} \to \mathsf{B}(\mathcal{C}_{/_{c'}})^{\mathcal{W}} \tag{27}$$

is an equivalence.

Now, by the notation of (22), an object $c \in \mathcal{C}$ is an object

$$\Big(\ (J_+,(V_j))\ ,\ (K_+(W_k))\ ,\ J_+ \wedge K_+ \xrightarrow{f} I_+\ \Big)\ \in\ \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\ .$$

Again unpacking the notation of (22), observe that the projections onto each factor define an equivalence between categories:

$$\begin{aligned}(\mathcal{C}_{/_c})^{\mathcal{W}} &:= \left(\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{/_{(J_+,(V_j)),(K_+,(W_k)),J_+\wedge K_+ \xrightarrow{f} I_+}}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}} \\ &\xrightarrow{\simeq} \left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}} \times \left(\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(K_+,(W_k))}\right)^{\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(K_+,(W_k))}}\ .\end{aligned}$$

We now restrict attention to one factor of this product. Consider a morphism $(J_+,(V_j)) \to (J'_+,(V'_j))$ in $\mathcal{J}(X)^{\otimes}_{\mathsf{c}}$. Note that the resulting functor

$$\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(V_j))}}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}} \hookrightarrow \left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}}$$

is a right adjoint with left adjoint given by using the inert-active factorization system of Fin_* (see Observation A.30). Therefore, this functor induces an equivalence of classifying spaces. Furthermore, note that

$$\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(V_j))}}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}} \simeq \prod_{j\in J}\mathcal{J}(V_j)\ .$$

Now, for r a finite cardinality, and for M a topological space, consider the topological space

$$\mathsf{Conf}_r(M)_{\Sigma_r} \ := \ \{S\subseteq M \mid \mathsf{Card}(S)=r\}$$

whose underlying set consists of subsets of M with cardinality r and whose topology is the finest with respect to which the map from the subspace of the product

$$M^{\times r}\supseteq \big\{(x_1,\dots,x_r)\in M^{\times r}\mid x_i=x_j \implies i=j\big\}\longrightarrow \mathsf{Conf}_r(M)_{\Sigma_r}\ , \qquad (x_1,\dots,x_r)\longmapsto\{x_1,\dots,x_r\}\ ,$$

is continuous. We use Theorem A.64 to identify the homotopy types:

$$\mathsf{B}\prod_{j\in J}\mathcal{J}(V_j)\ \simeq\ \prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}\ .$$

Before doing so, we indicate the spirit of this identification. Note that an object in $\prod_{j\in J}\mathcal{J}(V_j)$ is, for each $j\in J$, an open subset $D_j\subseteq V_j$ that is a finite disjoint union of (open) disks. The spirit of this identification associates to such an object $(D_j\subseteq V_j)_{j\in J}$ an element $(C_j)_{j\in J}\in\prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}$ such that, for each $j\in J$, there is containment $C_j\subseteq D_j$ for which the induced map on components $\pi_0(C_j)\to\pi_0(D_j)$ is a bijection. In other words, the identification $\mathsf{B}\prod_{j\in J}\mathcal{J}(V_j)\simeq\prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}$ associates to each collection of disks a choice of a point in each disk. Because disks are contractible, this association is homotopically well-defined. We now implement this spirit.

Consider the functor

$$\mathcal{U}:=\prod_{j\in J}\mathcal{J}(V_j)\to\prod_{j\in J}\mathsf{open}\left(\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}\right) \tag{28}$$

given by

$$(D_j\subset V_j)_{j\in J}\mapsto\left(\left\{S\subset V_j\mid S\subset D_j \ \text{ and }\ \pi_0S\xrightarrow{\cong}\pi_0D_j\right\}\right)_{j\in J}\ .$$

For each $j\in J$, and each $(D_j\subset V_j)\in\mathcal{J}(V_j)$, notice the homeomorphism

$$\{S\subset V_j\mid S\subset D_j\ \text{ and }\ \pi_0S\xrightarrow{\cong}\pi_0D_j\}\cong\prod_{\alpha\in\pi_0D_j}D_j^{\alpha}$$

between the indicated subspace of $\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}$ and the product of disks. In particular, each value of the functor (28) is a finite-fold product of disks, and is therefore contractible. Let $(S_j\subset V_j)_{j\in J}\in\prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}$. Recall the subscript notation of Theorem A.64. Observe the isomorphism between posets:

$$\mathcal{U}_{(S_j\subset V_j)_{j\in J}}=\prod_{j\in J}\{D_j\in\mathcal{J}(V_j)\mid S_j\subset D_j\ \text{ and }\ \pi_0S_j\xrightarrow{\cong}\pi_0D_j\}\ .$$

By Example A.6, open disks are a basis for the topology of a manifold. Therefore, each factor of this product of posets is a cofiltered poset (see Definition A.1). Because the product of cofiltered posets is cofiltered, we conclude that the poset $mcU_{(S_j\subset V_j)_{j\in J}}$ is cofiltered. In particular, the classifying space $\mathsf{B}mcU_{(S_j\subset V_j)_{j\in J}}\simeq *$ is contractible. Therefore, the hypotheses of Theorem A.64 are satisfied with respect to $\mathcal{U}$. So Theorem A.64 establishes an equivalence

$$\mathsf{B}\prod_{j\in J}\mathcal{J}(V_j)\simeq\prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}\ .$$

Next, earlier, we reduced the Segal condition to showing for each morphism $c \to c'$ in $\mathcal{W}$, the induced map between spaces

$$\mathsf{B}(\mathcal{C}_{/_c})^{\mathcal{W}} \to \mathsf{B}(\mathcal{C}_{/_{c'}})^{\mathcal{W}} \tag{29}$$

is an equivalence. So let

$$\Big(\ (J_+,(V_j))\ ,\ (K_+\,(W_k))\ ,\ J_+\wedge K_+ \xrightarrow{f} I_+\ \Big) \longrightarrow \Big(\ (J'_+,(V'_{j'}))\ ,\ (K'_+\,(W'_{k'}))\ ,\ J'_+\wedge K'_+ \xrightarrow{f'} I_+\ \Big)$$

be a morphism in $\mathcal{W} \overset{(22)}{=} \mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}$. By definition of $\mathcal{W}$, this morphism is a pair $J \xrightarrow{g} J'$ and $K \xrightarrow{h} K'$ of bijections such that $J_+ \wedge K_+ \xrightarrow{(g\times h)_+} J'_+ \wedge K'_+$ lies over I_+, and such that, for each $(j,k) \in J \times K$, there are inclusions of open disks $V_j \subset V'_{g(j)}$ in X and $W_k \subset W_{h(k)}$ in Y. Inspecting the equivalence established just above reveals that the map between spaces $\mathsf{B}(\mathcal{C}_{/_c})^{\mathcal{W}} \to \mathsf{B}(\mathcal{C}_{/_{c'}})^{\mathcal{W}}$ is homotopy equivalent with the map between spaces

$$\prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r} \times \prod_{k\in K}\coprod_{s\geq 0}\mathsf{Conf}_s(W_k)_{\Sigma_s} \longrightarrow \prod_{j'\in J'}\coprod_{r\geq 0}\mathsf{Conf}_r(V'_{j'})_{\Sigma_r} \times \prod_{k'\in K'}\coprod_{s\geq 0}\mathsf{Conf}_s(W'_{k'})_{\Sigma_s}$$

induced from the bijections $J \xrightarrow{g} J'$ and $K \xrightarrow{h} K'$ and, for each $(j,k) \in J \times K$, the inclusions $V_j \subset V_{g(j)}$ and $W_k \subset W_{h(k)}$. Therefore, to show the map between spaces $\mathsf{B}(\mathcal{C}_{/_c})^{\mathcal{W}} \to \mathsf{B}(\mathcal{C}_{/_{c'}})^{\mathcal{W}}$ is an equivalence, we are reduced to showing, for finite cardinality ℓ and for each open embedding $D \xrightarrow{\varphi} D'$ between (open) disks, the induced map

$$\mathsf{Conf}_\ell(D)_{\Sigma_\ell} \xrightarrow{\mathsf{Conf}_\ell(\varphi)_{\Sigma_\ell}} \mathsf{Conf}_\ell(D')_{\Sigma_\ell}\ , \qquad S \mapsto \varphi(S)\ ,$$

is a homotopy equivalence. Well, Theorem A.65 implies each open embedding of an open disk into another open disk is isotopic to a homeomorphism. So choose an isotopy φ_t from φ to a homeomorphism φ_1. Being an isotopy, for each t, the map $D \xrightarrow{\varphi_t} D'$ is injective. So, for each t, the map $\mathsf{Conf}_\ell(\varphi_t)_{\Sigma_\ell}$ is well-defined. Consequently, the maps $\mathsf{Conf}_\ell(\varphi_t)_{\Sigma_\ell}$ define a homotopy from $\mathsf{Conf}_\ell(\varphi_0)_{\Sigma_\ell}$ to $\mathsf{Conf}_\ell(\varphi_1)_{\Sigma_\ell}$. Because φ_1 is a homeomorphism, so is $\mathsf{Conf}_\ell(\varphi_1)_{\Sigma_\ell}$ – indeed, its inverse is $\mathsf{Conf}_\ell(\varphi^{-1})_{\Sigma_\ell}$. It follows that $\mathsf{Conf}_\ell(\varphi_t)_{\Sigma_\ell}$ is a homotopy equivalence, as desired. This concludes the proof that the map between spaces (29) is an equivalence, which, as established above, implies the simplicial space

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}}\Big([\bullet], \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big) \tag{30}$$

is a Segal space.

We now prove that the Segal space (30) is complete. That is, we will show that the map from the space of [0]-points

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}}\Big([0], \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)$$

into the space of [1]-points that are equivalences

$$\Big(\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}}\Big([1], \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)\Big)^{\mathsf{equiv}}$$

is an equivalence of spaces. A key observation to proving this is that a morphism in

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}$$

is in the isotopy equivalences

$$\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}$$

if and only if the underlying maps of finite sets are both bijections. Now, consider a [1]-point of the Segal space in equation (30) that is an equivalence. By definition, this is a point in the space

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}}\left([1],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)\ .$$

Such a point is represented by an object in

$$\mathsf{Fun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}}\left([1],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)\ ,$$

i.e. a functor

$$[1]\to\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\ .$$

Let's say this functor selects out the morphism

$$\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+\xrightarrow{f}I_+\big)\xrightarrow{\varphi,\psi}\big((K'_+,(V_{k'})),(L'_+,(W_{\ell'})),K'_+\wedge L'_+\xrightarrow{f'}I_+\big)\ . \qquad (31)$$

Since there is a natural morphism of Segal spaces

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}}\left([\bullet],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)\to\mathsf{Fin}_*\times\mathsf{Fin}_*\ ,$$

the equivalence in equation (31) gets carried to an equivalence in $\mathsf{Fin}_*\times\mathsf{Fin}_*$. This implies that φ and ψ are bijections, so by the key observation above, the equivalence in equation (31) lies in

$$\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\ .$$

Now, using the notation given in equation (22), observe the solid commutative diagram

$$\begin{tikzcd}
 & & {[1]} \arrow[dll, dashed, bend right] \arrow[dl, dashed] \arrow[ddl, bend left] \\
\mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] & \\
\mathsf{BFun}^{\mathcal{W}}([0],\mathcal{C}) \arrow[r] & \mathsf{BFun}^{\mathcal{W}}([\bullet],\mathcal{C}) &
\end{tikzcd}$$

where the solid arrow $[1]\to\mathsf{BFun}^{\mathcal{W}}([\bullet],\mathcal{C})$ is the assumed equivalence. We showed that this is represented by a dashed arrow $[1]\to\mathcal{C}$, and further that this dashed arrow actually factors through $\mathcal{W}$, hence the other dashed arrow $[1]\to\mathcal{W}$ in the diagram. This shows that the space of [0]-points is a deformation retract of the [1]-points that are equivalences, and is thus a homotopy equivalence as desired. ☐

Proposition 4.14. *An isotopy equivalence $(I_+,(U_i))\hookrightarrow(J_+,(V_j))$ in $\mathcal{J}(X\times Y)^{\otimes}_{\mathsf{c}}$, induces an equivalence of ∞-categories*

$$\begin{aligned}(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}&[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(I_+,(U_i))}]\\ &\xrightarrow{\simeq}(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(J_+,(V_j))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(J_+,(V_j))}]\ .\end{aligned}$$

Proof. We use Theorem A.26 to identify the localizations as complete Segal spaces in Lemma 4.13 above. By Observation A.18, the map between these complete Segal spaces is an equivalence if and only if the resulting maps between spaces of [0]- and [1]-points are both equivalences.

First, we establish the equivalence of [0]-points. To do this, we first identify the [0]-points, which is the classifying space of the localizing subcategory,

$$\mathsf{B}(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))} \;\simeq \coprod_{[K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+]}\left(\prod_{i\in I}\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)\right)_{/\mathsf{Aut}(K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+)}, \tag{32}$$

with a space we now explain. The coproduct is indexed by the set of isomorphism classes of objects in $(\mathsf{Fin}_*\times\mathsf{Fin}_*)_{/I_+}$. For each such isomorphism class, select a representative $(K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+)$. Given such a representative, for each $i\in I$, the topological space $\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)$ is defined as the following pullback

$$\begin{tikzcd}
\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\times K}\times Y^{\times K} \arrow[d, "\times"] \\
& (X\times Y)^{\times(K\times L)} \arrow[d, "\text{restriction}"] \\
\mathsf{Conf}_{f^{-1}(i)}(U_i) \arrow[r, hook, "\text{inclusion}"] & (X\times Y)^{\times f^{-1}(i)}
\end{tikzcd} \tag{33}$$

explained here. For J a set and Z a topological space, the topological space $Z^{\times J}$ is the J-fold product of Z; the map $X^{\times K}\times Y^{\times L}\xrightarrow{\times}(X\times Y)^{K\times L}$ is given by $\big((x_k)_{k\in K},(y_\ell)_{\ell\in L}\big)\mapsto\big((x_k),(y_\ell)\big)_{(k,\ell)\in K\times L}$. Note that $\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)$ carries a natural action by the group $\mathsf{Aut}(K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+)$ of automorphisms of the object $(K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+)\in(\mathsf{Fin}_*\times\mathsf{Fin}_*)_{/I_+}$. Indeed, this group is the subgroup of the product of permutation groups $\Sigma_K\times\Sigma_L$, which acts on $X^{\times K}\times Y^{\times L}$, consisting of those pairs of permutations (α,β) such that the permutation $\alpha\times\beta\in\Sigma_{K\times L}$ restricts as a permutation of $f^{-1}(i)$.

Now, by definition of $\mathcal{J}(X)^{\otimes}_{\mathsf{c}}$ and $\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}$ in terms of isotopy equivalences, the functor $\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\to(\mathsf{Fin}_*\times\mathsf{Fin}_*)_{/I_+}$ carries each morphism to an isomorphism. This is to say this functor factors through the maximal ∞-subgroupoid of $(\mathsf{Fin}_*\times\mathsf{Fin}_*)_{/I_+}$:

$$\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\longrightarrow\mathsf{Obj}\big((\mathsf{Fin}_*\times\mathsf{Fin}_*)_{/I_+}\big)\;\simeq\;\mathsf{Obj}(\mathsf{Fin}_*)\times\mathsf{Obj}(\mathsf{Fin}_*)_{/I_+}\;.$$

Every functor to an ∞-groupoid is a coCartesian fibration whose coCartesian monodromy functors are, necessarily, equivalences. Therefore, Theorem A.62 implies the following canonical diagram among spaces is a pullback:

$$\begin{tikzcd}
\mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|_{K_+\wedge L_+\xrightarrow{f}I_+}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right) \arrow[d] \\
\ast \arrow[r, "\langle K_+\wedge L_+\xrightarrow{f}I_+\rangle"] & \mathsf{B}\left(\mathsf{Obj}(\mathsf{Fin}_*)\times\mathsf{Obj}(\mathsf{Fin}_*)_{/I_+}\right)
\end{tikzcd}\;. \tag{34}$$

Note that the canonical map to the bottom right space in this diagram from the coproduct

$$\coprod_{[K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+]}\mathsf{BAut}(K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+)\;\simeq\;\mathsf{B}\left(\mathsf{Obj}(\mathsf{Fin}_*)\times\mathsf{Obj}(\mathsf{Fin}_*)_{/I_+}\right)$$

is an equivalence. Consequently, the top right space in (34) is a coproduct of its base-changes of each cofactor of the bottom right space in (34):

$$\coprod_{[K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+]}\mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|\mathsf{BAut}(K_+,L_+,K_+\wedge L_+\xrightarrow{f}I_+)}\;\simeq\;\mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right).$$

Therefore, the sought equivalence (32) is implied by an equivalence

$$\mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|\mathsf{BAut}(K_+,L_+,K_+\wedge L_+ \xrightarrow{f} I_+)} \;\simeq\; \left(\prod_{i\in I}\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)\right)_{/\mathsf{Aut}(K_+,L_+,K_+\wedge L_+\xrightarrow{f} I_+)} \tag{35}$$

for each $(K_+, L_+, K_+\wedge L_+ \xrightarrow{f} I_+)$. Such an equivalence is, in turn, implied by a $\mathsf{Aut}(K_+, L_+, K_+\wedge L_+ \xrightarrow{f} I_+)$-equivariant equivalence between spaces:

$$\mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|K_+\wedge L_+ \xrightarrow{f} I_+} \;\simeq\; \left(\prod_{i\in I}\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)\right) . \tag{36}$$

We use Theorem A.64 to establish such an equivalence (36). First, observe that

$$\mathcal{U} := \left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|_{K_+\wedge L_+ \xrightarrow{f} I_+}}$$

is a poset. The bottom horizontal morphism in the diagram in equation (33) is an open embedding, so by Proposition A.9 the top horizontal arrow is as well. Using this, we define a $\mathsf{Aut}(K_+, L_+, K_+ \wedge L_+ \xrightarrow{f} I_+)$-equivariant functor

$$\mathcal{U} \to \prod_{i\in I}\mathsf{open}\left(\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)\right) \subset \mathsf{open}\left(\prod_{i\in I}\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i\subset X\times Y)\right) \tag{37}$$

given by

$$(K_+,(V_k)),(L_+,(W_\ell)) \mapsto \{(v_k)_K \mid v_k\in V_k\}\times\{(w_\ell)_L \mid w_\ell\in W_L\} .$$

This is well-defined since it takes values in $\mathsf{open}(X^K\times Y^L)$ that evidently restrict to opens in $\mathsf{Conf}_{f^{-1}(i)}(U_i)$. To invoke Theorem A.64 we must verify that for $c\in\mathsf{Conf}_{f^{-1}(i)}(U_i\subset X\times Y)$, the classifying space $\mathsf{B}\mathcal{U}_c\simeq *$ is contractible. This is true since the category is cofiltered, which can be seen using the fact that disks form a basis for opens (Example A.6). Finally, observe that each value of the functor in equation (33) is a contractible subspace of configuration space, since it is a product of disks. Thus, by Theorem A.64, we identify the [0]-points of each space in the statement of this proposition in a functorial manner. Similar to the proof of Lemma 4.13, since the functor is induced by an isotopy equivalence, Theorem A.65 gives us a weak homotopy equivalence of [0]-points. It remains to verify an equivalence on [1]-points. Adopting the notation of Lemma 4.13, observe the following diagram

$$\begin{tikzcd}
 & & & \mathsf{B}(\mathcal{C}'_{/c'})^{\mathcal{W}} \arrow[r] \arrow[d] & \mathsf{BFun}^{\mathcal{W}}([1],\mathcal{C}') \arrow[d, "\mathsf{ev}_1"] \\
\mathsf{B}(\mathcal{C}_{/c})^{\mathcal{W}} \arrow[r] \arrow[urrr, "\simeq"] \arrow[d] & \mathsf{B}(\mathsf{Fun}^{\mathcal{W}}([1],\mathcal{C}) \arrow[d, "\mathsf{ev}_1"] \arrow[urrr, dashed] & & * \arrow[r, "\langle c'\rangle"] & \mathsf{B}(\mathsf{Fun}^{\mathcal{W}}([0],\mathcal{C}') \\
* \arrow[r, "\langle c\rangle"'] \arrow[urrr, "\simeq"] & \mathsf{BFun}^{\mathcal{W}}([0],\mathcal{C}) \arrow[urrr, "\simeq"] & & &
\end{tikzcd} .$$

The top left diagonal equivalence was established in Lemma 4.13, and we just established the bottom right diagonal equivalence. Thus, the dashed arrow between [1]-points is manifestly an equivalence. ☐

Corollary 4.15. *The adjunction*

$$\rho^* : \mathit{Fun}^{m,J_\infty,opd}(\mathit{disk}(X\times Y)^{\otimes},\mathcal{V}^{\otimes}) \rightleftarrows \mathit{BiFun}^{m,J_\infty}(\mathit{disk}(X)^{\otimes},\mathit{disk}(Y)^{\otimes};\mathcal{V}^{\otimes}) : \rho_!$$

is an equivalence. Further, the adjunction restricts to an adjunction between locally constant subcategories

$$\rho^* : \mathit{Fun}^{\mathsf{m},J_\infty,\mathsf{l.c.},\mathsf{opd}}(\mathit{disk}(X\times Y)^{\otimes},\mathcal{V}^{\otimes}) \rightleftarrows \mathit{BiFun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathit{disk}(X)^{\otimes},\mathit{disk}(Y)^{\otimes};\mathcal{V}^{\otimes}) : \rho_!.$$

Manifestly, the restricted adjunction is an equivalence.

Proof. The first statement is immediate from Theorem 3.1, Lemma 4.6, and Lemma 4.10. For the second statement, consider $\mathcal{F} \in \mathsf{BiFun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X)^{\otimes},\mathsf{disk}(Y)^{\otimes};\mathcal{V}^{\otimes})$. We wish to show that $\rho_!\mathcal{F} \in \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{l.c.},\mathsf{opd}}(\mathsf{disk}(X\times Y)^{\otimes},\mathcal{V}^{\otimes})$. To prove this, we restrict to the ∞-sub-operads of connected disks. That is, given an isotopy equivalence, $(I_+,(U_i)) \overset{\varphi}{\hookrightarrow} (J_+,(V_j))$ in $\mathsf{disk}(X\times Y)^{\otimes}_{\mathsf{c}}$, we show the morphism

$$\rho_!\mathcal{F}((I_+,(U_i))) \to \rho_!\mathcal{F}((J_+,(V_j)))$$

in $\mathcal{V}^{\otimes}$ is an equivalence. Recall that $\rho_!\mathcal{F}((I_+,(U_i)))$ is computed as the following colimit

$$\operatorname{colim}\left((\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))} \xrightarrow{\text{fgt}} (\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/I_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/I_+} \xrightarrow{(-)_!} \mathcal{V}^{\otimes}_{|I_+}\right).$$

Note that the following diagram commutes

$$\begin{tikzcd}
(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(I_+,(U_i))}] \arrow[d, "\text{fgt}"] & (\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))} \arrow[l, "\text{loc}"'] \arrow[d, "\text{fgt}"] \\
(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}] \arrow[dr, dashed] & \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}} \arrow[l, "\text{loc}"'] \arrow[d, "\mathcal{F}"] \\
 & \mathcal{V}^{\otimes}
\end{tikzcd}$$

The existence of the dashed arrow is given by the assumption that $\mathcal{F}$ is locally constant. By Proposition A.25 localizations are final, so

$$\rho_!\mathcal{F}((I_+,(U_i))) \simeq \operatorname{colim}\left((\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(I_+,(U_i))}] \to \mathcal{V}^{\otimes}|_{I_+}\right),$$

and likewise for $\rho_!\mathcal{F}((J_+,(V_j)))$. Thus, by Proposition 4.14, we have $\rho_!\mathcal{F}((I_+,(U_i))) \to \rho_!\mathcal{F}((J_+,(V_j)))$ is an equivalence. ☐

A Appendix

In this appendix we briefly recall the necessary definitions and notations that underlie this paper.

(Higher) Category theory

In this section we record some essential notation, definitions, and results from (∞-)category theory that we use freely in this paper. For more details on the foundations of ordinary category theory, we refer the reader to [Awo10] and [Rie16]. For early developments of ∞-category theory, we refer the reader to [Rez01], [Lur09], and/or [RV22].

Definition A.1. An ∞-category $\mathcal{C}$ is called *cofiltered* if every functor $\mathcal{K} \xrightarrow{F} \mathcal{C}$ between ∞-categories extends to a functor out of the left cone

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "F"] \arrow[d, hook] & \mathcal{C} \\
\mathcal{K}^{\triangleleft} \arrow[ur, dashed] &
\end{tikzcd}.$$

Proposition A.2. *Let P be a poset and $\mathcal{K}$ be an ∞-category. The restriction*

$$\mathsf{Fun}(\mathcal{K}, P) \to \mathsf{Fun}(\mathsf{Obj}(\mathcal{K}), P) \simeq \mathsf{Fun}(\pi_0 \mathsf{Obj}(\mathcal{K}), P)$$

is a monomorphism.

Corollary A.3. *Let P be a poset and $\mathcal{K}$ be an ∞-category. A functor $F : \mathcal{K} \to P$ admits an extension*

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] \arrow[d, hook] & P \\ \mathcal{K}^{\triangleleft} \arrow[ur, dashed] & \end{tikzcd}$$

if and only if there exists an extension

$$\begin{tikzcd} \pi_0(\mathsf{Obj}(\mathcal{K})) \arrow[r, "F"] \arrow[d, hook] & \mathcal{C} \\ \pi_0(\mathsf{Obj}(\mathcal{K})^{\triangleleft}) \arrow[ur, dashed] & \end{tikzcd} .$$

Corollary A.4. *A poset P is cofiltered (as an ∞-category) if and only if for all finite subsets $S \subset P$, there exists $p_{-\infty} \in P$ such that for all $s \in S$, we have $p_{-\infty} \leq s$.*

Example A.5. Let X be a topological space. By Corollary A.4, the poset $\mathsf{open}(X)$ is cofiltered since the finited intersection of open sets is open.

Example A.6. Let X be a topological manifold. Using Corollary A.4, the subposet $\mathsf{disk}(X)$ is cofiltered. Namely, the finite intersection of disks is open, and since X is a manifold, $\mathsf{disk}(X)$ is a basis, so we can find an element of $\mathsf{disk}(X)$ in the finite intersection.

Proposition A.7. *If a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\ \mathcal{B} \arrow[r, "F"] & \mathcal{B}' \end{tikzcd}$$

is a pullback, then for all $b \in \mathcal{B}$ the functor

$$F_{|_b} : \mathcal{E}_{|_b} \to \mathcal{E}'_{|_{F(b)}}$$

is an equivalence.

Proposition A.8. *A diagram of spaces*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\ \mathcal{B} \arrow[r, "F"] & \mathcal{B}' \end{tikzcd}$$

is a pullback, if and only if for all $[b] \in \pi_0\mathcal{B}$, there exists some $\widetilde{b} \in [b]$ for which the functor

$$F_{|_{\widetilde{b}}} : \mathcal{E}_{|_{\widetilde{b}}} \to \mathcal{E}'_{|_{F(\widetilde{b})}}$$

is an equivalence.

Proposition A.9. *Consider a pullback diagram in topological spaces*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & Y' . \end{tikzcd}$$

If f is an open embedding, then so is g.

There is a technical condition that we need to assume on our target symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ in this paper to enable us to compute colimits.

Definition A.10 ([AF15] Definition 3.4)**.** A symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ is *$\otimes$-presentable* if it is presentable, and if for each $V \in \mathcal{V}^{\otimes}$, the functor $V \otimes - : \mathcal{V} \to \mathcal{V}$ takes colimit diagrams to colimit diagrams.

In particular, $\mathcal{V}^{\otimes}$ being presentable means that all colimits exist. We also use that $V \otimes -$ distributes over colimits to compute certain colimits in some of our proofs.

Definition A.11. A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is *final* if for each functor $\mathcal{D} \to \mathcal{E}$ to another ∞-category the canonical morphism

$$\operatorname{colim}(\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathcal{E}) \to \operatorname{colim}(\mathcal{D} \to \mathcal{E})$$

is an equivalence, provided the colimits exist.

Note that if $\mathcal{D}$ has a final object, d, then the inclusion $* \xrightarrow{\langle d \rangle} \mathcal{D}$ is a final functor.

Complete Segal spaces and localization

Complete Segal spaces as developed by Rezk in [Rez01] are one model for the theory of ∞-categories. Though we work model independently in this paper, we explicitly use complete Segal spaces to use a theorem of Mazel-Gee [MG19] to identify localizations of ∞-categories. Here we recall the basics of complete Segal spaces.

Complete Segal spaces are simplicial presheaves of spaces satisfying two conditions. To describe simplicial objects, we recall the simplex category.

Definition A.12. The *simplex category* Δ is the category of finite nonempty linearly ordered sets and order preserving maps between them.

We denote objects in Δ by $[p] := \{0 < \cdots < p\}$ for $p \in \mathbb{Z}_{>0}$.

Now, we define what we mean by a space.

Definition A.13. The *∞-category of spaces* Spaces is the category of topological spaces that admit a CW structure localized on the weak homotopy equivalences.

Definition A.14. A *simplicial space* is a functor $\Delta^{\mathsf{op}} \to \mathsf{Spaces}$. The simplicial space represented by an object $[p] \in \Delta^{\mathsf{op}}$ is denoted simply

$$[p]\colon \Delta^{\mathsf{op}} \longrightarrow \mathsf{Spaces} , \qquad [q] \longmapsto \mathsf{Hom}_{\Delta}\big([q],[p]\big) .$$

There is a special class of simplicial spaces called the Segal spaces

Definition A.15. A simplicial space, $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ is a *Segal space* if for every integer $p > 1$ the diagram

$$\begin{tikzcd} F[p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F\{p-1<p\} \arrow[d] \\ F\{0<\cdots<p-1\} \arrow[r] & F\{p-1\} \end{tikzcd}$$

is a pullback of spaces.

Given a Segal space, there is a subspace of [1]-points that have both left and right inverses. We call these [1]-points equivalences.

Definition A.16. Let $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ be a Segal space. An *equivalence* in F is map between simplicial sets

$$[1] \xrightarrow{\langle x \xrightarrow{f} y \rangle} F$$

such that dashed arrows in the following two diagrams among simplicial sets exist

$$\begin{tikzcd} \{0<1\} \arrow[d, hook] \arrow[dr, bend left, "f"] & \\ {[2]} \arrow[r, dashed] & F \\ \{0<2\} \arrow[u, hook] \arrow[r] & * \arrow[u, "\langle x \rangle"'] \end{tikzcd} \tag{38}$$

$$\begin{tikzcd} \{1<2\} \arrow[d, hook] \arrow[dr, bend left, "f"] & \\ {[2]} \arrow[r, dashed] & F \\ \{0<2\} \arrow[u, hook] \arrow[r] & * \arrow[u, "\langle y \rangle"'] \end{tikzcd} \tag{39}$$

We denote the subspace of equivalences by $F^{\mathsf{equiv}}[1] \subset F[1]$.

The diagram in equation (38) asserts that f has a left inverse, and the diagram in equation (39) asserts that f has a right inverse.

Let $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ be a Segal space. The unique map from $[1] \to [0]$ induces a map $F[0] \to F[1]$ that uniquely factors through the equivalences F^{equiv}. This maps the [0]-points to degenerate [1]-points.

Definition A.17. A *complete Segal space* is a Segal space $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ for which the map $F[0] \to F^{\mathsf{equiv}}[1]$ is an equivalence of spaces.

The Segal condition says that the [0]-points and [1]-points determine the [p]-points. This is useful for identifying two complete Segal spaces, as codified in the following observation.

Observation A.18. Two complete Segal spaces $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is an equivalence between [0]-points and [1]-points.

Central to the proof of the Theorem 4.1 is the identification of the localization of an ∞-category via Theorem A.26 below. We define localizations using classifying spaces, or ∞-groupoid completions. The idea of localization of an ∞-category $\mathcal{C}$ is to formally invert a specific class of morphisms in $\mathcal{C}$. If we simply invert only the isomorphisms, then we obtain the original ∞-category $\mathcal{C}$. On the other hand, if we invert every morphism in $\mathcal{C}$, then we obtain the classifying space, or ∞-groupoid completion of $\mathcal{C}$.

Definition A.19. As developed in [Lur09], there exists a left adjoint to the inclusion

$$\begin{tikzcd} \mathsf{Cat}_{(\infty,1)} \arrow[r, bend left, "\mathsf{B}"] & \mathsf{Spaces} \arrow[l, hook] \end{tikzcd}$$

of the ∞-category of spaces into the ∞-category of ∞-categories. For $\mathcal{C}$ an ∞-category, we call the value of the left adjoint $\mathsf{B}\mathcal{C}$ the *classifying space of* $\mathcal{C}$.

Remark A.20. If one takes complete Segal spaces as a model for ∞-categories, then the classifying space of a complete Segal space $\mathcal{C} : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ is given by the colimit $\mathsf{B}\mathcal{C} := \operatorname{colim} \mathcal{C}$. If one takes quasicategories as a model for ∞-categories, then the classifying space of a quasicategory $\mathcal{C} : \Delta^{\mathsf{op}} \to \mathsf{Set}$ is given by the geometric realization $|\mathcal{C}|$.

Observation A.21. Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ possesses an initial object, then its classifying space $\mathsf{B}\mathcal{C}$ is contractible. Dually, if $\mathcal{C}$ possesses a final object, then its classifying space $\mathsf{B}\mathcal{C}$ is contractible.

Proposition A.22 ([MG19] Corollary 1.28)**.** *An adjunction between ∞-categories induces an equivalence between their classifying spaces.*

Using the notion of a classifying space, we can now define the localization of an ∞-category.

Definition A.23. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{W} \subset \mathcal{C}$ be an ∞-subcategory that contains all the equivalences in $\mathcal{C}$. The *localization of $\mathcal{C}$ at $\mathcal{W}$* is defined to be the pushout

$$\begin{tikzcd} \mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathsf{B}\mathcal{W} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{C}[\mathcal{W}^{-1}] \end{tikzcd} .$$

Example A.24. If we take $\mathcal{W} = \mathcal{C}^{\sim}$ to be the maximal ∞-subgroupoid of $\mathcal{C}$, then $\mathcal{C}[\mathcal{C}^{\sim}] \simeq \mathcal{C}$. At the other end of the spectrum, if we localize $\mathcal{C}$ on all morphisms, then we obtain the classifying space of $\mathcal{C}$. That is, $\mathcal{C}[\mathcal{C}^{-1}] \simeq \mathsf{B}\mathcal{C}$.

Proposition A.25 ([AF20a] Proposition 5.13)**.** *A localization of ∞-categories is both final and initial.*

Theorem A.26 ([MG19] Theorem 3.8)**.** *Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{W} \subset \mathcal{C}$ be an ∞-subcategory that contains the maximal ∞-subgroupoid of $\mathcal{C}$. If the classifying space $\mathsf{BFun}^{\mathcal{W}}([\bullet], \mathcal{C})$ is a complete Segal space, then there is an equivalence of ∞-categories*

$$\mathsf{BFun}^{\mathcal{W}}([\bullet], \mathcal{C}) \simeq \mathcal{C}[\mathcal{W}^{-1}] .$$

Here, $\mathsf{Fun}^{\mathcal{W}}([\bullet], \mathcal{C})$ denotes the simplicial category whose $[p]$-points are defined as the following pullback of ∞-categories

$$\begin{tikzcd} \mathsf{Fun}^{\mathcal{W}}([p], \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}([p], \mathcal{C}) \arrow[d] \\ \mathsf{Fun}([p]^{\sim}, \mathcal{W}) \arrow[r] & \mathsf{Fun}([p]^{\sim}, \mathcal{C}) \end{tikzcd}$$

where $[p]^{\sim}$ denotes the maximal ∞-subgroupoid of $[p]$.

Observation A.27. Note the $[0]$-points are equivalent to $\mathcal{W}$. Also, the $[1]$-points are given by natural transformations whose morphisms are drawn from $\mathcal{W}$.

∞-operads

We use the theory of ∞-operads as developed by Lurie in §2 of [Lur17]. The notion of ∞-operad is an ∞-categorical analog of a multicategory, or colored operad. We now recall the basic definitions and notation of this theory.

Colored operads can be thought of as symmetric monoidal categories where the symmetric monoidal product is not actually representable. The category of based finite sets is used to organize ∞-operads.

Definition A.28. Let Fin_* denote the category of based finite sets with based maps between them.

Typically, we will denote objects of Fin_* by I_+, where I is a finite set and $+$ is a disjoint basepoint. There are several special classes of morphisms in Fin_*.

Definition A.29. A morphism $I_+ \xrightarrow{f} J_+$ in Fin_* is called

- *inert* if $f^{-1}(j) \simeq *$ for all $j \in J$;
- *active* if $f^{-1}(+) = \{+\}$.

Observation A.30. The inert and active morphisms form a factorization system on Fin_*. By this we mean that every morphism $I_+ \to J_+$ in Fin_* can be uniquely factored as the composition $I_+ \to \widetilde{I}_+ \to J_+$ of an inert morphism followed by an active morphism.

Definition A.31. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. A morphism f in $\mathcal{D}$ is called F-*coCartesian* if there exists an initial filler for each solid diagram of categories

$$\begin{tikzcd} * \arrow[r, "\langle C \rangle"] \arrow[d, "\langle s \rangle"'] & \mathcal{C} \arrow[d, "F"] \\ \{s < t\} \arrow[r, "\langle f \rangle"'] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} .$$

We will denote the lift of f by $f_!$.

Definition A.32. Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an F-coCartesian morphism in $\mathcal{D}$, $f : D \to D'$, we can consider the *coCartesian monodromy functor of f*

$$f_! : \mathcal{C}_{|_D} \to \mathcal{C}_{|_{D'}}$$

that sends $C \in \mathcal{C}_{|_D}$ to $f_!(C)$, the coCartesian lift of f evaluated at C.

Definition A.33. A functor $F : \mathcal{C} \to \mathcal{D}$ is a *coCartesian fibration* if every morphism in $\mathcal{D}$ is F-coCartesian.

Proposition A.34. *Let $E \to B$ be a coCartesian fibration. For each $b \in B$, the canonical functor*

$$E_{|_b} \hookrightarrow E_{/_b}$$

is a right adjoint.

Definition A.35. Let $E \to B$ be a coCartesian fibration. For $b \xrightarrow{f} b'$ in B, the coCartesian monodromy functor is defined via the left adjoint to the above right adjoint:

$$\begin{tikzcd} E_{|_b} \arrow[r, dashed, "f_!"] \arrow[d, hook] & E_{|_{b'}} \arrow[d, hook, shift right] \\ E_{/_b} \arrow[r, "f\circ -"] & E_{/'_b} \arrow[u, shift right] \end{tikzcd} .$$

Definition A.36. A functor $F : \mathcal{C} \to \mathsf{Fin}_*$ is called *inert-coCartesian fibration* if each inert morphism in Fin_* is F-coCartesian.

Definition A.37. An ∞-*operad* is an ∞-category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathsf{Fin}_*$ such that

1. F is an inert-coCartesian fibration;
2. for all $I_+ \in \mathsf{Fin}_*$, the canonical functor

$$\mathcal{C}|_{I_+} \xrightarrow{((c_i)_!)_{i\in I}} \prod_{i\in I} \mathcal{C}|_{\{i\}_+}$$

is an equivalence of ∞-categories;

3. for every $f : I_+ \to J_+$ in Fin_* and every $O \in \mathcal{C}|_{I_+}$ and $P \in \mathcal{C}|_{J_+}$, the canonical map between spaces

$$\mathrm{Map}_{\mathcal{C}}(O, P)|_f \xrightarrow{((c_j)_!(O)\circ -)_{j\in J}} \prod_{j\in J} \mathrm{Map}_{\mathcal{C}}(O, P_j)|_{c_j \circ f}$$

is an equivalence of ∞-categories.

Say an ∞-operad $\mathcal{C} \xrightarrow{F} \mathsf{Fin}_*$ is *unital* if the fiber ∞-category $F^{-1}(1_+) \simeq \{\emptyset\}$ is a singleton, call it $\emptyset$, and this object $\emptyset \in \mathcal{C}$ is initial.

We now make precise the idea that ∞-operads generalize ordinary colored operads, or multicategories.

Construction A.38. Let $\mathcal{O}$ be a multicategory. There exists a category $\mathcal{O}^{\otimes} \xrightarrow{\pi_{\mathcal{O}}} \mathsf{Fin}_*$ over the category of non-empty based finite sets. An object in $\mathcal{O}^{\otimes}$ is a pair $(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O}))$ consisting of a based finite set I_+, and a map $O_- : I \to \mathrm{obj}(\mathcal{O})$, $i \mapsto O_i \in \mathcal{O}$ that selects out an object of $\mathcal{O}$ for each $i \in I$. We might suppress notation and refer to an object, $(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O}))$, as the list $(O_i)_{i\in I}$ or even just (O_i). A morphism of objects

$$(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O})) \to (J_+, J \xrightarrow{P_-} \mathrm{obj}(\mathcal{O}))$$

consists of a map of based finite sets, $I_+ \xrightarrow{f} J_+$, and for each $j \in J$, a multimorphism $g_j \in \mathcal{O}((O_i)_{i\in f^{-1}(j)}; P_j)$.

Observation A.39. Given a multicategory $\mathcal{O}$, the functor $\pi_{\mathcal{O}}$ from Construction A.38 is inert coCartesian. Namely, for $f : I_+ \to J_+$ an inert morphism, and $(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O})) \in \mathcal{O}^{\otimes}$, we have

$$f_!(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O})) \simeq (J_+, J \xrightarrow{O_{f^{-1}(-)}} \mathrm{obj}(\mathcal{O})).$$

That is, $f_!((O_i)_{i\in I}) \simeq (O_{f^{-1}(j)})_{j\in J}$. Furthermore, Construction A.38 actually produces an ∞-operad.

An extremely important example of Construction A.38 is the following.

Example A.40. Let X be a topological space. Let $\mathsf{open}(X)$ denote the poset whose objects are open sets in X with partial order given by inclusion of open sets. This can be thought of as a multicategory by declaring that the collection of multimorphisms from a list of opens $(U_i)_{i\in I}$ to another open V is a singleton if $U_i \subset V$ for each $i \in I$ and $U_i \cap U_{i'} = \emptyset$ for each $i \neq i' \in I$, and otherwise the collection of multimorphisms is the emptyset. Construction A.38 produces an ∞-operad $\mathsf{open}(X)^{\otimes}$. We can think of an object of $\mathsf{open}(X)^{\otimes}$ as an I indexed list of open sets in X. We will typically denote such objects by $(I_+, (U_i))$. Note that a morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ is a map of based finite sets $I_+ \xrightarrow{f} J_+$ satisfying the condition that for each $j \in J$, the collection $(U_i)_{f^{-1}(j)}$ is a pairwise disjoint collection of open sets of V_j.

Observation A.41. Note that $\mathsf{open}(X)^{\otimes}_{|1_+} \simeq \mathsf{open}(X)$.

There is a special class of ∞-operads that play an important role in our arguments.

Definition A.42. A *symmetric monoidal ∞-category* is an ∞-operad $\mathcal{O}^{\otimes} \xrightarrow{\pi} \mathsf{Fin}_*$ for which π is a coCartesian fibration.

Remark A.43. Ordinary symmetric monoidal categories can be thought of as multicategories where the collection of multimorphisms is given by the collection of maps out of the tensor product. In this way, one can again use Construction A.38 to produce a symmetric monoidal ∞-category from an ordinary symmetric monoidal category.

Tensor products and bifunctors

As mentioned in the introduction, one of the key ideas underlying the proof of general additivity is that the ∞-category of ∞-operads possesses a tensor product with the property that

$$\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})) \simeq \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes} \otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \ .$$

The defining feature of the tensor product of ∞-operads is such that there is an equivalence of ∞-categories

$$\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes} \otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

We now spell this out a little more, and in particular, define the ∞-category

$$\mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$$

of bifunctors between ∞-operads. We refer the interested reader to §2.2.5 of [Lur17] for details.

Define the smash product functor of based finite sets as follows:

$$\mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$$

$$I_+, J_+ \mapsto I_+ \wedge J_+ := (I \times J)_+$$

and for $f : I_+ \to K_+$ and $g : J_+ \to L_+$ based maps, define the based map

$$f \wedge g : I_+ \wedge J_+ \to K_+ \wedge L_+$$

given by

$$(f \wedge g)(u,v) \ := \ \begin{cases} \big(f(i), g(j)\big) \ , & \text{if } f(u) \neq + \text{ and } g(v) \neq + \\ + \ , \ \text{otherwise.} \end{cases}$$

Definition A.44. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$, and $\mathcal{Q}^{\otimes}$ be ∞-operads. A *bifunctor of* ∞*-operads* is a functor

$$\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \xrightarrow{\varphi} \mathcal{Q}^{\otimes}$$

such that

1. the following diagram commutes

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "\varphi"] \arrow[d] & \mathcal{Q}^{\otimes} \arrow[d] \\ \mathsf{Fin}_* \times \mathsf{Fin}_* \arrow[r, "\wedge"] & \mathsf{Fin}_*. \end{tikzcd}$$

2. φ takes pairs of inert coCartesian morphisms to inert coCartesian morphisms.

Definition A.45. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$, and $\mathcal{Q}^{\otimes}$ be ∞-operads, and let $\varphi : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ be a bifunctor. The bifunctor φ *exhibits* $\mathcal{Q}^{\otimes}$ *as a tensor product of* $\mathcal{O}^{\otimes}$ *and* $\mathcal{P}^{\otimes}$ if for any ∞-operad, $\mathcal{C}^{\otimes}$, the functor

$$\mathsf{Fun}^{\mathsf{opd}}(\mathcal{Q}^{\otimes}, \mathcal{C}^{\otimes}) \to \mathrm{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{C}^{\otimes})$$

given by precomposition with φ is an equivalence.

We attempted to directly work with the tensor product, but even for the relatively simple ∞-operads like $\mathsf{open}(X)^{\otimes}$ and $\mathsf{disk}(X)^{\otimes}$, we encountered trouble explicitly identifying the tensor product. The tensor product of ∞-operads as given above should be a generalization of the Boardman-Vogt tensor product of ordinary operads. This is another interesting aspect of the tensor product that we have yet to unravel.

Left Kan extension

We use left Kan extensions in a variety of contexts throughout the body of this work. In this section we recall the basic definitions of ordinary left Kan extension and operadic left Kan extension. Additionally, we provide basic results that we utilize.

Ordinary left Kan extension

Given a diagram of categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\rho"] & \mathcal{D} \\ \mathcal{E} & \end{tikzcd},$$

one might wish for an extension of F. That is, a functor $\widetilde{F} : \mathcal{E} \to \mathcal{D}$ that fills the above diagram. Often, such a filler will not exist. However, a left Kan extension is a natural approximation to a filler arrow. In fact, it is the initial approximation.

Definition A.46. Given a diagram of categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\rho"] & \mathcal{D} \\ \mathcal{E} & \end{tikzcd},$$

the *left Kan extension of F along ρ* is a functor $\rho_!F : \mathcal{E} \to \mathcal{D}$ and a natural transformation $\varepsilon : F \to \rho_!F \circ \rho$. This data satisfies the following universal property: given another functor $G : \mathcal{E} \to \mathcal{D}$ and natural transformation $\beta : F \to G \circ \rho$, there exists a unique natural transformation $\sigma : \rho_!F \to G$ such that the following diagram commutes

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[dr, "\beta"'] & \rho_!F \circ \rho \arrow[d, "\sigma_{\rho(-)}"] \\ & G \circ \rho \end{tikzcd}.$$

Example A.47. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If it exists, the left Kan extension of F along the unique functor $\mathcal{C} \to *$ is given by $\operatorname{colim} F$.

The following proposition tells us that when they exist, left Kan extensions compose.

Proposition A.48. *Assume we have a solid diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\alpha"] & \mathcal{D} \\ \mathcal{B} \arrow[d, "\beta"] & \\ \mathcal{C} & \end{tikzcd}.$$

If the left Kan extensions $\alpha_!F$ and $(\beta \circ \alpha)_!F$ exist, there is an equivalence of functors

$$(\beta \circ \alpha)_!F \simeq \beta_!(\alpha_!F) .$$

The following proposition is useful for working with left Kan extensions along coCartesian fibrations, such as projections. It tells us that the left Kan extension along a coCartesian fibration evaluates as a fiberwise colimit.

Proposition A.49 ([Lur09] Proposition 4.3.3.10)**.** *Given a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathcal{E} \\ \mathcal{D} \end{tikzcd}$$

for which π is a coCartesian fibration, if the left Kan extension $\pi_! F$ of F along π exists, then for all $D \in \mathcal{D}$, the left Kan extension evaluates as

$$\pi_! F(D) \simeq \operatorname{colim}\left(\mathcal{C}_{|_D} \xrightarrow{F} \mathcal{E}\right)$$

a colimit indexed by the fiber over D.

Proposition A.50. *Consider a commutative diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, "F"] \arrow[d, "\rho"] & \mathcal{V} \arrow[d, "\pi"] \\ \mathcal{E} \arrow[r, "p"] & \mathcal{B} . \end{tikzcd}$$

Provided that for all $e \in \mathcal{E}$, the canonical morphism in $\mathcal{B}$

$$\pi\left(\operatorname{colim}(\mathcal{E}_{0/e} \to \mathcal{E}_0 \xrightarrow{F} \mathcal{V})\right) \to p(e) \tag{40}$$

is an equivalence, then the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\rho_! F"] \arrow[d, "p"] & \mathcal{V} \arrow[dl, "\pi"] \\ \mathcal{B} \end{tikzcd}$$

canonically commutes. If, in addition, $\mathcal{V} \xrightarrow{\pi} \mathcal{B}$ is a coCartesian fibration, then for all $e \in \mathcal{E}$, there exists a canonical equivalence

$$\rho_! F(e) \simeq \operatorname{colim}\left(\mathcal{E}_{0/e} \to \mathcal{E}_{0/p(e)} \xrightarrow{F} \mathcal{V}_{/p(e)} \xrightarrow{(-)_!} \mathcal{V}|_{p(e)}\right) .$$

Proof. For the first statement, recall that for $e \in \mathcal{E}$, the left Kan extension is given by $\rho_! F(e) = \operatorname{colim}(\mathcal{E}_{0/e} \to \mathcal{E}_0 \xrightarrow{F} \mathcal{V})$. Observe the commutative diagram

$$\begin{tikzcd} \mathcal{E}_{0/e} \arrow[r] \arrow[dr] & \mathcal{E}_0 \arrow[r, "F"] & \mathcal{V} \\ & \mathcal{E}_{0/p(e)} \arrow[u] \arrow[r] & \mathcal{V}_{/p(e)} \arrow[u] \end{tikzcd} .$$

Since $\mathcal{V}_{/p(e)} \xrightarrow{\mathsf{fgt}} \mathcal{V}$ preserves and detects colimits, we see that

$$\operatorname{colim}\left(\mathcal{E}_{0/e} \to \mathcal{E}_{0/p(e)} \to \mathcal{V}_{/p(e)}\right) \simeq \left(\rho_! F(e), \pi(\rho_! F(e)) \to p(e)\right) .$$

The inclusion $\mathcal{V}|_{p(e)} \hookrightarrow \mathcal{V}_{/p(e)}$ is fully faithful with image consisting of those objects whose morphism to $p(e)$ is an equivalence. Thus, the hypothesis of equation (40) guarantees that $\rho_! F(e) \in \mathcal{V}|_{p(e)}$, which completes the first statement.

Now, since π is a coCartesian fibration, Proposition A.34 tells us that

$$\mathcal{V}_{|_b} \overset{\mathsf{R}}{\hookrightarrow} \mathcal{V}_{/_b}$$

is a right adjoint. Now, the first part of this proposition showed that $\rho_!\mathcal{F}(e) \in \mathcal{V}_{|_{p(e)}} \simeq \mathsf{im}(\mathsf{R})$, and thus $\rho_!\mathcal{F}(e) = \mathsf{R}(e')$ for some $e' \in \mathcal{V}_{|_{p(e)}}$. By definition of an adjunction, the following diagram commutes

$$\begin{tikzcd} \mathsf{R}(e') \arrow[r, "\mathsf{unit}"] \arrow[dr, "\mathsf{id}"'] & \mathsf{RLR}e' \arrow[d, "\mathsf{R(counit)}"] \\ & \mathsf{R}e' \ . \end{tikzcd}$$

Note that the right vertical arrow is an equivalence since R is a right adjoint. Since R is fully faithful, this implies that the unit

$$\rho_!\mathcal{F}(e) \to \mathsf{RL}\rho_!\mathcal{F}(e)$$

is an equivalence. Since left adjoints preserve colimits, we see that

$$\mathsf{L}\rho_!\mathcal{F}(e) \simeq \operatorname{colim}\left(\mathcal{E}_{0/e} \to \mathcal{E}_{0/p(e)} \xrightarrow{F} \mathcal{V}_{/p(e)} \xrightarrow{(-)_!} \mathcal{V}|_{p(e)}\right) \ ,$$

which completes the proof. $\square$

Operadic left Kan extension

Central to several of our proofs is an operadic version of left Kan extension. The general theory of operadic left Kan extension is detailed in section 3.1.2 of [Lur17]. In this subsection we establish a colimit formula for computing operadic left Kan extensions within the context of this paper. Namely, we prove the following formula:

Proposition A.51. *Let $\iota : \mathcal{D}^\otimes \hookrightarrow \mathcal{O}^\otimes$ be a fully faithful functor between ∞-operads with $\mathcal{D}^\otimes$ unital. Given a morphism of ∞-operads, $\mathcal{F} : \mathcal{D}^\otimes \to \mathcal{V}^\otimes$, with target a $\otimes$-presentable symmetric monoidal ∞-category, the operadic left Kan extension of $\mathcal{F}$ along ι evaluates as*

$$\iota_!\mathcal{F}((I_+,(O_i)) \simeq \operatorname{colim}\left(\mathcal{D}^\otimes_{/^{act}_{(I_+,(O_i))}} \xrightarrow{fgt} \mathcal{D}^\otimes_{/^{act}_{I_+}} \xrightarrow{\mathcal{F}_{/I_+}} \mathcal{V}^\otimes_{/^{act}_{I_+}} \xrightarrow{\otimes} \mathcal{V}^\otimes_{|_{I_+}}\right) \ \in \ \mathcal{V}^\otimes_{|I_+} \ .$$

Proof. We first prove the first statement concerning values of $\iota_!\mathcal{F}$. Proposition 4.3.2.17 in [Lur09] establishes the adjunction of functors over Fin_* :

$$\iota_! : \mathsf{Fun}_{\mathsf{Fin}_*}(\mathcal{D}^\otimes, \mathcal{V}^\otimes) \rightleftarrows \mathsf{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^\otimes, \mathcal{V}^\otimes) : \iota^*.$$

It remains to check that $\iota_!\mathcal{F}$ takes inert-coCartesian morphisms to inert-coCartesian morphisms. Let $f : I_+ \to J_+$ be an inert morphism and consider $(I_+,(O_i)_I) \xrightarrow{f_!} (J_+,(O_j)_J)$ the coCartesian morphism in $\mathcal{O}^\otimes$. Consider the diagram

$$\begin{tikzcd} \mathcal{D}^\otimes_{/^{\mathrm{act}}_{(I_+,(O_i))}} \arrow[r] \arrow[d, "f_!"] & \mathcal{D}^\otimes_{/^{\mathrm{act}}_{I_+}} \arrow[r] \arrow[d, "f_!"] & \mathcal{V}^\otimes_{/^{\mathrm{act}}_{I_+}} \arrow[r] \arrow[d, "f_!"] & \mathcal{V}^\otimes|_{I_+} \arrow[d, "f_!"] \\ \mathcal{D}^\otimes_{/^{\mathrm{act}}_{(J_+,(O_j))}} \arrow[r] & \mathcal{D}^\otimes_{/^{\mathrm{act}}_{J_+}} \arrow[r] & \mathcal{V}^\otimes_{/^{\mathrm{act}}_{J_+}} \arrow[r] & \mathcal{V}^\otimes|_{J_+}. \end{tikzcd}$$

Since f is inert, each of the vertical functors is a projection. Further, the leftmost vertical arrow is final, as justified through Quillen's Theorem A, which we have recorded as Theorem A.61. To invoke this theorem, we must show that for each $(K_+,(D_k)) \xrightarrow{\alpha} (J_+,(O_j))$ in $\mathcal{D}^\otimes_{/^{\mathsf{act}}_{(J_+,(O_j))}}$ the classifying space of the undercategory

$$\left(\mathcal{D}^\otimes_{/^{\mathrm{act}}_{(I_+,(O_i))}}\right)^{(K_+,(D_k))\xrightarrow{\alpha}(J_+,(O_j))/}$$

is contractible. Since $f : I_+ \to J_+$ is inert, $f^{-1}(j)$ is a singleton for each $j \in J$. This defines a section $\sigma : J_+ \to I_+$ of the map $I_+ \xrightarrow{f} J_+$. This enables us to canonically consider $(K_+, (D_k)) \xrightarrow{\alpha} (J_+, (O_j))$ as an object of $\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}}$ via the coCartesian lift along σ

$$(K_+, (D_k)) \xrightarrow{\alpha} (J_+, (O_j)) \xrightarrow{\sigma_!} (I_+, (U_i)) ,$$

where $U_i := O_j$ if $f(i) = j$ and $U_i := \emptyset$, the initial object of $\mathcal{D}^{\otimes}$, otherwise. This implies the undercategory

$$\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}}\right)^{(K_+,(D_k)) \xrightarrow{f} (J_+,(O_j))/}$$

has an initial object, and thus by Observation A.21, its classifying space is contractible. Therefore,

$$\begin{aligned}
\iota_!\mathcal{F}(f_!(I_+, (O_i))) &\simeq \iota_!\mathcal{F}(J_+, (O_j)) \\
&\simeq \mathrm{colim}\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{J_+,(O_j))}} \to \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{J_+}} \to \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{J_+}} \to \mathcal{V}^{\otimes}_{|_{J_+}}\right) \\
&\simeq \mathrm{colim}\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}} \to \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{|_{I_+}} \xrightarrow{f_!} \mathcal{V}^{\otimes}_{|_{J_+}}\right) \\
&\simeq f_!\left(\mathrm{colim}\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}} \to \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{|_{I_+}}\right)\right),
\end{aligned}$$

where the last equivalence follows because the projection

$$f_! : \mathcal{V}^J \simeq \mathcal{V}^{\otimes}_{|_{J_+}} \to \mathcal{V}^{\otimes}_{|_{I_+}} \simeq \mathcal{V}^I$$

preserves colimits.

□

Cosheaves

Factorization algebras are functors that satisfy a local-to-global property. This is codified in the idea of a cosheaf.

Informally, equipping a category, $\mathcal{C}$, with a Grothendieck topology specifies a notion of 'cover' for the objects in $\mathcal{C}$. This enables us to make sense of particular coherent systems of data on $\mathcal{C}$, namely (co)sheaves.

Definition A.52. For $C \in \mathcal{C}$, a *sieve* is a fully faithful functor $\mathcal{U} \hookrightarrow \mathcal{C}_{/C}$ such that for each $(D \xrightarrow{f} C) \in \mathcal{U}$ and $(E \xrightarrow{g} D) \in \mathcal{C}(E, D)$, we have $(E \xrightarrow{g} D \xrightarrow{f} C) \in \mathcal{U}$.

Intuitively, we think of a sieve as specifying the allowable ways of accessing the object C.

Observation A.53. Let $\mathcal{C}$ be a category. Let $C \in \mathcal{C}$ be an object. Let $\{U_\alpha \to C\}$ be a collection of objects in the overcategory $\mathcal{C}_{/C}$. Consider the full subcategory $\overline{\{U_\alpha\}} \subseteq \mathcal{C}_{/C}$ consisting of those $D \xrightarrow{f} C$ such that there exists a factorization

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, dashed, "\exists"'] & C \\
U_\alpha \arrow[ur] &
\end{tikzcd} .$$

for some $U_\alpha \in \{U_\alpha\}$. This full subcategory $\overline{\{U_\alpha\}} \subseteq \mathcal{C}_{/C}$ is a sieve.

Definition A.54. A *Grothendieck topology*, τ, on $\mathcal{C}$ is

- for each $C \in \mathcal{C}$, a collection of *covering sieves* for C, denoted $\tau(C)$,

such that

1. for each $C \in \mathcal{C}$, $\mathcal{C}_{/C} \overset{=}{\hookrightarrow} \mathcal{C}_{/C}$ is in $\tau(C)$;
2. for $\mathcal{U} \in \tau(C)$, and $f : D \to C$ a morphism in $\mathcal{C}$, we have $f^*\mathcal{U} \in \tau(D)$;
3. if $\mathcal{U}$ is any sieve on $C \in \mathcal{C}$ such that the sieve
$$\bigcup_D \{f : D \to C \mid f^*\mathcal{U} \in \tau(D)\} \in \tau(C),$$
then in fact, $\mathcal{U} \in \tau(C)$.

For τ a Grothendieck topology on $\mathcal{C}$, say a collection of objects $\{U_\alpha \to C\} \subseteq \mathcal{C}_{/C}$ is a *cover in the τ-topology* if the sieve $\overline{\{U_\alpha\}}$ of Observation A.53 is a covering sieve.

Example A.55. Let X be a topological space. There is a standard Grothendieck topology on $\mathsf{open}(X)$ the poset of open sets in X where for $O \in \mathsf{open}(X)$, a sieve $\mathcal{U} = \{U_\alpha \hookrightarrow O\} \in \tau_{\mathsf{std}}(\mathsf{open}(X))$ is a cover iff for each $x \in O$ there exists some U_α containing x. Thus, the covering sieves are precisely the (complete) standard open covers.

A category can be equipped various different Grothendieck topologies, similar to how a set can be endowed with various distinct topologies. For example, consider the following family of topologies on $\mathsf{open}(X)$.

Example A.56. Let X be a topological space and consider the category $\mathsf{open}(X)$. For each integer $r > 0$ there is a topology on $\mathsf{open}(X)$ called the J_r-topology, let us denote this by τ_{J_r}. Given an open set $O \in \mathsf{open}(X)$, if a collection of open sets $\mathcal{U} = \{U_\alpha\}$ is a cover of O in the J_r-topology on $\mathsf{open}(X)$, then for each subset $S \subset O$ with cardinality at most r, there exists some $U_\alpha \in \mathcal{U}$ that contains S. As an example of the distinction between these topologies, consider the collection $\mathcal{U} := \{(-\infty, 1), (-1, \infty)\}$ of open subsets of $\mathbb{R}$. This is a J_1-cover since it is an ordinary open cover, however it is *not* a J_2-cover. To see this, consider the cardinality 2 subset $S = \{-2, 2\}$ of $\mathbb{R}$. Note that S is not contained in either element of $\mathcal{U}$.

Definition A.57. Let $(\mathcal{C}, \tau)$ be a site. A *basis*, $\mathcal{B}$, is a full subcategory of $\mathcal{C}$ with the property that every $C \in \mathcal{C}$ admits a τ-covering by objects in $\mathcal{B}$.

Example A.58. Let X be a topological n-manifold. The full subcategory $\mathsf{disk}(X)_{\mathsf{c}} \subset \mathsf{open}(X)$ consisting of those $U \in \mathsf{open}(X)$ for which $U \cong \mathbb{R}^n$ is a basis for the standard topology on $\mathsf{open}(X)$. However, $\mathsf{disk}(X)_{\mathsf{c}}$ is in general *not* a basis for any J_r-topology on $\mathsf{open}(X)$ when $r > 1$. Rather, consider $\mathsf{disk}(X) \subset \mathsf{open}(X)$ the full subcategory consisting of those $U \in \mathsf{open}(X)$ for which $U \cong \coprod_{\text{finite}} \mathbb{R}^n$ is homeomorphic to a finite disjoint union of open disks. Then $\mathsf{disk}(X)$ is a basis for every J_r-topology on $\mathsf{open}(X)$.

Recall the *right cone* of a category, $\mathcal{U}$, is given by
$$\mathcal{U}^{\triangleright} := \mathcal{U} \times \{0, 1\} \coprod_{\mathcal{U} \times \{1\}} *.$$

For $\mathcal{U} \subset \mathcal{C}_{/C}$, observe the functor
$$\mathcal{U}^{\triangleright} \to \mathcal{C}_{/C}$$
given by sending the cone point to $(C \xrightarrow{\mathrm{id}} C)$, and sending each morphism $(C' \to C) \xrightarrow{!} *$ to the obvious square.

Definition A.59. Let $(\mathcal{C}, \tau)$ be a site. The category of *($\mathcal{S}$-valued) cosheaves (w.r.t. τ)* is the full subcategory

$$\mathsf{cShv}^{\tau}(\mathcal{C}) \subset \mathsf{Fun}(\mathcal{C}, \mathcal{S})$$

consisting of those functors that have the property that for all $C \in \mathcal{C}$ and all covering sieves $\mathcal{U} \subset \mathcal{C}_{/C}$, the composite

$$\mathcal{U}^{\triangleright} \to \mathcal{C}_{/C} \xrightarrow{\text{fgt}} \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{S}$$

is a colimit diagram.

Remark A.60. The right cone in the above definition allows us to keep track of the map that is part of the data of an object in $\mathcal{C}_{/C}$. Note that there is a terminal object in the above diagram, namely $\mathcal{F}(C)$, and by the diagram being a colimit, we mean that this terminal point is the colimit of the diagram with the terminal point removed.

Quillen's theorems

Quillen's Theorem A is a useful tool for computing (co)limits, as it provides a way to check if a functor is final or initial. This is relevant for us because we often must analyze colimits via the colimit formula for (operadic) left Kan extension. We refer the reader to [AF20a] for more information on an ∞-categorical treatment of Quillen's Theorems A and B.

Theorem A.61 (Quillen's Theorem A)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. The functor F is final if and only if for each $D \in \mathcal{D}$, the classifying space*

$$\mathsf{B}(\mathcal{C}^{D/}) \simeq *$$

is contractible. The functor F is initial if and only if for each $D \in \mathcal{D}$, the classifying space

$$\mathsf{B}(\mathcal{C}_{/D}) \simeq *$$

is contractible.

Quillen's Theorem B is designed precisely to check if the classifying space of a fiber sequence is again a fiber sequence.

Theorem A.62 (Quillen's Theorem B)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If for each morphism $D \to D'$ in $\mathcal{D}$, the functor $\mathcal{C}_{/D} \to \mathcal{C}_{/D'}$ induces an equivalence between classifying spaces $\mathsf{B}(\mathcal{C}_{/D} \xrightarrow{\simeq} \mathsf{B}(\mathcal{C}_{/D'}$, then for each $D \in \mathcal{D}$ the diagram of classifying spaces*

$$\begin{tikzcd}
\mathsf{B}(\mathcal{C}_{/D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{B}\mathcal{C} \arrow[d] \\
* \arrow[r, "\langle D \rangle"] & \mathsf{B}\mathcal{D}
\end{tikzcd}$$

is a pullback.

Lemma A.63 ([Cep19] Lemma 4.3.1)**.** *Let*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "\pi"] & \mathcal{E}' \arrow[d, "\pi'"] \\
\mathcal{B} \arrow[r] & \mathcal{B}'
\end{tikzcd}$$

be a pullback of ∞-categories. If π' satisfies the hypotheses of Quillen's Theorem B, then so does π.

Miscellaneous

In this subsection we compile a collection of miscellaneous facts that we reference in this paper.

Theorem A.64 ([Lur17] Theorem A.3.1)**.** *Let X be a paracompact Hausdorff topological space and let $\mathcal{U} \xrightarrow{F} \mathsf{open}(X)$ be a functor from a poset into the poset of open sets in X. For each $x \in X$, consider the full subcategory*
$$\mathcal{U}_x := \{U \in \mathcal{U} \mid x \in F(U)\} \subset \mathcal{U} \ .$$
If for all $x \in X$, the classifying space
$$\mathsf{B}\mathcal{U}_x \simeq *$$
is contractible, then the map
$$\mathsf{hocolim}\Big(\mathcal{U} \xrightarrow{F} \mathsf{open}(X) \to \mathsf{Top}\Big) \xrightarrow{\simeq} X$$
is a weak homotopy equivalence. Furthermore, if $F(U) \simeq *$ for each $U \in \mathcal{U}$, then
$$\mathsf{B}\mathcal{U} \xleftarrow{\simeq} \mathsf{hocolim}\Big(\mathcal{U} \xrightarrow{F} \mathsf{open}(X) \to \mathsf{Top}\Big) \xrightarrow{\simeq} X \ .$$

Theorem A.65 ([Kis64])**.** *The inclusion*
$$\mathsf{Homeo}(\mathbb{R}^n) \hookrightarrow \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$$
of the space of self-homeomorphisms of $\mathbb{R}^n$ into the space of self-embeddings is a homotopy equivalence.

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