ADDITIVITY OF FACTORIZATION ALGEBRAS

&

THE COHOMOLOGY OF REAL GRASSMANNIANS

by

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DEDICATION

This dissertation is dedicated to all who have participated in this process with me. I would not have persevered were it not for you.

“All flourishing is mutual.”

– Robin Wall Kimmerer

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ABSTRACT

This dissertation is composed of two separate projects. The first chapter proves two additivity results for factorization algebras. These provide a way to understand factorization algebras on the product of two spaces. Our results can be thought of as a generalization of Dunn's additivity for E_n-algebras. In particular, our methods provide a new proof of Dunn's additivity. The second chapter is an examination of the Schubert stratification of real Grassmann manifolds. We use this extra structure to identify the quasi-isomorphism type of the Schubert CW chain complex for real Grassmannians. We provide explicit computations using our methods.

INTRODUCTION

This dissertation is composed of two separate bodies of work. Chapter 2 proves two additivity theorems for factorization algebras. Somewhere along the way in proving the results in Chapter 2, we got distracted by some shiny new objects, not too dissimilar to a raccoon. Chapter 3 is a result of this distraction. In Chapter 3, we compute the cohomology of real Grassmannian manifolds via their Schubert stratifications.

Additivity of factorization algebras

Factorization algebras were developed by Costello and Gwilliam in [11] to understand the algebraic structure of observables in perturbative quantum field theory. They are related to chiral algebras which were developed by Beilinson and Drinfeld in [6] to understand vertex algebras in conformal field theory.

A typical source of examples of factorization algebras comes from perturbative σ-models. One such example is the Poisson σ-model. The classical Poisson σ-model aims to understand the mapping space $\mathsf{Map}(\Sigma, X)$ where X is a Poisson manifold and Σ is a compact oriented surface. In general, this mapping space is quite complicated. A first approximation to understanding $\mathsf{Map}(\Sigma, X)$ is to fix a map $\varphi \in \mathsf{Map}(\Sigma, X)$, such as a constant map, and consider an infinitesimal neighborhood of φ. This is known as the perturbative Poisson σ-model. In [11] Costello and Gwilliam laid the foundation for understanding field theories in ths way. Using their framework, the classical observables in perturbative σ-models possess the structure of a factorization algebra.

For example, in the classical Poisson σ-model, for X a target Poisson manifold, the space of fields on an open $U \subset \Sigma$ can be taken to be the smooth stack $\mathsf{Map}(U_{dR}, X)$. Here,

U_{dR} is the de Rham stack of U, whose global functions is the de Rham complex of U. Notice the embedding $X \xhookrightarrow{\mathsf{const}} \mathsf{Map}(U_{dR}, X)$ of constant maps. For perturbative σ-models, one is interested in the infinitesimal neighborhood of this embedding. For open $U \subset \Sigma$, the fields for this perturbative σ-model can be taken to be the dg Lie algebra $\Omega^*(U) \otimes \mathfrak{g}_X$, where $\mathfrak{g}_X$ is a curved L_∞-algebra defined from the Poisson structure on X. Therefore, the classical observables for this perturbative σ-model can be taken to be $\mathsf{C}_*^{\mathsf{Lie}}(\Omega_{\mathsf{c}}^*(U) \otimes \mathfrak{g}_X)$. This expression is functorial in U

$$\mathsf{Obs}^{\mathsf{cl}} : U \mapsto \mathsf{C}_*^{\mathsf{Lie}}(\Omega_{\mathsf{c}}^*(U) \otimes \mathfrak{g}_X)\ . \tag{1.1}$$

Further, this functor is a factorization algebra. We refer the reader to [**?**] for more details.

Similarly, given a based space $* \in Z$ and a manifold M, consider the functor

$$\mathsf{Map}_{\mathsf{c}}(-, Z) : \mathsf{open}(M) \to \mathsf{Spaces}\ , \qquad U \mapsto \mathsf{Map}_{\mathsf{c}}(U, Z)\ . \tag{1.2}$$

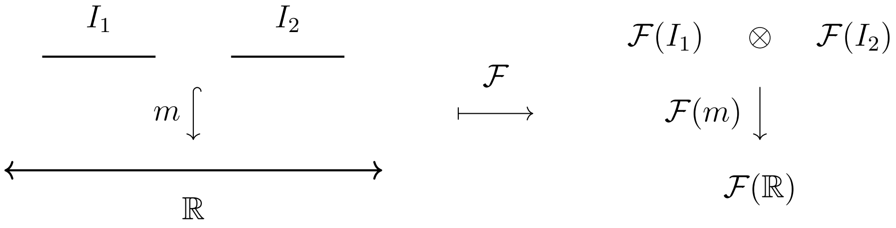
As shown in [2], if Z is $(n-1)$-connected, and $\mathsf{dim}(M) \leq n$, then $\mathsf{Map}_{\mathsf{c}}(-, Z)$ is a locally constant factorization algebra.

Heuristically, a factorization algebra on a topological space X valued in a symmetric monoidal category $(\mathcal{V}, \otimes)$ is a functor $\mathcal{F} : \mathsf{open}(X) \to \mathcal{V}$ that possesses a local-to-global property and that takes disjoint unions of open sets to tensor products in $\mathcal{V}$. Here, $\mathsf{open}(X)$ denotes the poset of open subsets of X. Note that disjoint union is only a partially defined operation on $\mathsf{open}(X)$, so we cannot simply require that $\mathcal{F}$ is a symmetric monoidal functor. There are algebraic gadgets called operads that capture this notion of a partial operation. We use this formalism to give a more precise description of factorization algebras later. There is a special class of factorization algebras called locally constant factorization algebras. A factorization algebra $\mathcal{F}$ is called locally constant if each isotopy equivalence $U \hookrightarrow V$ in $\mathsf{open}(X)$ is carried to an equivalence $\mathcal{F}(U) \xrightarrow{\simeq} \mathcal{F}(V)$ in $\mathcal{V}$. For example, the classical observables of the Poisson σ-model as given in equation (1.1) form a locally constant factorization algebra.

Factorization algebras also provide an approach to understanding (higher) algebraic structures. For instance, we will now show how a locally constant factorization algebra on $\mathbb{R}$ gives rise to an associative algebra. Consider a locally constant factorization algebra on $\mathbb{R}$

$$\mathcal{F} : \mathsf{open}(\mathbb{R}) \to \mathsf{Vect}_{\mathbb{R}}$$

valued in the category of real vector spaces equipped with the symmetric monoidal structure provided by $\otimes_{\mathbb{R}}$. Recall that the open subsets of $\mathbb{R}$ are generated by intervals. Consider the inclusion of two disjoint intervals $I_1 \amalg I_2 \overset{m}{\hookrightarrow} \mathbb{R}$. Since $\mathcal{F}$ takes disjoint unions to tensor products in $\mathcal{V}$, the map m gets carried to a linear map of vector spaces as depicted below.



Let A denote the vector space $\mathcal{F}(\mathbb{R})$. Since $\mathcal{F}$ is assumed to be locally constant, given any open interval $I \hookrightarrow \mathbb{R}$, the induced map $\mathcal{F}(I) \to \mathcal{F}(\mathbb{R}) =: A$ is an equivalence. Thus, $\mathcal{F}(I) \cong A$ for each interval $I \in \mathsf{open}(X)$. Therefore, the map m gets carried to a linear map $A \otimes_{\mathbb{R}} A \xrightarrow{\mathcal{F}(m)} A$. One can check this endows A with the structure of an associative algebra over $\mathbb{R}$.

As shown in [11], there is an equivalence of categories

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Vect}_k) \simeq \mathsf{AssocAlg}_k \ ,$$

between the category of locally constant factorization algebras on $\mathbb{R}$ valued in the symmetric monoidal (with respect to $\otimes_k$) category of vector spaces over a field, k, and the category of associative algebras over k. More generally, for a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, in [21] Lurie proves that there is an equivalence of ∞-categories

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^n}(\mathcal{V}^{\otimes}) \simeq \mathsf{Alg}_{\mathsf{E}_n}(\mathcal{V}^{\otimes}) \tag{1.3}$$

between the ∞-category of locally constant factorization algebras on $\mathbb{R}^n$ valued in $\mathcal{V}^{\otimes}$ and the ∞-category of E_n-algebras in $\mathcal{V}^{\otimes}$.

In [12] Dunn proved a celebrated theorem about the E_n-operads that is referred to as Dunn's additivity. Lurie generalized Dunn's additivity to the setting of ∞-operads in [21]. Dunn's additivity asserts that for nonnegative integers $n, m \geq 0$, the E_{n+m}-operad is a tensor product of the E_n-operad with the E_m-operad. In particular, for a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, there is an equivlence of ∞-categories

$$\mathsf{Alg}_{\mathsf{E}_{n+m}}(\mathcal{V}^{\otimes}) \simeq \mathsf{Alg}_{\mathsf{E}_n}(\mathsf{Alg}_{\mathsf{E}_m}(\mathcal{V}^{\otimes})) \tag{1.4}$$

between the ∞-category of E_{n+m}-algebras in $\mathcal{V}^{\otimes}$ and the ∞-category of E_n-algebras in the ∞-category of E_m-algebras in $\mathcal{V}^{\otimes}$. Note that the ∞-category $\mathsf{Alg}_{\mathsf{E}_m}(\mathcal{V}^{\otimes})$ is a symmetric monoidal ∞-category via pointwise tensor product in $\mathcal{V}^{\otimes}$, thus the right-hand side of equation (1.4) makes sense.

Using equation (1.3), we can reformulate the statement of Dunn's additivity as an equivalence of ∞-categories

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^{n+m}}(\mathcal{V}^{\otimes}) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^n}(\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^m}(\mathcal{V}^{\otimes})) \ . \tag{1.5}$$

There are several natural generalizations of this statement that we contemplate in this dissertation:

Question 1.0.1. Is there an analog of equation (1.5) for factorization algebras that are not necessarily locally constant?

Question 1.0.2. Is there an analog of equation (1.5) when one considers factorization algebras over topological spaces other than Euclidean space?

In Chapter 2, we provide solutions to both Question 1.0.1 and Question 1.0.2. A novelty of our approach is that we recover Dunn's additivity as a corollary. Lurie provides a highly non-trivial proof of Dunn's additivity in [21]. In particular, our methods provide a new proof.

We reformulate factorization algebras within the context of ∞-operads as developed by Lurie in [21]. The poset $\mathsf{open}(X)$ can be regarded as a multicategory, which captures the notion that disjoint union is only a partially defined operation. There is a standard way of regarding a multicategory as an ∞-operad, and we will denote the ∞-operad associated to $\mathsf{open}(X)$ by $\mathsf{open}(X)^{\otimes}$. An element of $\mathsf{open}(X)^{\otimes}$ can be thought of as a pair $(I_+, (U_i))$ consisting of a based fintie set I_+ and an I-indexed list (U_i) of open sets in X. Symmetric monoidal ∞-categories are also defined using the framework of ∞-operads. Using this language, a factorization algebra is then a functor of ∞-operads $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$, again satisfying a local-to-global principle and the condition that disjoint unions map to tensor products.

This operadic formulation of factorization algebras provides us with a natural approach to answering Question 1.0.1 and Question 1.0.2.

General additivity

We first provide the following answer to Question 1.0.1:

Theorem 1.0.3. *Let X and Y be topological spaces, and let $\mathcal{V}^{\otimes}$ be a $\otimes$-presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathit{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathit{Fact}_{X}(\mathit{Fact}_{Y}(\mathcal{V}^{\otimes})) \ .$$

<u>Key ideas</u> The ∞-category of factorization algebras is an ∞-subcategory

$$\mathsf{Fact}_{X}(\mathcal{V}^{\otimes}) \hookrightarrow \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})$$

of the functors of ∞-operads between the $\mathsf{open}(X)^{\otimes}$ and $\mathcal{V}^{\otimes}$. Thus, the statement of additivity is making a comparison between an ∞-subcategory of $\mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X \times Y)^{\otimes}, \mathcal{V}^{\otimes})$ and an ∞-subcategory of $\mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}))$. The category of ∞-operads possesses a tensor product with the property that

$$\mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})) \simeq \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes} \otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \ .$$

The defining feature of the tensor product of ∞-operads is such that there is an equivalence of ∞-categories

$$\mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes} \otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ ,$$

where the righthand side is the ∞-category of bifunctors of ∞-operads. A bifunctor is a special type of functor out of $\mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}$. There is a natural bifunctor

$$\rho : \mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes} \to \mathsf{open}(X \times Y)^{\otimes}$$

given by taking the product of open sets in X with open sets in Y to produce an open set in $X \times Y$. Restriction along ρ provides a comparison

$$\rho^* : \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X \times Y)^{\otimes}, \mathcal{V}^{\otimes}) \to \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

There is a left adjoint to ρ^* given by left Kan extension. The strategy for proving Theorem 1.0.3 is to show that this adjunction restricts to an equivalence between the ∞-subcategories of factorization algebras.

Locally constant additivity

Next, we provide an affirmative answer to Question 1.0.2 with the caveat that we now require the topological spaces X and Y to be topological manifolds:

Theorem 1.0.4. *Let X and Y be topological manifolds and let $\mathcal{V}^{\otimes}$ be a $\otimes$-presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathit{Fact}^{l.c.}_{X \times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathit{Fact}^{l.c.}_{X}(\mathit{Fact}^{l.c.}_{Y}(\mathcal{V}^{\otimes})) \ .$$

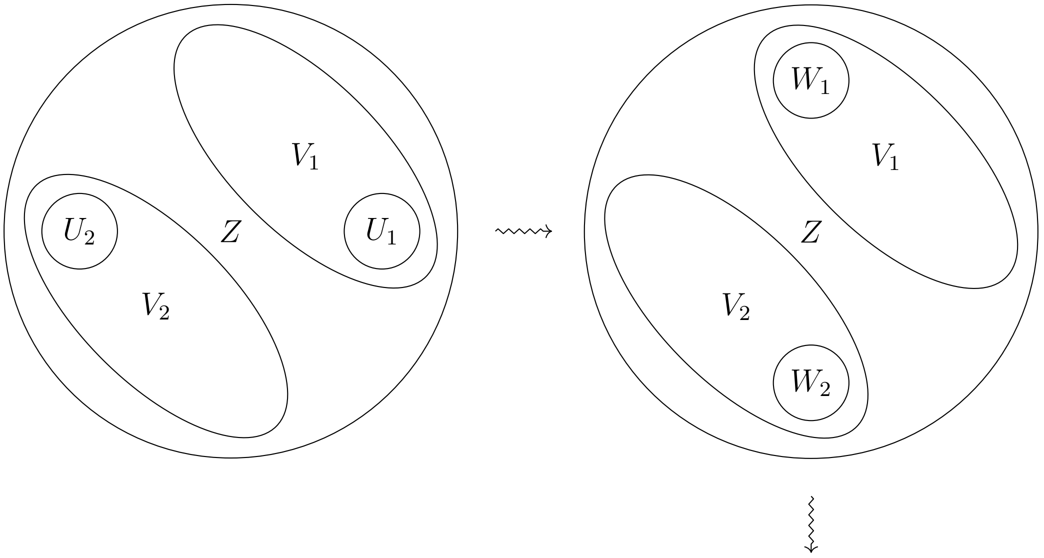
Before discussing the key ideas of the proof, we first mention an immediate implication of this theorem. Let $\mathsf{Ch}_k^{\otimes}$ denote the symmetric monoidal ∞-category of chain complexes over a fixed field. Theorem 1.0.4 implies that an algebra in $\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})$ does not simply possess two multiplication rules. Rather, there is a space of multiplication rules. We now describe how to see this space has an interesting topology to it. Namely, we can see a nontrivial loop of multiplications. Using equation (1.3), there is an equivalence

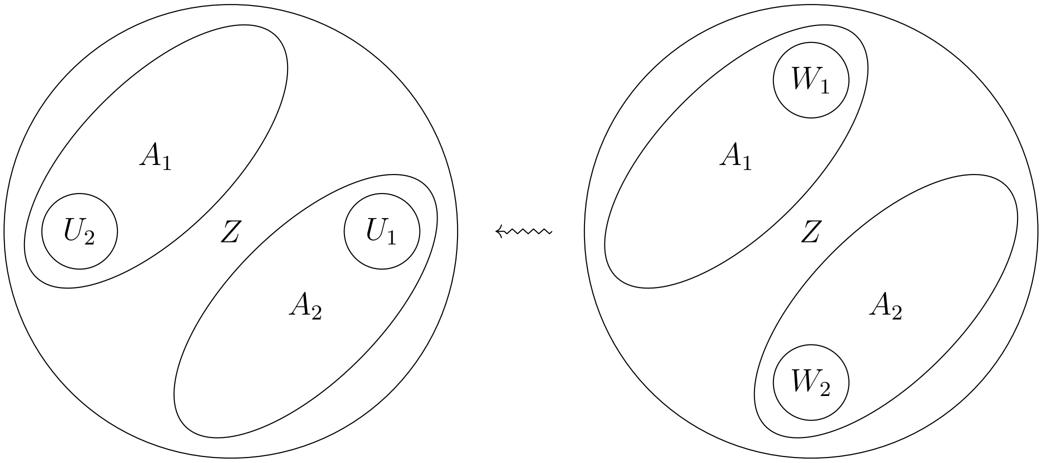
$$\mathsf{Alg}(\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Ch}_k^{\otimes})) \ .$$

Theorem 1.0.4 further asserts an equivalence

$$\mathsf{Alg}(\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^2}(\mathsf{Ch}_k^{\otimes})\ .$$

Now take $\mathcal{F} \in \mathsf{Alg}(\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}^2}(\mathsf{Ch}_k^{\otimes})$. Consider two disjoint disks $U_1 \amalg U_2 \overset{\varphi}{\hookrightarrow} Z \subset \mathbb{R}^2$ including into a larger disk, as depicted below. Note that each inclusion of a disk into $\mathbb{R}^2$ is an isotopy equivalence. Since $\mathcal{F}$ is assumed to be locally constant, this implies that $\mathcal{F}(U_1) \simeq \mathcal{F}(U_2) \simeq \mathcal{F}(Z) \simeq \mathcal{F}(\mathbb{R}^2)$. Therefore, $\mathcal{F}$ carries φ to a morphism $\mathcal{F}(\varphi) : \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \to \mathcal{F}(Z)$ in $\mathsf{Ch}_k^{\otimes}$. We now illustrate a zig-zag of inclusions that produces an example of a loop in the space of multiplications.





If we apply $\mathcal{F}$ to the morphisms indicated above, we get the following diagram in $\mathsf{Ch}_k^{\otimes}$

$$\begin{tikzcd}
& \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \arrow[dl, "\simeq"'] \arrow[d] \arrow[dr, "\simeq"] & \\
\mathcal{F}(V_1) \otimes \mathcal{F}(V_2) \arrow[r] & \mathcal{F}(Z) & \mathcal{F}(A_1) \otimes \mathcal{F}(A_2) \arrow[l] \ . \\
& \mathcal{F}(W_1) \otimes \mathcal{F}(W_2) \arrow[ul, "\simeq"] \arrow[u] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Key ideas Recall that a factorization algebra $\mathcal{F}$ is called locally constant if it carries an isotopy equivalence of open sets in X to an equivalence in $\mathcal{V}$. We now place the locally constant condition within the operadic formulation of factorization algebras. Note that there is an ∞-subcategory $\mathcal{I}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ consiting of the same objects as $\mathsf{open}(X)^{\otimes}$, but only those morphisms $(I_+, (U_i)) \to (J_+, (V_j))$ for which the map of based finite sets $I_+ \xrightarrow{f} J_+$ is a bijection such that for all $i \in I$, the inclusion $U_i \hookrightarrow V_{f(i)}$ is an isotopy equivalence. The condition for a factorization algebra $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ to be locally constant can be phrased as the condition that $\mathcal{F}$ factors

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathsf{loc}"] & \mathcal{V}^{\otimes} \\
\mathsf{open}(X)^{\otimes}[\mathcal{I}(X)^{\otimes -1}] \arrow[ur, dashed] &
\end{tikzcd}$$

through the localization of $\mathsf{open}(X)^{\otimes}$ at the isotopy equivalences $\mathcal{I}(X)^{\otimes}$. The localization $\mathsf{open}(X)^{\otimes}[\mathcal{I}(X)^{\otimes -1}]$ is a little intractable. This is precisely where we employ the additional requirement that X and Y are topological manifolds. As such, we can reduce the situation to analyzing functors out of $\mathsf{disk}(X)^{\otimes}$, the full ∞-sub-operad consisting of those open sets that are homeomorphic to a finite disjoint union of disks. Let $\mathcal{J}(X)^{\otimes}$ denote the full ∞-subcategory of $\mathcal{I}(X)^{\otimes}$ consisting of those objects that lie in $\mathsf{disk}(X)^{\otimes}$. The locally constant condition can then be reduced to analyzing the localization $\mathsf{disk}(X)^{\otimes}[\mathcal{J}(X)^{\otimes -1}]$. By evaluating disks at their centers, we can understand this localization in terms of configuration spaces.

An application

Consider the moduli space

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right)$$

of $U(1)$-bundles on the space $\mathbb{R} \times (S^1)^{\amalg r}$ that are trivialized outside of a compact set. We can use Theorem 1.0.4 to identify the algebra of chains on this moduli space. There is a homotopy equivalence

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R} \times (S^1)^{\amalg r}\right) \simeq \mathsf{Map}_{\mathsf{c}}(\mathbb{R} \times (S^1)^{\amalg r}, \mathsf{B}U(1))$$

between the moduli space of $U(1)$-bundles and the space of compactly supported maps into $\mathsf{B}U(1)$. Note that $\mathsf{B}U(1)$ is 1-connected, since $U(1)$ is 0-connected, and $\mathbb{R} \times (S^1)^{\amalg r}$ is 2-dimensional. Consider the functor

$$C_*\left(\mathsf{Map}_{\mathsf{c}}(-, \mathsf{B}U(1))\right) : \mathsf{open}\left(\mathbb{R} \times (S^1)^{\amalg r}\right) \to \mathsf{Ch}_k^{\otimes}$$

where Ch_k denotes the ∞-category of chain complexes over a field k. This defines a locally constant factorization algebra, per the discussion surrounding equation (1.2). That is,

$$C_*\left(\mathsf{Map}_{\mathsf{c}}(-, \mathsf{B}U(1))\right) \in \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}\times(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes}) \ . \tag{1.6}$$

By Theorem 1.0.4, we know

$$\mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}\times(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes}) \simeq \mathsf{Fact}^{\mathsf{l.c.}}_{\mathbb{R}}(\mathsf{Fact}^{\mathsf{l.c.}}_{(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes})) \simeq \mathsf{Alg}(\mathsf{Fact}^{\mathsf{l.c.}}_{(S^1)^{\amalg r}}(\mathsf{Ch}_k^{\otimes})) \ .$$

Therefore, if we evaluate the factorization algebra in equation (1.6) on the total space $\mathbb{R} \times (S^1)^{\amalg r}$ we then obtain an object in $\mathsf{Alg}(\mathsf{Ch}_k^{\otimes})$. Further, this algebra is

$$C_*\left(\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R}\times(S^1)^{\amalg r}\right)\right) \ .$$

In this case, we can explicitly identify this algebra by other means. Namely, note that for any space Y and pointed space Z

$$\begin{aligned}
\mathsf{Map}_{\mathsf{c}}(\mathbb{R}\times Y, Z) &:= \mathsf{Map}_*((\mathbb{R}\times Y)^+, Z) \\
&\cong \mathsf{Map}_*(\mathbb{R}^+ \wedge Y^+, Z) \\
&\cong \mathsf{Map}_*(\mathbb{R}^+, \mathsf{Map}_*(Y^+, Z)) \\
&\cong \Omega\mathsf{Map}_{\mathsf{c}}(Y, Z) \\
&\cong \mathsf{Map}_{\mathsf{c}}(Y, \Omega Z) \ .
\end{aligned}$$

Here, $\mathsf{Map}_*(-,-)$ denotes the space of based maps, Y^+ denotes the one-point compactification, and ΩZ denotes the based loop space. There, if we take $Y = (S^1)^{\amalg r}$ and $Z = \mathsf{B}U(1)$, we have

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R}\times(S^1)^{\amalg r}\right)\simeq \mathsf{Map}_{\mathsf{c}}(\mathbb{R}\times(S^1)^{\amalg r},\mathsf{B}U(1))\cong \mathsf{Map}_{\mathsf{c}}((S^1)^{\amalg r},U(1))\ .$$

By the universal property of coproducts,

$$\mathsf{Map}_{\mathsf{c}}((S^1)^{\amalg r},U(1))\cong \mathsf{Map}_{\mathsf{c}}(S^1,U(1))^{\times r}\ .$$

Further, there is a homeomorphism

$$\mathsf{Map}_{\mathsf{c}}(S^1,U(1))^{\times r}\xrightarrow{\cong}\left(\mathsf{Map}_{*}(S^1,U(1))\times U(1)\right)^{\times r}$$

given factorwise by

$$(S^1\xrightarrow{f}U(1))\mapsto\left(S^1\xrightarrow{f(1)^{-1}\cdot f}U(1),f(1)\right)\ .$$

Noting that $\mathsf{Map}_{*}(S^1,U(1))\simeq\mathbb{Z}$, we see

$$\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R}\times(S^1)^{\amalg r}\right)\simeq(\mathbb{Z}\times U(1))^{\times r}\ .$$

All told,

$$C_*\left(\mathcal{M}_{U(1),\mathsf{c}}\left(\mathbb{R}\times(S^1)^{\amalg r}\right);k\right)\simeq k[x^{\pm1}]^{\otimes r}\otimes\left(k[\varepsilon]_{/(\varepsilon^2)}\right)^{\otimes}r\simeq\left(k[x^{\pm1},\varepsilon]_{/(\varepsilon^2)}\right)^{\otimes r}\ ,$$

where $\mathsf{deg}(\varepsilon)=1$. Therefore, we have identified the algebra of global sections of the locally constant factorization algebra given in equation (1.6) with the algebra $\left(k[x^{\pm1},\varepsilon]_{/(\varepsilon^2)}\right)^{\otimes r}$.

Future

Recently, the theory of stratified spaces has been given a solid foundation in the context of ∞-categories by Ayala-Francis-Tanaka in [5]. There is a class of factorization algebras on stratified spaces called constructible factorization algebras. A constructible factorization algebra $\mathcal{F}$ on a stratified space $X \to \mathcal{P}$ is a factorization algebra on X such that it is locally constant when restricted to each stratum. We believe that our methods can be used to provide an additivity statement for constructible factorization algebras on stratified spaces. Namely, we conjecture:

Conjecture 1.0.5. *For nice stratified spaces X and Y, and $\mathcal{V}^{\otimes}$ a $\otimes$-presentable ∞-category, there is an equivalence of ∞-categories*

$$\mathsf{Fact}^{\mathsf{cbl}}_{X\times Y}(\mathcal{V}^{\otimes}) \to \mathsf{Fact}^{\mathsf{cbl}}_{X}(\mathsf{Fact}^{\mathsf{cbl}}_{Y}(\mathcal{V}^{\otimes})) \ .$$

The cohomology of real Grassmannians

As the moduli space of k-dimensional subvector spaces of $\mathbb{R}^n$, the real Grassmannians, $\mathsf{Gr}_k(\mathbb{R}^n)$, play an important role in the study of manifolds. Namely, we can interpret $H^*(\mathsf{Gr}_k(\mathbb{R}^n))$ as measuring obstructions to natural geometric questions about manifolds. For instance, let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold of $\mathbb{R}^n$. Consider the Gauss map

$$\tau_M : M \to \mathsf{Gr}_k(\mathbb{R}^n) \ , \qquad x \mapsto T_xM \ ,$$

that sends a point $x \in M$ to the tangent space of M at x. This induces a map on the level of cohomology

$$\tau_M^* : H^*(\mathsf{Gr}_k(\mathbb{R}^n)) \to H^*(M) \ .$$

For M a compact, orientable submanifold, there exists an element $e \in H^k(\mathsf{Gr}_k(\mathbb{R}^n))$ such that $\tau_M(e) = 0$ if and only if M admits a non-vanishing vector field [24].

To aid in the computation of the cohomology of $\mathsf{Gr}_k(\mathbb{R}^n)$, there is a natural CW structure that one can place on the Grassmannian called the Schubert CW structure [24]. The complex Grassmannian, $\mathsf{Gr}_k(\mathbb{C}^n)$, can also be endowed with this structure. Since each cell in the Schubert CW strucutre of $\mathsf{Gr}_k(\mathbb{C}^n)$ is even dimensional, the boundary maps in the corresponding chain complex are zero. Thus, $H^i(\mathsf{Gr}_k(\mathbb{C}^n))$ is freely generated by the Schubert cells of dimension i [13]. A similar situation occurs in the cohomology of $\mathsf{Gr}_k(\mathbb{R}^n)$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Namely, all the boundary maps in the chain complex are zero. So again, $H^i(\mathsf{Gr}_k(\mathbb{R}^n); \mathbb{Z}/2\mathbb{Z})$ is freely generated by the Schubert cells of dimension i [24]. Borel computed the rational cohomology algebra of the odd dimensional real Grassmannians [7], and Takeuchi computed the rational cohomology algebra of the even dimensional real Grassmannians [28]. More recently, equivariant versions of the rational cohomology have been considered in [8], [16], [27].

The difficulty in computing the integral cohomology of real Grassmannians lies in the computation of the attaching maps in the Schubert CW structure. Using $\mathbb{Z}/2\mathbb{Z}$ coefficients, one is able to bypass this difficulty, but to understand the integral cohomology, we must have a clear understanding of the attaching maps. In [9], the authors apply a combinatorial approach via Young diagrams, and in [18], Jungkind provides a closed formula for the differentials in the CW chain complex of $\mathsf{Gr}_k(n)$ with integer coefficients. Using our framework, we recover this formula as Lemma 1.0.6 below. We now outline our approach and contributions.

Approach

In Chapter 3, we provide a complete description of the additive structure of the R-cohomology of $\mathsf{Gr}_k(\mathbb{R}^n)$. Fix $0 \leq k \leq n$, and consider the poset

$$\genfrac\{\}{0pt}{}{n}{k} := \mathsf{Fun}^{\mathsf{inj}}\big(\{1 < \cdots < k\}, \{1 < \cdots < n\}\big)$$

of injective functors. For each $0 \leq r \leq k$, consider the map

$$d_r : \genfrac\{\}{0pt}{}{n}{k} \xrightarrow{\{s_1 < \cdots < s_k\} \mapsto \sum_{r \leq i \leq k} s_i - i} \mathbb{Z}_{\geq 0} \ .$$

First, we prove the following lemma.

Lemma 1.0.6. *Let R be a commutative ring. The R-valued cohomology $H^*\big(\mathsf{Gr}_k(n); R\big)$ is isomorphic with the cohomology of the chain complex $\Big(R\left\langle \genfrac\{\}{0pt}{}{n}{k} \right\rangle, \delta\Big)$ over R whose underlying graded R-module is free on the graded set $\genfrac\{\}{0pt}{}{n}{k} \xrightarrow{d_1} \mathbb{Z}_{\geq 0}$ and whose differential evaluates as*

$$\delta : S = \{s_1 < \cdots < s_k\} \ \mapsto \sum_{r \in \{1 \leq r \leq k \ \mid \ s_{r+1} - s_r > 1 \ and \ k - s_r \ is \ odd\}} (-1)^{d_{r-1}(S)} 2 \cdot S_r \ , \qquad (1.7)$$

where $S_r := \{s_1 < \cdots < s_{r-1} < s_r + 1 < s_{r+1} < \cdots < s_k\} \in \genfrac\{\}{0pt}{}{n}{k}$.

Then, we use Lemma 1.0.6 to provide the following remarkable decomposition of the Schubert CW chain complex $C_*^{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z})$:

Theorem 1.0.7. *Let R be a commutative ring. There is an isomorphism of chain complexes*

$$\begin{aligned} C_*^{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z}) &\cong \bigoplus_{S \in \genfrac\{\}{0pt}{}{n}{k}_{\mathsf{Out}}} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\otimes \mathsf{In}(S)}[d_1(S) - \mathsf{card}(\mathsf{In}(S))] \\ &\cong \left(\bigoplus_{S \in \genfrac\{\}{0pt}{}{n}{k}_{\mathsf{Out}} \setminus \genfrac\{\}{0pt}{}{n}{k}_{\mathsf{In}}} \mathbb{Z}[d_1(S)] \right) \oplus \left(\bigoplus_{S \in \genfrac\{\}{0pt}{}{n}{k}_{\mathsf{Min}}} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[d_1(S) - 1] \right) \end{aligned}$$

where $\left\{ {n \atop k} \right\}_{\textsf{\textit{Out}}}, \left\{ {n \atop k} \right\}_{\textsf{\textit{In}}}, \left\{ {n \atop k} \right\}_{\textsf{\textit{Min}}} \subset \left\{ {n \atop k} \right\}$, *and* $\textsf{\textit{In}}(S) \subset \{1, \ldots, k\}$.

A consequence of Theorem 1.0.7 is the following closed formula for the R-cohomology of $\mathsf{Gr}_k(n)$:

Corollary 1.0.8. *There is an isomorphism of graded* R*-modules*

$$H^*(\textsf{\textit{Gr}}_k(n); R) \cong \bigoplus_{S \in \left\{ {n \atop k} \right\}} V_S[d_1(S)] \ ,$$

where

$$V_S := \begin{cases} R, & \text{if } \textsf{\textit{In}}(S) = \emptyset = \textsf{\textit{Out}}(S) \\ \textsf{\textit{ker}}(R \xrightarrow{2} R), & \text{if } \textsf{\textit{Min}}(\textsf{\textit{In}}(S) \cup \textsf{\textit{Out}}(S)) \in \textsf{\textit{Out}}(S) \\ \textsf{\textit{coker}}(R \xrightarrow{2} R), & \text{if } \textsf{\textit{Min}}(\textsf{\textit{In}}(S) \cup \textsf{\textit{Out}}(S)) \in \textsf{\textit{In}}(S) \end{cases} ,$$

for $\textsf{\textit{In}}(S), \textsf{\textit{Out}}(S) \subset \{1, \ldots, k\}$.

We now provide some immedaite consequences of Corollary 1.0.8:

Corollary 1.0.9. *Let* R *be a ring in which* 2 *is not a zero-divisor, for example* $R = \mathbb{Z}$ *or* $R = \mathbb{Z}_2$*, the* 2*-adics. Then*

$$H^*(\textsf{\textit{Gr}}_k(n); R) \cong \left(\bigoplus_{\substack{S \in \left\{ {n \atop k} \right\} \\ \textsf{\textit{In}}(S) = \emptyset = \textsf{\textit{Out}}(S)}} R \right) \oplus \left(\bigoplus_{\substack{S \in \left\{ {n \atop k} \right\} \\ \textsf{\textit{Min}}(\textsf{\textit{In}}(S) \cup \textsf{\textit{Out}}(S)) \in \textsf{\textit{In}}(S)}} R/2R \right) .$$

Corollary 1.0.10. *Let* R *be a ring in which* 2 *is invertible, for example* $R = \mathbb{Q}$*,* $R = \mathbb{R}$*,* $R = \mathbb{Z}/q\mathbb{Z}$*, or* $R = \mathbb{F}_q$*, for some* q *relatively prime to* 2*. Then*

$$H^*(\textsf{\textit{Gr}}_k(n); R) \cong \left(\bigoplus_{\substack{S \in \left\{ {n \atop k} \right\} \\ \textsf{\textit{In}}(S) = \emptyset = \textsf{\textit{Out}}(S)}} R \right) .$$

Corollary 1.0.11. *Let R be a commutative ring such that $R = \mathbb{Z}/2^a\mathbb{Z}$ then*

$$H^*(\mathsf{Gr}_k(n); R) \cong \left(\bigoplus_{\substack{S \in \left\{ n \atop k \right\} \\ \mathsf{In}(S) = \emptyset = \mathsf{Out}(S)}} \mathbb{Z} \right) \oplus \left(\bigoplus_{\substack{S \in \left\{ n \atop k \right\} \\ \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{Out}(S)}} \mathbb{Z}/2^{a-1}\mathbb{Z} \right) \oplus \left(\bigoplus_{\substack{S \in \left\{ n \atop k \right\} \\ \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{In}(S)}} \mathbb{Z}/2\mathbb{Z} \right).$$

ADDITIVITY OF FACTORIZATION ALGEBRAS

In this chapter, we prove the additivity results of Theorem 1.0.3 and Theorem 1.0.4.

Background

In this section we establish preliminary conventions, notation, and definitions. Throughout this chapter we use the theory of ∞-categories. There are various models for the foundations of ∞-category theory, for example quasicategories as developed by Joyal in [17] and complete Segal spaces as developed by Rezk in [25]. In this chapter we work model independently, with the notable exception being our explicit use of complete Segal spaces in the proof of Theorem 2.0.36.

As indicated in the introduction, a factorization algebra on a topological space X is a functor that assigns data to each open subset $U \subset X$. Furthermore, a factorization algebra satisfies a particular local-to-global property and behaves nicely with respect to disjoint unions of open sets. A good example to keep in mind is the following:

Example 2.0.1. There is an interesting class of factorization algebras that one can associate to a Lie algebra called the universal enveloping E_n-algebras. Let $n \geq 1$ be an integer and $\mathfrak{g}$ be a Lie algebra over a field k. Consider the functor

$$\mathcal{U}_n\mathfrak{g} : \mathsf{open}(\mathbb{R}^n) \to \mathsf{Ch}_k$$

from the poset of open sets in $\mathbb{R}^n$ to the category of chain complexes over a field k given by sending $U \in \mathsf{open}(\mathbb{R}^n)$ to

$$U \mapsto \mathsf{C}_*^{\mathsf{Lie}}(\Omega_{\mathsf{c}}^*(U) \otimes \mathfrak{g}) \ ,$$

the Lie algebra chains on the dgla of compactly supported de Rham forms on U with values in $\mathfrak{g}$. This defines a locally constant factorization algebra on $\mathbb{R}^n$.

We formulate factorization algebras within the framework of ∞-operads, and freely use this theory. The data of an ∞-operad is an ∞-category $\mathcal{O}^{\otimes}$ and a functor $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ to the ∞-category of based finite sets. As ∞-operads form the base of this chapter, we have recalled the basic definitions in the appendix. The appendix also contains other foundational definitions and results that we do our best to cite as we use them.

Conventions

Here we compile a list of basic notation and conventions that we use throughout this chapter. Note that many of these items are discussed in more detail in the appendix, so we recommend looking there if more information is desired.

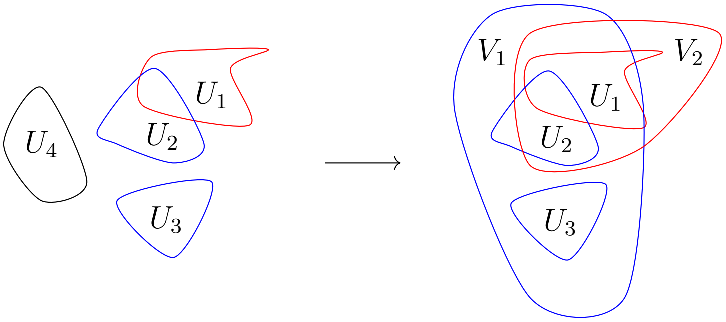
- Fin_* denotes the category of based finite sets and based maps between them.
- $[p]$ dentotes the poset $\{0 < 1 < \cdots < p\}$.
- For $\mathcal{C} \xrightarrow{F} \mathcal{D}$ a functor between categories and $d \in \mathcal{D}$, we let
  - $\mathcal{C}_{/_d}$ denote the overcategory consisting of objects $c \in \mathcal{C}$ equipped with a morphism $F(c) \to d$ in $\mathcal{D}$. See Definition A.0.1.
  - $\mathcal{C}_{|_d}$ denote the fiber of $\mathcal{C}$ over d. This consists of objects $c \in \mathcal{C}$ for which $F(c) = d$. See Definition A.0.2.

Preliminary definitions

Throughout the remainder of this chapter, unless otherwise specified, we will let $\mathcal{V}^{\otimes}$ denote a $\otimes$-presentable symmetric monoidal ∞-category. See Definition A.0.12 for a precise definition of what $\otimes$-presentable means. Note that this is not too restrictive of a condition though, and encompasses the prototypical codomains of factorization algebras. In particular, the ∞-category of chain complexes over a fixed ring is $\otimes$-presentable.

Definition 2.0.2. For X a topological space, let $\mathsf{open}(X)$ denote the poset of open sets in X with partial order given by inclusion.

As we recall in Example A.0.42, the poset $\mathsf{open}(X)$ gives rise to an ∞-operad in a standard way. We denote the resulting ∞-operad by $\mathsf{open}(X)^{\otimes}$. An object in $\mathsf{open}(X)^{\otimes}$ is a pair $(I_+, (U_i))$ consisting of a based finite set I_+ and an I-indexed list of open sets in X. A morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ in $\mathsf{open}(X)^{\otimes}$ is a map of based finte sets $f : I_+ \to J_+$ such that for each $j \in J$, the set $\{U_i \mid i \in f^{-1}(j)\}$ is a collection of pairwise disjoint open subsets of V_j. Below is an example of a morphism in $\mathsf{open}(\mathbb{R}^2)^{\otimes}$ given by the map of based finite sets $\{1, 2, 3, 4\}_+ \to \{1, 2\}_+$ that sends $2, 3 \mapsto 1$, $1 \mapsto 2$, and $4 \mapsto +$.



Remark 2.0.3. We emphasize the fact that $\mathsf{open}(X)^{\otimes}$ is an ordinary category. Additionally, so is the full ∞-sub-operad $\mathsf{disk}(X)^{\otimes}$ consisting of open sets that are homeomorphic to a disjoint union of open disks, as defined in Defininition 2.0.37. Throughout the remainder of this section, we will use a number of variations on $\mathsf{open}(X)^{\otimes}$ and $\mathsf{disk}(X)^{\otimes}$. Note that these are also ordinary categories. This is an important fact that enables us to do explicit constructions.

We now make precise the idea that factorization algebras behave nicely with respect to disjoint unions. First, note that disjoint union is only a partially defined operation on $\mathsf{open}(X)$. Indeed, if $U, V \in \mathsf{open}(X)$ such that $U \cap V \neq \emptyset$, then $U \amalg V \notin \mathsf{open}(X)$. The next observation characterizes disjoint unions as a particular class of morphisms in $\mathsf{open}(X)^{\otimes}$.

Observation 2.0.4. Note that coCartesian morphisms in $\mathsf{open}(X)^{\otimes}$ are of the form

$$(I_+, (U_i)) \xrightarrow{f} \left(J_+, \left(\coprod_{i \in f^{-1}(j)} U_i \right) \right) .$$

Further, these coCartesian morphisms exist precisely when for each $j \in J$, the collection $(U_i)_{i \in f^{-1}(j)}$ is a pairwise disjoint collection of open subsets.

Definition 2.0.5. We say that a functor of ∞-operads $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ is *multiplicative* if $\mathcal{F}$ carries all coCartesian morphisms in $\mathsf{open}(X)^{\otimes}$ to coCartesian morphisms in $\mathcal{V}^{\otimes}$. Define the ∞-category $\mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}) \subset \mathsf{Fun}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})$ to be the full ∞-subcategory consisting of the multiplicative functors of ∞-operads.

Observation 2.0.6. Recall that $\mathcal{V}^{\otimes}$ is a symmetric monoidal ∞-category (see Definition A.0.44). In particular, this means that $\mathcal{V}^{\otimes} \to \mathsf{Fin}_*$ is a coCartesian fibration. Analogous to Observation 2.0.4, a coCartesian morphism in $\mathcal{V}^{\otimes}$ is of the form

$$(I_+, (V_i)) \xrightarrow{f} \left(J_+, \left(\bigotimes_{i \in f^{-1}(j)} V_i \right) \right) .$$

In other words, coCartesian morphisms in $\mathcal{V}^{\otimes}$ are given by tensor products.

In light of Observations 2.0.4 and 2.0.6, the condition for a functor of ∞-operads

$$\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$$

to be multiplicative is an articulation of the idea that disjoint unions of open sets get carried to tensor products in $\mathcal{V}$.

Next, we describe the type of local-to-global condition that factorization algebras satisfy. There is the standard Grothendieck topology on $\mathsf{open}(X)$ where a cover corresponds to an ordinary open cover. However, this is not the correct form of descent for factorization algebras. As discussed in the introduction, a prototypical example of a factorization algebra on a topological space X looks like the functor

$$\mathsf{Map_c}(-, Z) : \mathsf{open}(X) \to \mathsf{Spaces}\ , \qquad U \mapsto \mathsf{Map_c}(U, Z)$$

that sends an open set U to the space of compactly supported maps valued in a pointed space Z. Observe that this is *not* an ordinary cosheaf. However, this functor does enjoy a different type of local-to-global property, namely that of a J_∞-cosheaf. We note that it is not obvious that $\mathsf{Map_c}(-, Z)$ is a J_∞-cosheaf. This can be seen as a consequence of non-abelian Poincare duality as proven in [2], for instance.

Definition 2.0.7. Let $U \subset X$ be an open subset of a manifold. We declare a subset $\mathcal{U} \subset \mathsf{open}(X)_{/U} = \mathsf{open}(U)$ to be a *naive* J_∞*-cover* of U if for all finite subsets $S \subset U$, there exists some $U_S \in \mathcal{U}$ such that $S \subset U_S$. A naive J_∞-cover $\mathcal{U}$ is a J_∞*-cover* if for any finite subset $\{U_1, \ldots, U_n\} \subset \mathcal{U}$, the subset $\mathcal{U}_{/U_1 \cap \cdots \cap U_n} \subset \mathsf{open}(U_1 \cap \cdots \cap U_n)$ is a naive J_∞-cover of $U_1 \cap \cdots \cap U_n$. We call the induced topology on $\mathsf{open}(X)$ the J_∞ *(or Weiss) topology.*

Remark 2.0.8. A naive J_∞-cover determines a J_∞-cover. Indeed, for $\mathcal{U}$ a naive J_∞-cover, the subset consisting of all finite fold intersections of members of $\mathcal{U}$ is a J_∞-cover.

Example 2.0.9. For M a d-manifold, consider $\mathsf{disk_c}(M) \subset \mathsf{open}(M)$ the subposet consisting of those open subsets $U \subset M$ for which $U \cong \mathbb{R}^d$. While $\mathsf{disk_c}(M)$ is an ordinary cover of M, it is *not* a J_∞-cover of M. In fact, it is not even a naive J_infty-cover.

Recall that the *right cone* of an ∞-category $\mathcal{U}$ is defined by

$$\mathcal{U}^{\triangleright} := \mathcal{U} \times \{0, 1\} \coprod_{\mathcal{U} \times \{1\}} * \ .$$

Thus, the objects of $\mathcal{U}^{\triangleright}$ consist of the same objects as $\mathcal{U}$ together with an additional object $*$ that recieves a unique morphism from every other object in $\mathcal{U}$.

Definition 2.0.10. We call a functor $F : \mathsf{open}(X) \to \mathcal{V}$ a J_∞*-cosheaf* if for all $O \in \mathsf{open}(X)$ and J_∞-covers $\mathcal{U}$ of O, the composite functor

$$\mathcal{U}^{\triangleright} \to \mathsf{open}(X)_{/O} \xrightarrow{\mathsf{fgt}} \mathsf{open}(X) \xrightarrow{F} \mathcal{V}$$

is a colimit diagram. We let $\mathsf{Fun}^{J_\infty}(\mathsf{open}(X), \mathcal{V}) \hookrightarrow \mathsf{Fun}(\mathsf{open}(X), \mathcal{V})$ denote the full ∞-subcategory consisting of those functors that are J_∞-cosheaves.

Definition 2.0.11. We let $\mathsf{Fun}^{J_\infty}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}) \hookrightarrow \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})$ denote the full ∞-subcategory consisting of those functors of ∞-operads $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ for which the restriction $\mathcal{F}_{|1_+} : \mathsf{open}(X)^{\otimes}_{|1_+} \to \mathcal{V}^{\otimes}_{|1_+}$ is a J_∞-cosheaf.

We now define the ∞-category of factorization algebras on a topological space X.

Definition 2.0.12. The ∞-category of *factorization algebras on* X is defined as the pullback

$$\begin{tikzcd}
\mathsf{Fact}_X(\mathcal{V}^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}) \arrow[d] \\
\mathsf{Fun}^{J_\infty}(\mathsf{open}(X), \mathcal{V}) \arrow[r, hook] & \mathsf{Fun}(\mathsf{open}(X), \mathcal{V})
\end{tikzcd}.$$

That is, a factorization algebra on X is a functor of ∞-operads

$$\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$$

that restricts to a J_∞-cosheaf and that takes coCartesian morphisms in $\mathsf{open}(X)^{\otimes}$ to coCartesian morphisms in $\mathcal{V}^{\otimes}$.

There is a special class of factorization algebras that will be of interest in the second half of this chapter. These are the locally constant factorization algebras.

Definition 2.0.13. Let $\mathcal{F} : \mathsf{open}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ be a factorization algebra. We say that $\mathcal{F}$ is *locally constant* if the restriction

$$\mathcal{F}_{|1_+} : \mathsf{open}(X)^{\otimes}_{|1_+} \to \mathcal{V}^{\otimes}_{|1_+}$$

carries isotopy equivalences of open sets to equivalences in $\mathcal{V}$. We let $\mathsf{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^{\otimes}) \hookrightarrow \mathsf{Fact}_X(\mathcal{V}^{\otimes})$ denote the full ∞-subcategory consisting of the locally constant factorization algebras.

There is another useful way of thinking about the locally constant condition.

Definition 2.0.14. Consider the full subcategory $\mathcal{I}(X) \subset \mathsf{open}(X)$ consisting of the same objects as $\mathsf{open}(X)$, but only those morphisms that are isotopy equivalences. Additionally, define the ∞-subcategory $\mathcal{I}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ over Fin_* that consists of the same objects as $\mathsf{open}(X)^{\otimes}$, but only those morphisms $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ that are a bijection of based such that for all $i \in I$ the inclusion $U_i \hookrightarrow V_{f(i)}$ is an isotopy equivalence.

In Lemma 2.0.16 below, we give alternate characterizations of what it means for a factorization algebra to be locally constant. These use the idea of localization of ∞-categories. We refer the unfamiliar reader to Definition A.0.26 for a discussion of localization. To prove Lemma 2.0.16, we make use of the following lemma.

Lemma 2.0.15. *The inclusion $\mathcal{I}(X) \hookrightarrow \mathcal{I}(X)^{\otimes}$ witnesses $\mathcal{I}(X)^{\otimes}$ as the free ∞-operad on $\mathcal{I}(X)$.*

Proof. By Construction 2.4.3.1 in [21], the free ∞-operad on $\mathcal{I}(X)$ is $\mathcal{I}(X)^{\amalg}$. Therefore, there is a unique functor between ∞-operads under $\mathcal{I}(X)$ from $\mathcal{I}(X)^{\amalg}$ to $\mathcal{I}(X)^{\otimes}$. The proof is complete by showing this functor is an equivalence. To show this, it is sufficient to show the functor is an equivalence over each object $I_+ \in \mathsf{Fin}_*$ and that defines an equivalence between spaces of morphisms over each $f : I_+ \to J_+$ in Fin_*. By inspection of Construction

2.4.3.1 in [21], $\mathcal{I}(X)^{\amalg}$ is an ordinary category. Further, the fiber $\mathcal{I}(X)^{\amalg}_{|I_+}$ is precisely $\mathcal{I}(X)^{\otimes}_{|I_+}$. Now, fix $f : I_+ \to J_+$ in Fin_*. By construction, the morphisms from $(I_+, (U_i))$ to $(J_+, (V_j))$ in $\mathcal{I}(X)^{\amalg}$ are

$$\prod_{j \in J} \prod_{i \in f^{-1}(j)} \mathsf{Hom}_{\mathcal{I}(X)}(U_i, V_j) \ .$$

These are precisely the morphisms in $\mathcal{I}^{\otimes}$. ☐

Lemma 2.0.16. *Let $\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$. The following are equivalent:*

1. *$\mathcal{F}$ is locally constant.*

2. *the induced functor between underlying ∞-categories*

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{|1_+} \arrow[r, "\mathcal{F}_{|1_+}"] \arrow[d] & \mathcal{V}^{\otimes}_{|1_+} \\
\mathsf{open}(X)^{\otimes}_{|1_+}[\mathcal{I}(X)^{-1}] \arrow[ur, dashed] &
\end{tikzcd}$$

uniquely factors through the localization on isotopy equivalences, $\mathcal{I}(X)$.

3. *$\mathcal{F}$ uniquely factors*

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{V}^{\otimes} \\
\mathsf{open}(X)^{\otimes}[\mathcal{I}(X)^{\otimes -1}] \arrow[ur, dashed] &
\end{tikzcd}$$

through the localization about $\mathcal{I}(X)^{\otimes}$.

Proof. The equivalence of conditions 1 and 2 follows immediately from the definition of localization. Condition 3 immediately implies condition 2. We now show that condition 2 implies condition 3. From the definition of localization (Definition A.0.26), the dotted arrow in condition 2 is equivalent to a unique filler

$$\begin{tikzcd}
\mathcal{I}(X) \arrow[r] \arrow[d] & \mathsf{open}(X)^{\otimes}_{|1_+} \arrow[r, "\mathcal{F}_{|1_+}"] & \mathcal{V}^{\otimes}_{|1_+} \\
\mathsf{B}\mathcal{I}(X) \arrow[urr, dashed, bend right] & & .
\end{tikzcd} \tag{2.1}$$

Now, there is a forgetful functor $(-)_{|1_+} : \mathsf{Op}_\infty \to \mathsf{Cat}_\infty$ from the ∞-category of ∞-operads to the ∞-category of ∞-categories. This functor sends an ∞-operad $\mathcal{O}^{\otimes}$ to its underlying ∞-category $\mathcal{O}^{\otimes}_{|1_+}$. This functor is a right adjoint with left adjoint given by $(-)^{\amalg}$, as in Construction 2.4.3.1 in [21]. Thus we see that

$$\mathsf{open}(X)^{\otimes}_{|1_+} \xrightarrow{\mathcal{F}_{|1_+}} \mathcal{V}^{\otimes}_{|1_+}$$

is the right adjoint $(-)_{|1_+}$ applied to the functor of ∞-operads

$$\mathsf{open}(X)^{\otimes} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes} .$$

As such, a functor $\mathcal{I}(X) \to \mathsf{open}(X)^{\otimes}_{|1_+}$ is equivalent to a functor of ∞-operads $\mathcal{I}(X)^{\amalg} \to \mathsf{open}(X)^{\otimes}$. Similarly, a functor $\mathsf{B}\mathcal{I}(X) \to \mathcal{V}^{\otimes}_{|1_+}$ is equivalent to a functor of ∞-operads $\mathsf{B}\mathcal{I}(X)^{\amalg} \to \mathcal{V}^{\otimes}$. Recall the classifying space B is defined as a left adjoint (Definition A.0.22). Since left adjoints commute, we can unambiguously write $\mathsf{B}\mathcal{I}(X)^{\amalg}$. Therefore, the diagram in equation (2.1) is equivalent to the following

$$\begin{tikzcd}
\mathcal{I}(X)^{\otimes} \arrow[r, "\simeq", "\text{Lemma 2.0.15}"'] \arrow[d] & \mathcal{I}(X)^{\amalg} \arrow[r] \arrow[d] & \mathsf{open}(X)^{\otimes} \arrow[r, "\mathcal{F}"] & \mathcal{V}^{\otimes} & . \\
\mathsf{B}\mathcal{I}(X)^{\otimes} \arrow[r, "\simeq", "\text{Lemma 2.0.15}"'] & \mathsf{B}\mathcal{I}(X)^{\amalg} \arrow[urr, dashed, bend right] & & &
\end{tikzcd} \tag{2.2}$$

Again, by the definition of localization, this is precisely condition 3 in the statement of this lemma. □

For the duration of this chapter we will further assume that the unit $\mathbb{1} \in \mathcal{V}$ is intial. This is not the case for most $\mathcal{V}^{\otimes}$ of interest, such as chain complexes; however, we justify this assumption using the following proposition.

Proposition 2.0.17 ([21] Proposition 2.3.1.11)**.** *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and let $\mathcal{V}^{\otimes}$ be a symmetric monoidal ∞-category. The forgetful functor $(\mathcal{V}^{\otimes})^{\mathbb{1}/} \to \mathcal{V}^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathsf{Fun}_{\mathsf{opd}}(\mathcal{O}^{\otimes}, (\mathcal{V}^{\otimes})^{\mathbb{1}/}) \xrightarrow{\simeq} \mathsf{Fun}_{\mathsf{opd}}(\mathcal{O}^{\otimes}, \mathcal{V}^{\otimes}) \ .$$

Note that both $\mathsf{open}(X)^{\otimes}$ and $\mathsf{disk}(X)^{\otimes}$ (see Definition 2.0.37 below) are unital, with the empty set $\emptyset$ as the unit. The above proposition then justifies our assumption that the unit $\mathbb{1} \in \mathcal{V}^{\otimes}$ is initial. Below we will also need to work with the ∞-category of bifunctors. We note that a similar statement also holds for bifunctors:

Proposition 2.0.18. *Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be unital ∞-operads and let $\mathcal{V}^{\otimes}$ be a symmetric monoidal ∞-category. The forgetful functor $(\mathcal{V}^{\otimes})^{\mathbb{1}/} \to \mathcal{V}^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathsf{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; (\mathcal{V}^{\otimes})^{\mathbb{1}/}) \xrightarrow{\simeq} \mathsf{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

Proof. This follows by the same logic used in the proof of Proposition 2.0.17 in [21]. In particular, using Lemma 2.3.1.12 therein. □

Additivity of factorization algebras

In this section we prove the following additivity statement for factorization algebras. Its proof can be found after Lemma 2.0.35 below.

Theorem 2.0.19. *Let X and Y be topological spaces and let $\mathcal{V}^{\otimes}$ be a $\otimes$-presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathsf{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes})) \ . \tag{2.3}$$

Before delving into proving this theorem, we first give a brief outline of the logic involved. The ∞-category $\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes}))$ is the more complicated object in the statement of Theorem 2.0.19. To understand this ∞-category, we are inspired by the tensor-hom adjunction. A classical version of this adjunction is the following. Let R be a commutative ring. Recall that given two R-modules O and P, we can form a new R-module $O \otimes_R P$. Now let V be a third R-module. In this setting, the tensor-hom adjunction asserts there is an isomorphism

$$\mathsf{Hom}_R(O \otimes_R P, V) \cong \mathsf{Hom}_R(O, \mathsf{Hom}_R(P, V)) \ .$$

As shown in [21], the ∞-category of ∞-operads possess a tensor product. Similarly, for ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$, and a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, there is an equivalence

$$\mathsf{Fun}_{\mathsf{opd}}(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{Fun}_{\mathsf{opd}}(\mathcal{O}^{\otimes}, \mathsf{Fun}_{\mathsf{opd}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes})) \ .$$

Recall that an R-linear map $O \otimes_R P \to V$ is the same as an R-bilinear map $O \times P \to V$. We again have an analogous statement for ∞-operads, where the role of bilinear maps is played by bifunctors of ∞-operads. (We refer the unfamiliar reader to Definition A.0.46 and the surrounding discussion in the appendix.) The statement in this setting is that there is an equivalence of ∞-categories

$$\mathsf{Fun}_{\mathsf{opd}}(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

Thus, we use the theory of bifunctors of ∞-operads to make sense of the ∞-category $\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))$.

The proof of Theorem 2.0.19 now consists of two major components. First, we identify $\mathsf{Fact}_{X\times Y}(\mathcal{V}^\otimes)$ with $\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$. This is the statement of Proposition 2.0.34. Then we identify $\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$ with $\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))$. This is the statement of Lemma 2.0.35. Lemma 2.0.35 involves simply verifying that the aforementioned tensor-hom adjunction respects the two additional conditions of being a factorization algebra: multiplicativity and J_∞-cosheaf. The main content of this section lies in establishing Proposition 2.0.34.

The beginning of this section mostly addresses more technical issues. In particular, Lemma 2.0.27, Lemma 2.0.28, and Corollary 2.0.29 allow us to reduce our consideration of functors out of the category of open sets to functors out of a simpler category consisting of open sets that have finitely many connected components. Then, Corollary 2.0.30 provides an explcit formula that enables us to prove the logical crux of Proposition 2.0.32. We now proceed towards the proof of Theorem 2.0.19.

Consider the natural bifunctor

$$\rho : \mathsf{open}(X)^\otimes \times \mathsf{open}(Y)^\otimes \to \mathsf{open}(X\times Y)^\otimes ~, \quad (I_+,(U_i)),(J_+,(V_j)) \mapsto (I_+\wedge J_+,(U_i\times V_j)) ~.$$

Restriction along ρ has a left adjoint given formally by left Kan extension

$$\rho_! : \mathsf{BiFun}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \rightleftarrows \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X\times Y)^\otimes, \mathcal{V}^\otimes) : \rho^* ~. \tag{2.4}$$

We provide a formula for how this left Kan extension evaluates in Proposition 2.0.30 below. See Definition A.0.48 for a definition and discussion of left Kan extensions.

Definition 2.0.20. Let $\mathcal{F} \in \mathsf{BiFun}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$ be a bifunctor.

- We say $\mathcal{F}$ is a *multiplicative bifunctor* if $\mathcal{F}$ takes all pairs of coCartesian morphisms in $\mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}$ to coCartesian morphisms in $\mathcal{V}^{\otimes}$. Let

  $$\mathsf{BiFun}^{\mathsf{m}}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$$

  denote the full sub ∞-category consisting of the multiplicative bifunctors.

- We say $\mathcal{F}$ is a J_∞-*bifunctor* if the restriction $\mathcal{F}_{|_{1_+}} \to \mathcal{V}^{\otimes}_{|_{1_+}}$ is a J_∞-cosheaf separately in each variable. Let

  $$\mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$$

  denote the full ∞-subcategory consisting of the J_∞-bifunctors.

Proposition 2.0.34 below establishes that the adjunction in equation (2.4) restricts as an equivalence

$$\rho_! : \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fun}^{\mathsf{m},J_\infty}_{\mathsf{opd}}(\mathsf{open}(X \times Y)^{\otimes}, \mathcal{V}^{\otimes}) : \rho^* \ .$$

Key to the proof of Proposition 2.0.34 is the colimit expression for left Kan extension along a bifunctor that we provide in Corollary 2.0.30. To establish this, we use an alternate characterization of multiplicative factorization algebras that we now describe.

Definition 2.0.21. Let $\mathsf{open}(X)^{\otimes}_{\mathsf{fin}} \hookrightarrow \mathsf{open}(X)^{\otimes}$ denote the full ∞-sub-operad consisting of those objects $(I_+, (U_i))$ for which each U_i has finitely many connected components.

Definition 2.0.22. Let $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \hookrightarrow \mathsf{open}(X)^{\otimes}_{\mathsf{fin}}$ denote the full ∞-sub-operad consisting of those objects $(I_+, (U_i))$ for which each U_i has a single connected component.

Observation 2.0.23. Note that for connected $U \in \mathsf{open}(X)$, the composite functor

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}\,/^{\mathsf{act}}_{(1_+,U)}} \hookrightarrow \mathsf{open}(X)^{\otimes}_{\mathsf{fin}/^{\mathsf{act}}_{(1_+,U)}} \xrightarrow{(-)_!} \mathsf{open}(X)_{\mathsf{fin}/_U} \simeq \mathsf{open}(U)_{\mathsf{fin}}$$

is an equivalence. Here, $(-)_!$ denotes the coCartesian monodromy functor which exists in light of Observation 2.0.4.

Definition 2.0.24. We define $\mathsf{Fun}^{J_\infty}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes}) \hookrightarrow \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes})$ to be the full ∞-subcategory consisting of those morphisms of operads $\mathcal{F}$ for which for each $U \in \mathsf{open}(X)$, the composite

$$\mathsf{open}(X)_{\mathsf{fin}/_U} \xrightarrow{\simeq} \mathsf{open}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(1_+,U)}} \to \mathcal{V}^{\otimes}_{/1_+} \xrightarrow{\otimes} \mathcal{V} \ ,$$

is a J_∞-cosheaf.

Observation 2.0.25. Note that $\mathsf{open}(X)_{\mathsf{fin}} \to \mathsf{open}(X)$ is a basis for the J_∞ Grothendieck topology on $\mathsf{open}(X)$. Since we are only interested in J_∞-cosheaves, this observation justifies our restriction to $\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}$.

There is a functor $\mathsf{expand} : \mathsf{open}(X)^{\otimes}_{\mathsf{fin}} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}}$ given by

$$(I_+, (U_i)) \mapsto \left(\left(\coprod_{i \in I} \pi_0(U_i) \right)_+ , (U_{i_\alpha})_{\alpha \in \pi_0(U_i)} \right) \ ,$$

which expands an I-indexed list of open sets. This functor is right adjoint to the inclusion

$$\iota_{\mathsf{c}} : \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \rightleftarrows \mathsf{open}(X)^{\otimes}_{\mathsf{fin}} : \mathsf{expand} \ . \tag{2.5}$$

Observation 2.0.26. A morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ in $\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}$ is coCartesian if and only if $\mathsf{expand}(f)$ is coCartesian in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}}$.

Lemma 2.0.27. *Given a functor of ∞-operads $\mathcal{F} : \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \to \mathcal{V}^{\otimes}$, there exists a unique multiplicative filler to the following diagram*

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \arrow[r, "\mathcal{F}"] \arrow[d, "\iota_{\mathsf{c}}"'] & \mathcal{V}^{\otimes} \\
\mathsf{open}(X)^{\otimes}_{\mathsf{fin}} \arrow[ur, dashed] &
\end{tikzcd} .$$

Proof. The counit of the adjunction in equation (2.5) defines a functor

$$\varepsilon : \mathsf{open}(X)^{\otimes} \to \mathsf{Ar}(\mathsf{open}(X)^{\otimes}) \ .$$

Define the composite

$$\widetilde{\mathcal{F}} : \mathsf{open}(X)^{\otimes}_{\mathsf{fin}} \xrightarrow{\varepsilon} \mathsf{Ar}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}) \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times_{\mathsf{Fin}_*} \mathsf{Ar}(\mathsf{Fin}_*) \xrightarrow{\mathcal{F}\times\mathsf{id}} \mathcal{V}^{\otimes} \times_{\mathsf{Fin}_*} \mathsf{Ar}(\mathsf{Fin}_*) \xrightarrow{(-)_!} \mathcal{V}^{\otimes} \ .$$

Here, the fiber product $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times_{\mathsf{Fin}_*} \mathsf{Ar}(\mathsf{Fin}_*)$ is taken with respect to the functor $\mathsf{Ar}(\mathsf{Fin}_*) \xrightarrow{\langle s\rangle} \mathsf{Fin}_*$. Using Observation 2.0.26, one can show that $\widetilde{\mathcal{F}}$ is indeed the unique multiplicative filler to the diagram. □

Lemma 2.0.28. *There is an equiavelnce of* ∞*-categories*

$$\mathsf{Fun}^{m}_{opd}(\mathsf{open}(X)^{\otimes}_{fin}, \mathcal{V}^{\otimes}) \underset{\iota_c^*}{\xrightarrow{\simeq}} \mathsf{Fun}_{opd}(\mathsf{open}(X)^{\otimes}_{c}, \mathcal{V}^{\otimes}) \ .$$

Furthermore, this equivalence restricts to an equivalence between the J_∞ *subcategories*

$$\mathsf{Fun}^{m,J_\infty}_{opd}(\mathsf{open}(X)^{\otimes}_{fin}, \mathcal{V}^{\otimes}) \underset{\iota_c^*}{\xrightarrow{\simeq}} \mathsf{Fun}^{J_\infty}_{opd}(\mathsf{open}(X)^{\otimes}_{c}, \mathcal{V}^{\otimes}) \ .$$

Proof. Lemma 2.0.27 defines a functor $\mathsf{Fun}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes}) \to \mathsf{Fun}^{\mathsf{m}}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}, \mathcal{V}^{\otimes})$ which is inverse to the restriction ι^*_{c}. The J_∞ statement is readily verified since $\mathsf{open}(X)^{\otimes}_{\mathsf{c}}$ is a J_∞-basis for $\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}$. □

By applying Lemma 2.0.28 in each factor, we obtain the following corollary.

Corollary 2.0.29. *There is an equivalence of* ∞*-categories*

$$\mathsf{BiFun}^{m}(\mathsf{open}(X)^{\otimes}_{fin}, \mathsf{open}(Y)^{\otimes}_{fin}; \mathcal{V}^{\otimes}) \underset{\iota_c^*}{\xrightarrow{\simeq}} \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}_{c}, \mathsf{open}(Y)^{\otimes}_{c}; \mathcal{V}^{\otimes}) \ .$$

Furthermore, this equivalence restricts as an equivalence between the J_∞ subcategories

$$\mathsf{BiFun}^{m,J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}, \mathsf{open}(Y)^{\otimes}_{\mathsf{fin}}; \mathcal{V}^{\otimes}) \xrightarrow[\iota_c^*]{\simeq} \mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)^{\otimes}_{c}, \mathsf{open}(Y)^{\otimes}_{c}; \mathcal{V}^{\otimes}) \ .$$

In light of Observation 2.0.25, Lemma 2.0.28, and Corollary 2.0.29 we restrict attention to the bifunctor

$$\rho_{\mathsf{c}} : \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}} \to \mathsf{open}(X \times Y)^{\otimes}_{\mathsf{c}}$$

defined in the same way as ρ. Using Proposition A.0.52, we obtain the following.

Corollary 2.0.30. *For $\mathcal{F} \in \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}_{c}, \mathsf{open}(Y)^{\otimes}_{c}; \mathcal{V}^{\otimes})$, the left adjoint $(\rho_{\mathsf{c}})_!\mathcal{F}$ evaluates on $(I_+, (U_i)) \in \mathsf{open}(X \times Y)^{\otimes}$ as the colimit*

$$\operatorname{colim}\left(\mathsf{open}(X)^{\otimes}_{c} \times \mathsf{open}(Y)^{\otimes}_{c}{}_{/(I_+,(U_i))} \to \mathsf{open}(X)^{\otimes}_{c} \times \mathsf{open}(Y)^{\otimes}_{c}{}_{/I_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/I_+} \xrightarrow{(-)_!} \mathcal{V}^{\otimes}_{|I_+}\right) \ .$$

Further, if $\mathcal{F}$ is J_∞, then $(\rho_{\mathsf{c}})_!\mathcal{F}$ is J_∞.

Proof. Proposition A.0.52 tells us this colimit expression defines a functor over Fin_*. It remains to check that $(\rho_{\mathsf{c}})_!$ carries inert-coCartesian morphisms to inert-coCartesian morphisms, and that $(\rho_{\mathsf{c}})_!\mathcal{F}$ is J_∞ if $\mathcal{F}$ is a J_∞ bifunctor. First we will verify that $(\rho_{\mathsf{c}})_!$ carries inert-coCartesian morphisms to inert-coCartesian morphisms. Let $(I_+, (U_i)) \xrightarrow{f} (J_+, (U_j))$ be an inert-coCartesian morphism in $\mathsf{open}(X \times Y)^{\otimes}_{\mathsf{c}}$. Since f is inert and $\mathcal{V}^{\otimes}$ is a coCartesian fibration, the monodromy functor $\mathcal{V}^{\otimes}_{|I_+} \to \mathcal{V}^{\otimes}_{|J_+}$ is given by projection. As such, the monodromy functor preserves colimits. This implies that

$$\begin{aligned}(\rho_{\mathsf{c}})_!\mathcal{F}((I_+, (U_i))) &\simeq \operatorname{colim}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathcal{V}^{\otimes}_{|I_+}) \\ &\simeq \operatorname{colim}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathcal{V}^{\otimes}_{|I_+} \xrightarrow{\mathsf{pr}} \mathcal{V}^{\otimes}_{|J_+}) \ .\end{aligned}$$

Now, observe the commutative diagram

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \arrow[r] \arrow[d] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/I_+} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{V}^{\otimes}_{|I_+} \arrow[d, "\mathsf{pr}"] \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))} \arrow[r] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/J_+} \arrow[r, "\mathcal{F}"] & \mathcal{V}^{\otimes}_{|J_+}
\end{tikzcd} .$$

This implies $(\rho_{\mathsf{c}})_!\mathcal{F}((I_+,(U_i)))$ is equivalent to

$$\operatorname{colim}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))} \to \mathcal{V}^{\otimes}_{|J_+}) \ .$$

Finally, we use Quillen's Theorem A (Theorem A.0.62) to show the functor

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))}$$

is final. The hypothesis of Theorem A.0.62 requires us to verify that for any object $\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+ \xrightarrow{\alpha} J_+\big)$ in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(U_j))}$, the classifying space of the undercategory

$$\Big(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)^{\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+ \xrightarrow{\alpha} J_+\big)/} \tag{2.6}$$

is contractible. To see this, note the assumption that $f : I_+ \to J_+$ is inert allows us to define a map $K_+ \wedge L_+ \xrightarrow{\beta} I_+$ via $(k,\ell) \mapsto f^{-1}(\alpha(k,\ell))$. The object $\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+ \xrightarrow{\beta} I_+\big)$ is then seen to be initial in the undercategory of equation (2.6). Hence its classifying space is contractible by Observation A.0.24. This completes the proof that $(\rho_{\mathsf{c}})_!\mathcal{F}$ is a functor of ∞-operads.

Now, for $\mathcal{F} \in \mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}, \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}; \mathcal{V}^{\otimes})$, we will show that $(\rho_{\mathsf{c}})_!\mathcal{F} \in \mathsf{Fun}^{J_\infty}_{\mathsf{opd}}(\mathsf{open}(X \times Y)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes})$. Note that products of open sets form a basis for $\mathsf{open}(X \times Y)$. Thus, we only need to check this statement holds for products. This is precisely what is shown in Proposition 2.0.32 below. ☐

In Proposition 2.0.32 below, we verify that $(\rho_{\mathsf{c}})_!\mathcal{F}(1_+, U \times V) \simeq \mathcal{F}((1_+, U), (1_+, V))$. Corollary 2.0.30 tells us that we can compute $(\rho_{\mathsf{c}})_!\mathcal{F}(1_+, U \times V)$ as the colimit of the following functor

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}\ /(1_+, U\times V)} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}\ /1_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/!_+} \xrightarrow{(-)_!} \mathcal{V}\ . \tag{2.7}$$

The general strategy of the proof of Proposition 2.0.32 exploits some additional functoriality coming from the unitality of $\mathcal{V}^{\otimes}$. We do this by extending the domain of the functor in equation (2.7) to a larger category. Using this extension, we then compute $(\rho_c)_!\mathcal{F}$ as an iterated left Kan extension. First, we lay out some necessary definitions.

Define the category $\mathsf{Fin}_* \times \mathsf{Fin}_{*/^{\mathsf{min}}_{1_+}}$ to consist of the same objects as $\mathsf{Fin}_* \times \mathsf{Fin}_{*/1_+}$ but with a morhpism $(I_+, J_+, I_+ \wedge J_+ \xrightarrow{f} 1_+) \to (K_+, L_+, K_+ \wedge L_+ \xrightarrow{g} 1_+)$ given by a map of based finite sets $I_+ \wedge J_+ \xrightarrow{\alpha} K_+ \wedge L_+$ such that the following conditions hold:

1. the diagram

$$\begin{tikzcd} I_+ \wedge J_+ \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & K_+ \wedge L_+ \arrow[dl, "g"] \\ & 1_+ & \end{tikzcd}$$

   commutes;

2. $\alpha^{-1}(K \times L) = f^{-1}(1)$;

3. for all $(i, j) \in \alpha^{-1}(K \times L)$, the projection $\mathsf{pr}_K(\alpha(i, j))$ is independent of j, and the projection $\mathsf{pr}_L(\alpha(i, j))$ is independent of i.

For notational purposes, let us now define

$$\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}\ /^{\mathsf{min}}_{1_+}} := \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}} \times_{\mathsf{Fin}_*} \mathsf{Fin}_* \times \mathsf{Fin}_{*/^{\mathsf{min}}_{1_+}}\ .$$

Definition 2.0.31. We define the category $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}}$ to consist of the same objects as $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+, U\times V)}$ but with a morphism

$$\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \to \big((K_+, (D_k)), (L_+, (E_\ell)), K_+ \wedge L_+ \xrightarrow{g} 1_+\big)$$

given by a morphism $\alpha \in \mathsf{Fin}_* \times \mathsf{Fin}_{*/^{\mathsf{min}}_{1_+}}$ satisfying the following conditions for all $(k, \ell) \in K \times L$:

1. the diagram

$$\begin{tikzcd} I_+ \wedge J_+ \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & K_+ \wedge L_+ \arrow[dl, "g"] \\ & 1_+ & \end{tikzcd}$$

   commutes;

2. $\alpha^{-1}(K \times L) = f^{-1}(1)$;

3. for all $(i, j) \in \alpha^{-1}(K \times L)$, the projection $\mathsf{pr}_K(\alpha(i, j))$ is independent of j, and the projection $\mathsf{pr}_L(\alpha(i, j))$ is independent of i.

4. the collection of the sets A_i indexed over all $i \in I$ for which there exists $j \in J$ such that $\alpha(i, j) = (k, \ell)$ form a pairwise disjoint collection of open susbsets of D_k;

5. the collection of the sets B_j indexed over all $j \in J$ for which there exists $i \in I$ such that $\alpha(i, j) = (k, \ell)$ form a pairwise disjoint collection of open susbsets of E_ℓ;

The idea of the category $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}}$ is that it selects out 'minimal' morphisms between partial grids in the open set $U \times V$. We give some visual intuition for what we mean by this below. Namely, condition 2 eliminates flexibility of the underlying morphisms of based finite sets by forcing the morphism of sets to collapse everything possible to the basepoint. This is illustrated in Figure 2.2. Condition 3 ensures that grids do not get split up, as illustrated in Figure 2.3. Conditions 4 and 5 ensure that we can only include grids, as illustrated in Figure 2.4.

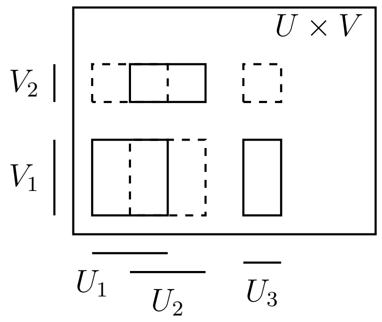


Figure 2.1: A typical object in $\mathsf{open}(\mathbb{R})^{\otimes}_{\mathsf{c}} \times \mathsf{open}(\mathbb{R})^{\otimes}_{\mathsf{c}}{}_{/(1_+, U\times V)}$. We have indicated the map $\{1,2,3\}_+ \wedge \{1,2\}_+ \to 1_+$ by the style of line used. Namely, those boxes that are sent to 1 are styled with solid boundary, and those that are sent to + are styled with dashed boundary.

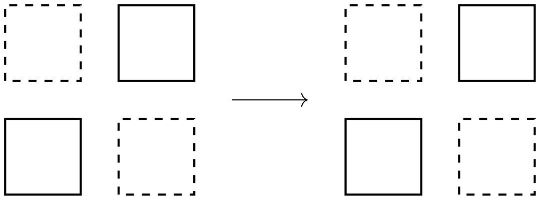


Figure 2.2: Condition 2) of Definition 2.0.31 disallows, for example, the identity morphism on underlying finite sets such as the one indicated.

Proposition 2.0.32. *For $\mathcal{F} \in \mathit{BiFun}^{J_\infty}(\mathit{open}(X)^{\otimes}_{\mathit{c}}, \mathit{open}(Y)^{\otimes}_{\mathit{c}}; \mathcal{V}^{\otimes})$ and connected open subsets $U \in \mathit{open}(X)_{\mathit{c}}$ and $V \in \mathit{open}(Y)_{\mathit{c}}$, there is an equivalence in $\mathcal{V}$*

$$\mathcal{F}((1_+, U), (1_+, V)) \xrightarrow{\simeq} (\rho_{\mathsf{c}})_! \mathcal{F}(1_+, U \times V) \ .$$

Proof. Observe the evident functor

$$\Phi : \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+, U\times V)} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}} \ .$$

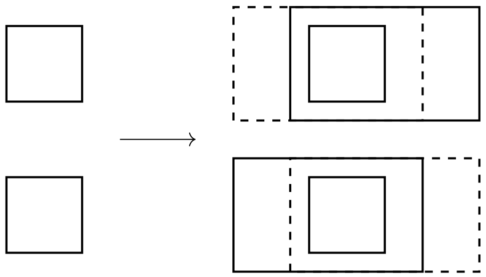


Figure 2.3: Condition 3) of Definition 2.0.31 disallows, for example, a morphism that splits up gridded elements such as the one indicated.

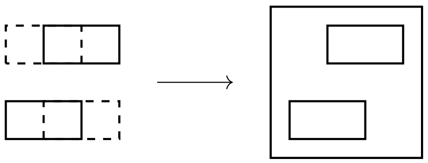


Figure 2.4: Condition 4) of Definition 2.0.31 disallows, for example, an inclusion of overlapping elements such as the one indicated.

To define the aforementioned extension, observe the solid commutative diagram

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+,U\times V)} \arrow[r] \arrow[d, "\Phi"] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/1_+} \arrow[r, "\mathcal{F}_{/1_+}"] \arrow[d] & \mathcal{V}^{\otimes}_{/1_+} \arrow[r, "(-)_!"] & \mathcal{V} \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}} \arrow[r] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{1_+}} \arrow[ur, dashed, bend right, "\mathcal{F}^{\mathsf{min}}_{/1_+}"'] & & .
\end{tikzcd}$$

We now define the dashed arrow, $\mathcal{F}^{\mathsf{min}}_{/1_+}$. On objects, $\mathcal{F}^{\mathsf{min}}_{/1_+}$ evaluates the same as $\mathcal{F}_{/1_+}$. Consider a morphism

$$\big((I_+,(A_i)),(J_+,(B_j),I_+\wedge J_+\xrightarrow{f}1_+\big)\xrightarrow{\alpha}\big((K_+,(D_k)),(L_+,(E_\ell)),K_+\wedge L_+\xrightarrow{g}1_+\big)$$

in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{1_+}}$. We define a morphism in $\mathcal{V}^{\otimes}_{/1_+}$ between their images as the following composite

$$\big(\mathcal{F}(A_i, B_j)\big)_{I\times J} \xrightarrow{\mathsf{inert}} \big(\mathcal{F}(A_i, B_j)\big)_{f^{-1}(1)} \xrightarrow{(-)_!} \left(\bigotimes_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j)\right)_{K\times L} \to \big(\mathcal{F}(D_k, E_\ell)\big)_{K\times L}\ . \tag{2.8}$$

The middle morphism is the coCartesian monodromy functor applied to the morphism of based finite sets $f^{-1}(1) \xrightarrow{\alpha_|} K \times L$. We now describe the third morphism in equation (2.8). This is a morphism over the identity $K \times L \to K \times L$, so we describe this morphism for each $(k, \ell) \in K \times L$. Fix $(k, \ell) \in K \times L$. If $\alpha^{-1}(k, \ell) = \emptyset$, then we take the empty tensor product

$$\bigotimes_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j)$$

to be the unit $\mathbb{1} \in \mathcal{V}$. By the universal property of the unit, there is a unique morphism $\mathbb{1} \to \mathcal{F}(D_k, E_\ell)$. If $\alpha^{-1}(k, \ell) \neq \emptyset$ then we claim there is a morphism in $\mathcal{V}$

$$\bigotimes_{(i,j)\in\alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j) \to \mathcal{F}(D_k, E_\ell)\ .$$

This follows precisely from the conditions that we placed on morphisms in the overcategory $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{1_+}}$.

Now, $(\rho_{\mathsf{c}})_!\mathcal{F}(1_+, U \times V)$ is given by the colimit of the top composite arrow in the following diagram

$$\begin{tikzcd}
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+, U\times V)} \arrow[r] \arrow[d, "\Phi"] & \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/1_+} \arrow[r, "\mathcal{F}_{/1_+}"] & \mathcal{V}^{\otimes}_{/1_+} \arrow[r, "(-)_!"] & \mathcal{V} \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}} \arrow[d, "!"] & & & \\
* \, . & & &
\end{tikzcd}$$

We will henceforth refer to this top composite functor as $\mathcal{F}_{/1_+}$. Note that the colimit of $\mathcal{F}_{/1_+}$ is equivalent to the left Kan extension of $\mathcal{F}_{/1_+}$ along the unique functor to $*$. By Proposition A.0.50, this is equivalent to the colimit of the left Kan extension $\Phi_! \mathcal{F}_{/1_+}$. That is,

$$(\rho_{\mathsf{c}})_! \mathcal{F}(1_+, U \times V) \simeq \operatorname{colim} \Phi_! \mathcal{F}_{/1_+} \ .$$

Next, we will identify our constructed extension $\mathcal{F}^{\mathsf{min}}_{/1_+}$ with $\Phi_! \mathcal{F}_{/1_+}$. To do this, we construct a section $\mathcal{F}^{\mathsf{min}}_{/1_+} \xrightarrow{\Psi} \Phi_! \mathcal{F}_{/1_+}$ under $\mathcal{F}_{/1_+}$. By Proposition A.0.36, the coCartesian monodromy functor $\mathcal{V}^{\otimes}_{/1_+} \to \mathcal{V}$ is a left adjoint and thus preserves colimits. Thus, we will work in $\mathcal{V}^{\otimes}_{/1_+}$. On objects $\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big)$, the definition of the section is clear since $\mathcal{F}^{\mathsf{min}}_{/1_+}$ evaluates the same as $\mathcal{F}_{/1_+}$. There is a natural morphism in $\mathcal{V}^{\otimes}_{/1_+}$

$$\mathcal{F}_{/1_+}\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \to \Phi_! \mathcal{F}_{/1_+}\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big)$$

since left Kan extensions are defined as initial extensions. Now, consider a morphism

$$\big((I_+, (A_i)), (J_+, (B_j), I_+ \wedge J_+ \xrightarrow{f} 1_+\big) \xrightarrow{\alpha} \big((K_+, (D_k)), (L_+, (E_\ell)), K_+ \wedge L_+ \xrightarrow{g} 1_+\big)$$

in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U \times V)}}$. Observe the factorization in $\mathcal{V}^{\otimes}_{/1_+}$ from equation (2.8) also allows us the following factorization

$$\begin{tikzcd}
\big(\mathcal{F}(A_i, B_j)\big)_{I \times J} \arrow[r, "\mathsf{inert}"] \arrow[d, "\Psi"] & \big(\mathcal{F}(A_i, B_j)\big)_{f^{-1}(1)} \arrow[r, "(-)_!"] \arrow[d] & \left(\bigotimes\limits_{(i,j) \in \alpha^{-1}(k,\ell)} \mathcal{F}(A_i, B_j)\right)_{K \times L} \arrow[r] \arrow[d, "\mathsf{canon}"] & \big(\mathcal{F}(D_k, E_\ell)\big)_{K \times L} \arrow[d, "\Psi"] \\
\big(\Phi_! \mathcal{F}_{/1_+}(A_i, B_j)\big)_{I \times J} \arrow[r, "\mathsf{inert}"] & \big(\Phi_! \mathcal{F}_{/1_+}(A_i, B_j)\big)_{f^{-1}(1)} \arrow[r, "(-)_!"] & \left(\bigotimes\limits_{(i,j) \in \alpha^{-1}(k,\ell)} \Phi_! \mathcal{F}_{/1_+}(A_i, B_j)\right)_{K \times L} \arrow[r] & \big(\Phi_! \mathcal{F}_{/1_+}(D_k, E_\ell)\big)_{K \times L} \ .
\end{tikzcd}$$

Note the bottom left object is equivalent to

$$\Phi_! \mathcal{F}_{/1_+}((I_+,(A_i)),(J_+,(B_j)),I_+ \wedge J_+ \xrightarrow{f} 1_+) \ .$$

Similarly, the bottom right object is equivalent to

$$\Phi_! \mathcal{F}_{/1_+}((K_+,(D_k)),(L_+,(E_\ell)),K_+ \wedge L_+ \xrightarrow{g} 1_+) \ .$$

The two left-most squares commute because coCartesian monodromy is a functor (see Definition A.0.37). We now explain why the right hand square commutes. Note that both right-most horizontal arrows come from morphisms in $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/(1_+,U\times V)}$. Further, the vertical arrow labeled canon is from the definiton of left Kan extension as the initial extension of $\mathcal{F}_{/1_+}$. Thus, the right hand square commutes since it is the evaluation of the canonical natural transformation from the definition of left Kan extension. This completes the construction of the natural transformation

$$\Psi : \mathcal{F}^{\mathsf{min}}_{/1_+} \to \Phi_! \mathcal{F}_{/1_+} \ .$$

Note that this evidently lies under $\mathcal{F}_{/1_+}$, and clearly defines a section of $\Phi_! \mathcal{F}_{/1_+}$. By initiality of the left Kan extension, this implies the equivalence $\Phi_! \mathcal{F}_{/1_+} \simeq \mathcal{F}^{\mathsf{min}}_{/1_+}$. Therefore, we have shown

$$(\rho_{\mathsf{c}})_! \mathcal{F}(1_+, U \times V) \simeq \operatorname{colim} \Phi_! \mathcal{F}_{/1_+} \simeq \operatorname{colim} \mathcal{F}^{\mathsf{min}}_{/1_+} \ . \tag{2.9}$$

We will now show that $\mathcal{F}((1_+,U),(1_+,V)) \simeq \operatorname{colim} \mathcal{F}^{\mathsf{min}}_{/1_+}$, which will complete the proof. Consider the object $\big((1_+,U),(1_+,V),\mathsf{id}\big) \in \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}$. Observe the forgetful functor

$$\nabla : \left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})} \to \mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}} \ .$$

We will show the counit $\nabla_! \nabla^* \mathcal{F}^{\mathsf{min}}_{/1_+} \to \mathcal{F}^{\mathsf{min}}_{/1_+}$ is an equivalence. Before doing so, we explain why this implies $\mathcal{F}((1_+, U), (1_+, V)) \simeq (\rho_{\mathsf{c}})_! \mathcal{F}(1_+, U \times V)$. Note that the category

$$\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+, U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})}$$

has a final object $((1_+, U), (1_+, V), \mathsf{id})$. This can be verified using Quillen's Theorem A (Theorem A.0.62). Thus,

$$\operatorname{colim} \nabla^* \mathcal{F}^{\mathsf{min}}_{/1_+} \simeq \mathcal{F}^{\mathsf{min}}_{/1_+}((1_+, U), (1_+, V), \mathsf{id}) \simeq \mathcal{F}((1_+, U), (1_+, V)) \ .$$

Further, since left Kan extensions compose by Proposition A.0.50, we have

$$\operatorname{colim} \nabla^* \mathcal{F}^{\mathsf{min}}_{/1_+} \simeq \operatorname{colim} \nabla_! \nabla^* \mathcal{F}^{\mathsf{min}}_{/1_+} \ .$$

Using equation (2.9), upon taking the colimit of the counit, we have

$$\mathcal{F}((1_+, U), (1_+, V)) \simeq \operatorname{colim} \nabla_! \nabla^* \mathcal{F}^{\mathsf{min}}_{/1_+} \to \operatorname{colim} \mathcal{F}^{\mathsf{min}}_{/1_+} \simeq (\rho_{\mathsf{c}})_! \mathcal{F}(1_+, U \times V) \ .$$

Finally, we now complete the proof by showing the counit $\nabla_! \nabla^* \mathcal{F}^{\mathsf{min}}_{/1_+} \to \mathcal{F}^{\mathsf{min}}_{/1_+}$ is an equivalence. To do this, we will use the assumption that $\mathcal{F}$ is a J_∞-cosheaf in each factor. We show this using Lemma 2.0.33 below. First, note that for

$$((I_+, (A_i)), (J_+, (B_j)), I_+ \wedge J_+ \xrightarrow{f} 1_+) \in \mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}_{/^{\mathsf{min}}_{(1_+, U\times V)}} \ ,$$

the functor $\nabla_!\nabla^*\mathcal{F}^{\mathsf{min}}_{/1_+}$ evaluates as the colimit of the composite

$$\begin{tikzcd}
\left(\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})}\right)_{/((I_+,(A_i)),(J_+,(B_j)),I_+\wedge J_+\xrightarrow{f}1_+)} \arrow[d] \\
\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})} \arrow[d, "\nabla"] \\
\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}} \arrow[d, "\mathcal{F}^{\mathsf{min}}_{/1_+}"] \\
\mathcal{V}\ .
\end{tikzcd}$$

Further, since $\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}$ is a subposet of $\mathsf{open}(U) \times \mathsf{open}(V)$, so is the domain of this composite. Now, we verify that

$$\left(\left(\mathsf{open}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{open}(Y)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{min}}_{(1_+,U\times V)}}\right)_{/((1_+,U),(1_+,V),\mathsf{id})}\right)_{/((I_+,(A_i)),(J_+,(B_j)),I_+\wedge J_+\xrightarrow{f}1_+)} \tag{2.10}$$

satisfies the hypotheses of Lemma 2.0.33. First, note that the projection of the overcategory in equation (2.10) is simply identified as $\mathsf{open}^{\mathsf{fin}}(\amalg A_i)$. Next, observe the subposet $\mathsf{open}^{\mathsf{surj}}(\amalg A_i) \subset \mathsf{open}^{\mathsf{fin}}(\amalg A_i)$ consisting of those open sets that are surjective on connected components. Using Theorem A.0.62, this is checked to be final. Further note that for $D \in \mathsf{open}^{\mathsf{surj}}(\amalg A_i)$, the fiber of equation (2.10) is evidently a J_∞-cover of V. Finally, the coCartesian condition in Lemma 2.0.33 is also readily verified, which completes the proof. □

Lemma 2.0.33. *Let $U \in \mathsf{open}(X)_{\mathsf{c}}$ and $V \in \mathsf{open}(Y)_{\mathsf{c}}$. Assume $\mathcal{F} : \mathsf{open}(U)\times\mathsf{open}(V) \to \mathcal{V}$ is a J_∞-cosheaf in each factor. For $\mathcal{U} \subset \mathsf{open}(U)\times\mathsf{open}(V)$ a full subposet, let $\mathcal{U}_U := \mathsf{pr}_U\mathcal{U} \subset \mathsf{open}(U)$ denote the full subposet given by projection onto U. If*

1. $\mathcal{U}_U$ is a J_∞ cover,

2. there exists a final full subcategory $\mathcal{U}_U^0 \subset \mathcal{U}_U$ such that for each $D \in \mathcal{U}_U^0$, the fiber $\mathcal{U}_{|_D} \subset \mathsf{open}(V)$ is a J_∞-cover, and

3. the functor $\mathcal{U}_{|_{\mathcal{U}_U^0}} \to \mathcal{U}_U^0$ is a coCartesian fibration,

then

$$\operatorname{colim}\left(\mathcal{U} \hookrightarrow \mathsf{open}(U) \times \mathsf{open}(V) \xrightarrow{\mathcal{F}} \mathcal{V}\right) \simeq \mathcal{F}(U, V)\ .$$

Proof. Note the functor $\mathcal{U} \xrightarrow{\mathsf{pr}} \mathcal{U}_U$. By Proposition A.0.50 and the assumption that $\mathcal{U}_U^0$ is final, we have

$$\operatorname{colim}\left(\mathcal{U} \hookrightarrow \mathsf{open}(U) \times \mathsf{open}(V) \xrightarrow{\mathcal{F}} \mathcal{V}\right) \simeq \operatorname{colim}\left(\mathsf{pr}_!\mathcal{F}\right) \simeq \operatorname{colim}\left(\mathcal{U}_U^0 \hookrightarrow \mathcal{U}_U \xrightarrow{\mathsf{pr}_!\mathcal{F}} \mathcal{V}\right)\ .$$

By assumption, $\mathcal{U}_{|_{\mathcal{U}_U^0}} \to \mathcal{U}_U^0$ is a coCartesian fibration. Therefore, by Proposition A.0.51, the left Kan extension can be computed as a fiberwise colimit. That is, the functor

$$\mathcal{U}_U^0 \hookrightarrow \mathcal{U}_U \xrightarrow{\mathsf{pr}_!\mathcal{F}} \mathcal{V}$$

evaluates on $D \in \mathcal{U}_U^0$ as $D \mapsto \operatorname{colim}\left(\mathcal{U}_{|_D} \xrightarrow{\mathcal{F}} \mathcal{V}\right) \simeq \mathcal{F}(D, V)$ The equivalence is the assumption that $\mathcal{F}$ is a J_∞-cosheaf in the second factor. Thus, since $\mathcal{F}$ is assumed to be a J_∞-cosheaf in the first factor, taking the colimit of the fiberwise evaluations is equivalent to $\mathcal{F}(U, V)$, as desired. □

Proposition 2.0.34. *Let $\mathcal{V}^\otimes$ be a $\otimes$-presentable ∞-category. The adjunction in equation (2.4) restricts to an equivalence of ∞-categories*

$$\rho_! : \mathsf{BiFun}^{m, J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathsf{Fact}_{X \times Y}(\mathcal{V}^\otimes) : \rho^*\ .$$

Proof. Observe the commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}^{\mathsf{m},J_\infty}_{\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}_{\mathsf{fin}},\mathcal{V}^{\otimes}) \arrow[r] \arrow[d, "\rho^*"] & \mathsf{Fun}^{J_\infty}_{\mathsf{opd}}(\mathsf{open}(X\times Y)^{\otimes}_{\mathsf{c}},\mathcal{V}^{\otimes}) \arrow[d, "\rho^*_{\mathsf{c}}"] \\
\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{fin}}\times\mathsf{open}(Y)^{\otimes}_{\mathsf{fin}},\mathcal{V}^{\otimes}) \arrow[r] & \mathsf{BiFun}^{J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{open}(Y)^{\otimes}_{\mathsf{c}},\mathcal{V}^{\otimes})
\end{tikzcd} . \tag{2.11}$$

Lemma 2.0.28 established that the top horizontal functor is an equivalence. Corollary 2.0.29 established that the bottom horizontal functor is an equivalence. Thus, it suffices to show that ρ^*_{c} is an equivalence. To prove this, we show that the unit and counit of the adjunction evaluate as equivalences.

First, we show the counit $(\rho_{\mathsf{c}})_!\rho^*_{\mathsf{c}}\mathcal{F}\to\mathcal{F}$ evaluates as an equivalence. Since $\mathsf{pr}:\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ preserves colimits, we have the equivalence $(\rho_{\mathsf{c}})_!\rho^*_{\mathsf{c}}\mathcal{F}(I_+,(U_i))\simeq(I_+,((\rho_{\mathsf{c}})_!\rho^*_{\mathsf{c}}\mathcal{F}(1_+,U_i)))$. Further, since $(\rho_{\mathsf{c}})_!\rho^*_{\mathsf{c}}\mathcal{F}$ is a J_∞-cosheaf, and products of open sets form a basis for the J_∞-topology on $\mathsf{open}(X\times Y)$, its values are determined by its values on products of opens. Proposition 2.0.32 establishes this.

Similarly, for $\mathcal{G}\in\mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}_{\mathsf{c}},\mathsf{open}(Y)^{\otimes}_{\mathsf{c}};\mathcal{V}^{\otimes})$, we can use the fact that $\rho^*_{\mathsf{c}}(\rho_{\mathsf{c}})_!\mathcal{G}$ is J_∞ in each factor to reduce the computation to an evaluation on unary objects. Again, since products of opens form a basis, we invoke Proposition 2.0.32 to show the unit evaluates as equivalences. ☐

There is a natural functor

$$\mathsf{ev}:\mathsf{BiFun}(\mathsf{open}(X)^{\otimes},\mathsf{open}(Y)^{\otimes};\mathcal{V}^{\otimes})\to\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes},\mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes},\mathcal{V}^{\otimes})^{\otimes})\ , \tag{2.12}$$

given by evaluation. As shown in [21], this functor is an equivalence. More precisely, by definition a functor $\mathcal{K} \to \mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ is a dashed arrow filling the diagram

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{O}^{\otimes} \arrow[r, dashed] \arrow[d, "\mathsf{pr}"] & \mathcal{P}^{\otimes} \arrow[dd] \\
\mathcal{O}^{\otimes} \arrow[d] & \\
\mathsf{Fin}_* \arrow[r, "\mathsf{id}"] & \mathsf{Fin}_*
\end{tikzcd} .$$

For $\mathcal{K} \to \mathsf{Fin}_*$ and $\mathcal{V}^{\otimes}$ a symmetric monoidal ∞-category, a functor $\mathcal{K} \to \mathsf{Fun}^{\mathsf{opd}}(\mathcal{O}^{\otimes}, \mathcal{V}^{\otimes})^{\otimes}$ over Fin_* is a dashed arrow filling

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{O}^{\otimes} \arrow[r, dashed] \arrow[d] & \mathcal{V}^{\otimes} \arrow[dd] \\
\mathsf{Fin}_* \times \mathsf{Fin}_* \arrow[d, "\wedge"] & \\
\mathsf{Fin}_* \arrow[r] & \mathsf{Fin}_*
\end{tikzcd} .$$

Thus, a functor

$$\mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \to \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}^{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})^{\otimes})$$

is a filler

$$\begin{tikzcd}
\big(\mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \times \mathsf{open}(X)^{\otimes}\big) \times \mathsf{open}(Y)^{\otimes} \arrow[r, dashed] \arrow[d, "\mathsf{pr}"] & \mathcal{V}^{\otimes} \arrow[dd] \\
\mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes} \arrow[d] & \\
\mathsf{Fin}_* \times \mathsf{Fin}_* \arrow[r, "\wedge"] & \mathsf{Fin}_*
\end{tikzcd} .$$

The natural such filler is what we are denoting by ev.

Lemma 2.0.35. *The equivalence in equation (2.12) restricts as an equiavelnce*

$$\mathit{ev} : \mathit{BiFun}^{\mathsf{m},J_\infty}(\mathit{open}(X)^\otimes, \mathit{open}(Y)^\otimes; \mathcal{V}^\otimes) \to \mathit{Fact}_X(\mathit{Fact}_Y(\mathcal{V}^\otimes)) \ .$$

Proof. Since equation (2.12) is an equivalence, it suffices to show

$$\mathsf{ev}^{-1}(\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))) \simeq \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \ .$$

So consider

$$\mathcal{G} \in \mathsf{ev}^{-1}(\mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))) \subset \mathsf{BiFun}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes) \ .$$

Since $\mathsf{ev}\mathcal{G} \in \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))$ is a factorization algebra in each factor, it folllows directly that $\mathcal{G}$ takes pairs of coCartesian morphisms to coCartesian morphisms. Also, it follows that $\mathcal{G}$ restricts as a J_∞-cosheaf in each factor. In other words, $\mathcal{G} \in \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$.

It remains to show that if $\mathcal{G} \in \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^\otimes, \mathsf{open}(Y)^\otimes; \mathcal{V}^\otimes)$, then $\mathsf{ev}\mathcal{G} \in \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^\otimes))$. Note that $\mathcal{G}$ being multiplicative is equivalent to saying that it takes pairs of morphisms of the form (id, α) and (α, id) to coCartesian morphisms, for α a coCartesian morphism. This implies that $\mathsf{ev}\mathcal{G} \in \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(X)^\otimes, \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(Y)^\otimes, \mathcal{V}^\otimes))$. The J_∞ condition on the Y variable follows immediately. Thus, we have $\mathsf{ev}\mathcal{G} \in \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(X)^\otimes, \mathsf{Fact}_Y(\mathcal{V}^\otimes))$. The J_∞ condition on the X variable is a little subtle. This is because it requires taking a colimit in the ∞-category $\mathsf{Fact}_Y(\mathcal{V}^\otimes)$. It's not clear colimits in this category look like, or if they even exist. To bypass this difficulty, note that

$$\mathsf{Fact}_Y(\mathcal{V}^\otimes) \hookrightarrow \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(Y)^\otimes, \mathcal{V}^\otimes) \tag{2.13}$$

is a full ∞-subcategory. Therefore, we can compute the colimit in $\mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})$ and then check that this colimit satisfies the J_∞ condition. Since the functor in equation (2.13) is full, this implies that the colimit in $\mathsf{Fact}_Y(\mathcal{V}^{\otimes})$ does exist, and is equivalent to the colimit as computed in $\mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})$. □

We now use the results established in this section to prove Theorem 2.0.19 stated at the beginning of this section.

Proof of Theorem 2.0.19. Proposition 2.0.34 asserts that restriction along ρ is an equivalence of ∞-categories

$$\rho^* : \mathsf{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

Lemma 2.0.35 then establishes that the evaluation functor is an equivalence of ∞-categories

$$\mathsf{ev} : \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes})) \ .$$

The composition of these two results establishes the desired equivalence

$$\mathsf{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \xrightarrow{\simeq} \mathsf{Fact}_X(\mathsf{Fact}_Y(\mathcal{V}^{\otimes})) \ .$$

□

Additivity of locally constant factorization algebras

In this section, we prove the following:

Theorem 2.0.36. *There is an equivalence of ∞-categories*

$$\mathit{Fact}^{l.c.}_{X\times Y}(\mathcal{V}) \simeq \mathit{Fact}^{l.c.}_X(\mathit{Fact}^{l.c.}_Y(\mathcal{V}^{\otimes})).$$

Proof. We organize the structure of this proof into establishing the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Fact}^{\mathsf{l.c.}}_{X\times Y}(\mathcal{V}^{\otimes}) \arrow[rr, dashed] \arrow[dd, shift left] \arrow[dr, hook] & & \mathsf{BiFun}^{\mathsf{m},\mathcal{J}_{\infty},\mathsf{l.c.}}(\mathsf{open}(X)^{\otimes},\mathsf{open}(Y)^{\otimes};\mathcal{V}^{\otimes}) \arrow[dd, shift left] \arrow[dr, hook] & \\
& \mathsf{Fact}_{X\times Y}(\mathcal{V}^{\otimes}) \arrow[rr, shift left] \arrow[dd, shift left] & & \mathsf{BiFun}^{\mathsf{m},\mathcal{J}_{\infty}}(\mathsf{open}(X)^{\otimes},\mathsf{open}(Y)^{\otimes};\mathcal{V}^{\otimes}) \arrow[ll, shift left] \arrow[dd, shift left] \\
\mathsf{Fun}^{\mathsf{m},\mathcal{J}_{\infty},\mathsf{l.c.}}(\mathsf{disk}(X\times Y)^{\otimes},\mathcal{V}^{\otimes}) \arrow[rr, dashed] \arrow[uu, shift left] \arrow[dr, hook] & & \mathsf{BiFun}^{\mathsf{m},\mathcal{J}_{\infty},\mathsf{l.c.}}(\mathsf{disk}(X)^{\otimes},\mathsf{disk}(Y)^{\otimes};\mathcal{V}^{\otimes}) \arrow[uu, shift left] \arrow[dr, hook] & \\
& \mathsf{Fun}^{\mathsf{m},\mathcal{J}_{\infty}}(\mathsf{disk}(X\times Y)^{\otimes},\mathcal{V}^{\otimes}) \arrow[rr, shift left] \arrow[uu, shift left] & & \mathsf{BiFun}^{\mathsf{m},\mathcal{J}_{\infty}}(\mathsf{disk}(X)^{\otimes},\mathsf{disk}(Y)^{\otimes};\mathcal{V}^{\otimes}) \arrow[ll, shift left] \arrow[uu, shift left]
\end{tikzcd}$$

Proving the statement amounts to showing that the top dashed arrow is an equivalence. To deduce this, we show that the bottom dashed arrow is an equivalence. Theorem 2.0.19 shows the top adjunction is an equivalence. Lemma 2.0.41 establishes that the second vertical adjunction from the left is an equivalence and Lemma 2.0.42 establishes that the leftmost viertical adjunction is an equivalence. Lemma 2.0.45 establishes that the rightmost vertical adjunction is an equivalence and Lemma 2.0.46 establishes the vertical adjunction second from the right is an equivalence. In particular, the bottom adjunction is an equivalence. Next, we show that this bottom adjunction restricts to an adjunction between the locally constant ∞-subcategories. This shows the bottom adjunction restricts, and thus is manifestly an equivalence. This statement is Corollary 2.0.50. This completes the proof. $\square$

We now turn our attention to proving the above mentioned lemmas.

Definition 2.0.37. For X a topological manifold of dimension n, there is a full ∞-sub-operad $\mathsf{disk}(X)^{\otimes} \overset{\iota}{\hookrightarrow} \mathsf{open}(X)^{\otimes}$ consisting of those $(I_+,(U_i))$ for which each open $U_i \cong \coprod \mathbb{R}^n$ is homeomorphic to a finte disjoint union of Euclidean space. Further, we let $\mathcal{J}(X) \subset \mathcal{I}(X)$ denote the full subcategory consisting of those opens that are homeomorphic to a finite disjoint union of disks.

The functor $\iota : \mathsf{disk}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ induces an adjunction of ∞-categories

$$\iota_! : \mathsf{Fun}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fun}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes}) : \iota^* \tag{2.14}$$

with left adjoint given by operadic left Kan extension. In Proposition A.0.53 we provide a colimit expression for computing the values of $\iota_!$.

Definition 2.0.38. For notational purposes, we define

$$\mathsf{Fun}^{J_\infty}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) := \iota_!^{-1}(\mathsf{Fun}^{J_\infty}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})) \ .$$

Definition 2.0.39. Let $\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \hookrightarrow \mathsf{disk}(X)^{\otimes}$ denote the full ∞-suboperad consisting of those $(I_+, (U_i))$ for which each U_i has a single connected component.

Lemma 2.0.40. *There is an equiavelnce of ∞-categories*

$$\mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \xrightarrow[\mathsf{rest}]{\simeq} \mathsf{Fun}_{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}, \mathcal{V}^{\otimes}) \ .$$

Proof. This follows the proof of Lemma 2.0.28. □

Lemma 2.0.41. *The adjunction in equation (2.14) restricts to an equivalence*

$$\iota_! : \mathsf{Fun}^{\mathsf{m},J_\infty}_{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fact}_X(\mathcal{V}^{\otimes}) : \iota^* \ .$$

Proof. Note that $\mathsf{disk}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ takes coCartesian morphisms to coCartesian morphisms, so for $\mathcal{F} \in \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathcal{V}^{\otimes})$, we see that $\iota^*\mathcal{F}$ is multiplicative. Let $\mathcal{G} \in \mathsf{Fun}^{\mathsf{m}}_{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes})$. By Lemma 2.0.40, this is equivalent to a functor of ∞-operads

$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \to \mathcal{V}^{\otimes}$, which we will also call $\mathcal{G}$. An active coCartesian morphism

$$(I_+, (U_i)) \xrightarrow{f} \left(J_+, \left(\coprod_{i \in f^{-1}(j)} U_i\right)\right)$$

in $\mathsf{open}(X)^{\otimes}$ induces a functor

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}\,/^{\mathsf{act}}_{(I_+,(U_i))}} \to \mathsf{disk}(X)^{\otimes}_{\mathsf{c}\,/^{\mathsf{act}}_{(J_+,(\coprod U_i))}}$$

given by postcomposing with f. We claim this functor is final. We verify this through Quillen's Theorem A (Theorem A.0.62). To use this theorem we must verify that for

$$\left((K_+, (V_k)), K_+ \xrightarrow{\alpha} J_+\right) \in \mathsf{disk}(X)^{\otimes}_{\mathsf{c}\,/^{\mathsf{act}}_{(J_+,(\coprod U_i))}} ,$$

the classifying space of the undercategory

$$\mathsf{disk}(X)^{\otimes \; \left((K_+,(V_k)),K_+ \xrightarrow{\alpha} J_+\right)/}_{\mathsf{c}\,/^{\mathsf{act}}_{(I_+,(U_i))}}$$

is contractible. To show contractibility, we note there is an initial object in the undercategory. The existence of an initial object comes from the fact that each V_k is connected. This implies that for $k \in \alpha^{-1}(J)$, the connected open set V_k is a subset of a unique U_i. This allows us to define a map $K_+ \xrightarrow{\beta} I_+$. The object $\left((K_+, (V_k)), K_+ \xrightarrow{\beta} I_+\right)$ is intial in the undercategory. Thus,

$$\begin{aligned} \iota_! \mathcal{G}(f_!(I_+, (U_i))) &\simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}\,/^{\mathsf{act}}_{(J_+,(\coprod U_i))}} \to \mathcal{V}^{\otimes}_{|J_+}\right) \\ &\simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}\,/^{\mathsf{act}}_{(I_+,(U_i))}} \to \mathcal{V}^{\otimes}_{|I_+} \xrightarrow{f_!} \mathcal{V}^{\otimes}_{|J_+}\right) . \end{aligned}$$

Since the canonical projection maps $\mathsf{pr}_j : J_+ \to \{j\}_+$ preserve colimits, we will compute the colimit separately in each factor of $\mathcal{V}^{\otimes}_{|_{J_+}} \simeq \mathcal{V}^{\times J}$. So without loss of generality, let us consider the case of an active coCartesian morphism $(I_+, (U_i)) \to (1_+, \coprod U_i)$. This general case of an active coCartesian morphism will follow completely analogously to the case $(2_+, (U_1, U_2)) \xrightarrow{f} (1_+, U_1 \amalg U_2)$, and this latter case will ease the notational burden tremendously. Note the canonical functor

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(2_+,(U_1,U_2))}} \xrightarrow{\simeq} \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \times \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}}$$

is an equivalence. Further, the functor

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \times \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}} \xrightarrow{\mathsf{pr}_1} \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}}$$

given by the projection onto the first factor is a coCartesian fibration. Proposition A.0.51 says that left Kan extension along a coCartesian fibration evaluates as a fiberwise colimit. Therefore,

$$\begin{aligned}
&\iota_!\mathcal{G}(f_!(2_+,(U_1,U_2))) \\
&\qquad \simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(2_+,(U_1,U_2))}} \xrightarrow{\mathcal{G}} \mathcal{V}^{\otimes}_{|_{2_+}} \xrightarrow{\otimes} \mathcal{V}^{\otimes}_{|_{1_+}} \simeq \mathcal{V}\right) \\
&\qquad \simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \times \mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}} \xrightarrow{\mathcal{G}} \mathcal{V}^{\otimes}_{|_{2_+}} \xrightarrow{\otimes} \mathcal{V}\right) \\
&\qquad \simeq \operatorname{colim}\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}} \xrightarrow{\mathsf{pr}_!\mathcal{G}} \mathcal{V}\right) \\
&\qquad \simeq \operatorname*{colim}_{(K_+,(D_k))\in\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}}} \left(\operatorname{colim}\left(\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}}\right)_{|_{(K_+,(D_k))}} \xrightarrow{\mathcal{G}} \mathcal{V}\right)\right) \\
&\qquad \simeq \operatorname*{colim}_{(K_+,(D_k))\in\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{1\}_+,U_1)}}} \; \operatorname*{colim}_{(L_+,(E_\ell))\in\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(\{2\}_+,U_2)}}} \mathcal{G}((K_+,(D_k))) \otimes \mathcal{G}((L_+,(E_\ell))) \\
&\qquad \simeq \iota_!\mathcal{G}(U_1) \otimes \iota_!\mathcal{G}(U_2)\ ,
\end{aligned}$$

as desired. Note that the last equivalence invokes the assumption that $\mathcal{V}^{\otimes}$ is $\otimes$-presentable. In particular, the $\otimes$-presentability of $\mathcal{V}^{\otimes}$ means that for any $V \in \mathcal{V}$, the functor $V \otimes - : \mathcal{V} \to \mathcal{V}$ preserves colimits.

By definition, $\mathcal{G} : \mathsf{disk}(X)^{\otimes} \to \mathcal{V}^{\otimes}$ is J_∞ if $\iota_!\mathcal{G}$ is J_∞. Thus, to show the adjunction restricts, it remains to show that if $\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$ then $\iota^*\mathcal{F} \in \mathsf{Fun}^{\mathsf{m},J_\infty}_{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes})$. That is, we must show that $\iota_!\iota^*\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$. Note that $\mathsf{disk}(X)$ is a basis for the J_∞ Grothendieck topology on $\mathsf{open}(X)$, as discussed in Example A.0.59. Therefore, we can check that $\iota_!\iota^*\mathcal{F}$ satisfies the J_∞-cosheaf condition by checking that it is a J_∞-cosheaf with respect to covers in $\mathsf{disk}(X)$. Observing that for $U \in \mathsf{disk}(X)$, $\iota_!\iota^*\mathcal{F}(U) \simeq \mathcal{F}(U)$, this follows from the fact that $\mathcal{F}$ is a J_∞-cosheaf.

To verify this adjunction is an equivalence, we check that the unit and counit evaluate as equivalences. Take $\mathcal{F} \in \mathsf{Fact}_X(\mathcal{V}^{\otimes})$ and consider the counit $\iota_!\iota^*\mathcal{F} \to \mathcal{F}$. Since both functors are J_∞ cosheaves, it suffices to check the counit is an equivalence evaluated on unary elements. Further, since $\mathsf{disk}(X)$ is a J_∞ basis for $\mathsf{open}(X)$, as just elucidated above, it suffices to check this equivalence on elements of $\mathsf{disk}(X)$ which follows immediately. A similar argument shows that the unit evaluates as equivalences. □

Lemma 2.0.42. *The adjunction in Lemma 2.0.41 restricts to an adjunction between the respective locally constant subcategories*

$$\iota_! : \mathit{Fun}^{m,J_\infty,l.c.}_{opd}(\mathit{disk}(X)^{\otimes}, \mathcal{V}^{\otimes}) \rightleftarrows \mathit{Fact}^{l.c.}_X(\mathcal{V}^{\otimes}) : \iota^* \ . \quad .$$

In particular, this adjunction is an equivalence.

Proof. First, for $\mathcal{G} \in \mathsf{Fun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}_{\mathsf{opd}}(\mathsf{disk}(X)^{\otimes}, \mathcal{V}^{\otimes})$, we show $\iota_!\mathcal{G} \in \mathsf{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^{\otimes})$. That is, for $U \overset{\varphi}{\hookrightarrow} V$ in $\mathcal{I}(X)$, we show the canonical morphism induced by φ

$$\iota_!\mathcal{G}(U) \to \iota_!\mathcal{G}(V)$$

is an equivalence. Observe the commutative diagram The colimit of the top horizontal line

$$\begin{tikzcd}
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}} \arrow[r] \arrow[d, "\mathrm{loc}"] & \mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}} \arrow[r, "\mathrm{fgt}"] \arrow[d, "\mathrm{loc}"] & \mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "\mathcal{G}"] & \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "\otimes"] & \mathcal{V} \\
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}}[(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}})^{-1}] \arrow[r] & \mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}}[(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}})^{-1}] \arrow[urrr, dashed, bend right] & & &
\end{tikzcd}$$

is the definition of $\iota_!\mathcal{G}(U)$ and the existence of the dashed arrow follows from $\mathcal{G}$ being locally constant. There are canonical identifications $\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}} \simeq \mathcal{J}(U)$ and $\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,U)}} \simeq \mathsf{disk}(U)$, where $\mathsf{disk}(U)$ is the poset. By Proposition A.0.14 localizations are final, so

$$\iota_!\mathcal{G}(U) \simeq \operatorname{colim}\left(\mathsf{disk}(U)[\mathcal{J}(U)^{-1}] \to \mathsf{disk}(X)[\mathcal{J}(X)^{-1}] \to \mathcal{V}\right).$$

By a similar analysis, we see

$$\iota_!\mathcal{G}(V) \simeq \operatorname{colim}\left(\mathsf{disk}(V)[\mathcal{J}(V)^{-1}] \to \mathsf{disk}(X)[\mathcal{J}(X)^{-1}] \to \mathcal{V}\right).$$

By Proposition 2.19 in [3], we see $\mathsf{disk}(U)[\mathcal{J}(U)^{-1}] \simeq \mathsf{Disk}(U)$, and likewise for V. Here, by $\mathsf{Disk}(X)$ we mean the topological category of embedded disks in X. (We choose to not dwell on $\mathsf{Disk}(X)$ because we make no further use of it beyond this paragraph. We refer the interested reader to [3] for further details.) Finally, we claim that the isotopy equivalence φ induces an equivalence $\mathsf{Disk}(U) \xrightarrow{\simeq} \mathsf{Disk}(V)$. To see this, choose an isotopy inverse to φ, say ψ. Then φ and ψ induce an adjunction that is checked to be an equivalence.

It remains to show that $\iota^*\mathcal{F}$ is locally constant, for $\mathcal{F} \in \mathsf{Fact}^{\mathsf{l.c.}}_X(\mathcal{V}^{\otimes})$. This follows from the following diagram

$$\begin{tikzcd}
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}} \arrow[r, "\iota"] \arrow[dd, "\mathrm{loc}"] & \mathsf{open}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}} \arrow[r] \arrow[d, "(-)_!"] & \mathsf{open}(X)^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "\mathcal{F}"] & \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{1_+}} \arrow[r, "(-)_!"] & \mathcal{V} \\
& \mathsf{open}(X) \arrow[d, "\mathrm{loc}"] \arrow[urrr, bend right=10, "\mathcal{F}"] & & & \\
\mathsf{disk}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}}[(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}})^{-1}] \arrow[r, dashed] & \mathsf{open}(X)[\mathcal{I}(X)^{-1}] \arrow[uurrr, bend right=20] & & &
\end{tikzcd}$$

The bottom right most arrow is the condition of $\mathcal{F}$ being multiplicative, and the existence of the dashed arrow is given by the fact that $((-)_! \circ \iota)(\mathcal{J}(X)^{\otimes}_{/^{\mathrm{act}}_{(1_+,X)}}) \subset \mathcal{I}(X)$. $\square$

Lemma 2.0.43. *The functor of ∞-operads $\mathsf{disk}(X)^{\otimes} \hookrightarrow \mathsf{open}(X)^{\otimes}$ induces an adjunction*

$$\iota_! : \mathsf{BiFun}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes};, \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) : \iota^* \tag{2.15}$$

in which the left adjoint evaluates on $((I_+, (U_i)), (J_+, (V_j))) \in \mathsf{open}(X)^{\otimes} \times \mathsf{open}(Y)^{\otimes}$ as the following colimit

$$\operatorname{colim}\left(\mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/((I_+,(U_i)),(J_+,(V_j)))} \to \mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/I_+ \wedge J_+} \xrightarrow{\mathcal{F}} \mathcal{V}^{\otimes}_{/I_+ \wedge J_+} \xrightarrow{(-)_!} \mathcal{V}^{\otimes}_{|I_+ \wedge J_+}\right) .$$

Proof. Proposition A.0.52 establishes the formula for the left adjoint, so it remains to verify that $\iota_! \mathcal{F}$ is a bifunctor. To see this, take an inert coCartesian morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (U_j))$. There is a commutative square

$$\begin{tikzcd}
\mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/((I_+,(U_i)),(K_+,(V_k)))} \arrow[r] \arrow[d] & \mathcal{V}^{\otimes}_{|I_+ \wedge K_+} \arrow[d] \\
\mathsf{disk}(X)^{\otimes} \times \mathsf{disk}(Y)^{\otimes}_{/((J_+,(U_j)),(K_+,(V_k)))} \arrow[r] & \mathcal{V}^{\otimes}_{|J_+ \wedge K_+}
\end{tikzcd} .$$

Note the left vertical functor is final, as verified using Quillen's Theorem A (Theorem A.0.62). One can use the fact that $I_+ \xrightarrow{f} J_+$ is inert to show the relevant undercategory in the statement of Theorem A.0.62 has an initial object. The result then follows from the fact that the projection $\mathcal{V}^{\otimes}_{|_{I_+\wedge K_+}} \to \mathcal{V}^{\otimes}_{|_{J_+\wedge K_+}}$ preserves colimits. ☐

To prove Lemma 2.0.45, we need an analogue of Lemma 2.0.40 for bifunctors:

Lemma 2.0.44. *There is an equiavelnce of ∞-categories*

$$\mathsf{BiFun}^{m}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \xrightarrow[\mathsf{rest}]{\simeq} \mathsf{BiFun}(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}, \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}; \mathcal{V}^{\otimes}) \ .$$

Proof. This directly follows the proof of Lemma 2.0.28. ☐

Lemma 2.0.45. *The adjunction in equation (2.15) restricts to an equivalence*

$$\iota_! : \mathsf{BiFun}^{m,J_\infty}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{BiFun}^{m,J_\infty}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}) : \iota^* \ .$$

Proof. This is proved in the same manner as Lemma 2.0.41. ☐

Lemma 2.0.46. *The adjunction in Lemma 2.0.45 restricts to an adjunction between the respective locally constant subcategories*

$$\iota_! : \mathsf{BiFun}^{m,J_\infty,l.c.}(\mathsf{disk}(X)^{\otimes}, \mathsf{disk}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \rightleftarrows \mathsf{BiFun}^{m,J_\infty,l.c.}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}) : \iota^* \ .$$

In particular, this adjunction is an equivalence.

Proof. The proof of this follows analogous to Lemma 2.0.42. ☐

To prove Proposition 2.0.49 below, we employ a result from [23] that enables us to identify localizations of ∞-categories via complete Segal spaces. This result is recorded as Theorem A.0.28. For the reader unfamiliar with complete Segal spaces, we devote a section in the appendix to the basic definitions and ideas that we use.

Definition 2.0.47. We let $\mathcal{J}(X)^{\otimes}_{\mathsf{c}}$ denote the full ∞-subcategory of $\mathcal{J}(X)^{\otimes}$ over Fin_* consisting of those $(I_+, (U_i))$ for which U_i is connected.

Lemma 2.0.48. *Let* $(I_+, (U_i)) \in \textsf{\textit{disk}}(X \times Y)^{\otimes}_{\textsf{\textit{c}}}$. *The classifying space*

$$\textsf{\textit{BFun}}^{\mathcal{J}(X)^{\otimes}_{c} \times \mathcal{J}(Y)^{\otimes}_{c}{}_{/(I_+,(U_i))}} \left([\bullet], \textsf{\textit{disk}}(X)^{\otimes}_{\textsf{\textit{c}}} \times \textsf{\textit{disk}}(Y)^{\otimes}_{\textsf{\textit{c}}}{}_{/(I_+,(U_i))}\right)$$

is a complete Segal space.

Proof. Let us adopt the notational conventions

$$\mathcal{C} := \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\ , \qquad \mathcal{W} := \mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\ . \tag{2.16}$$

First, we establish the Segal condition. That is, for all $p \geq 0$, we show the diagram of classifying spaces

$$\begin{tikzcd}
\mathsf{BFun}^{\mathcal{W}}([p], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{BFun}^{\mathcal{W}}(\{0 < 1\}, \mathcal{C}) \arrow[d] \\
\mathsf{BFun}^{\mathcal{W}}(\{1 < \cdots < p\}, \mathcal{C}) \arrow[r] & \mathsf{BFun}^{\mathcal{W}}(\{1\}, \mathcal{C})
\end{tikzcd} \tag{2.17}$$

is a pullback. To show the diagram in equation (2.17) is a pullback, we make use of Proposition A.0.10 and show an equivalence of vertical fibers. Note that the diagram prior to taking classifying spaces

$$\begin{tikzcd}
\mathsf{Fun}^{\mathcal{W}}([p], \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}^{\mathcal{W}}(\{0 < 1\}, \mathcal{C}) \arrow[d] \\
\mathsf{Fun}^{\mathcal{W}}(\{1 < \cdots < p\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}^{\mathcal{W}}(\{1\}, \mathcal{C})
\end{tikzcd} \tag{2.18}$$

is a pullback. Thus by Proposition A.0.9, the vertical fibers of the diagram in equation (2.18) are equivalent, hence their classifying spaces are equivalent. Therefore, we must show that the classifying space of the vertical fibers in the diagram in equation (2.18) are the vertical fibers of the diagram in equation (2.17). Quillen's Theorem B (Theorem A.0.63) provides a method to verify this. By Lemma A.0.64, we only need to show this for the rightmost vertical functor, so we restrict our attention there. All told, proving the Segal condition reduces to verifying the hypothesis of Quillen's Theorem B for the functor

$$\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C}) \to \mathsf{Fun}^{\mathcal{W}}(\{1\},\mathcal{C}) \ .$$

By Observation A.0.29, $\mathsf{Fun}^{\mathcal{W}}(\{1\},\mathcal{C}) \simeq \mathcal{W}$, so to invoke Quillen's Theorem B, we need to show that for each $c \xrightarrow{f} c'$ in $\mathcal{W}$, the induced functor

$$\mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{/c} \xrightarrow{\mathsf{B}(f\circ -)} \mathsf{BFun}^{\mathsf{W}}(\{0<1\},\mathcal{C})_{/c'} \tag{2.19}$$

is an equivalence. The vertical functors in the diagram in equation (2.18) are coCartesian fibrations, so by Proposition A.0.36 the canonical functor

$$\mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|c} \hookrightarrow \mathsf{Fun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{/c}$$

is a right adjoint, with left adjoint given by the coCartesian monodromy functor. Since, an adjunction induces an equivalence between classifying spaces by Proposition A.0.25, showing that equation (2.19) is an equivalence is equivalent to showing that

$$\mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})|_{c} \xrightarrow{\mathsf{B}(f\circ -)} \mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})|_{c'} \tag{2.20}$$

is an equivalence. Observe the equivalence

$$\mathsf{BFun}^{\mathcal{W}}(\{0<1\},\mathcal{C})_{|c} \simeq \mathsf{B}(\mathcal{C}_{/c})^{\mathcal{W}}.$$

This implies that showing equation (2.20) is an equivalence is equivalent to showing that the functor

$$\mathsf{B}(\mathcal{C}_{/c})^{\mathcal{W}} \to \mathsf{B}(\mathcal{C}_{/c'})^{\mathcal{W}} \tag{2.21}$$

is an equivalence. Observe the equivalence

$$\begin{aligned}(\mathcal{C}_{/c})^{\mathcal{W}} &:= \left(\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{/_{(J_+,(V_j)),(K_+,(W_k)),J_+\wedge K_+\xrightarrow{f} I_+}}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}} \\ &\simeq \left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}} \times \left(\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(K_+,(W_k))}\right)^{\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(K_+,(W_k))}} .\end{aligned}$$

We now restrict attention to one factor of this product. Namely, consider $(J_+,(V_j)) \to (J'_+,(V'_j))$ in $\mathcal{J}(X)^{\otimes}_{\mathsf{c}}$. Note that

$$\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(V_j))}}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}} \hookrightarrow \left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}}$$

is a right adjoint with left adjoint given by using the inert-active factorization system of Fin_* (see Observation A.0.32), and thus induces an equivalence of classifying spaces. Furthermore, note that

$$\left(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}{}_{/^{\mathsf{act}}_{(J_+,(V_j))}}\right)^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}{}_{/(J_+,(V_j))}} \simeq \prod_{j\in J} \mathcal{J}(V_j) .$$

Now, we use Theorem A.0.65 to identify $\mathsf{B}\prod_{j\in J}\mathcal{J}(V_j)$ with $\prod_{j\in J}\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}$. Consider the functor

$$\mathcal{U} := \prod_{j\in J}\mathcal{J}(V_j) \to \prod_{j\in J}\mathsf{open}\left(\coprod_{r\geq 0}\mathsf{Conf}_r(V_j)_{\Sigma_r}\right)$$

given by

$$(D_j \subset V_j)_{j\in J} \mapsto \Big(\{S \subset V_j \mid \mathsf{card}(S) < \infty\ ,\ S \subset D_j\ ,\ \pi_0 S \xrightarrow{\cong} \pi_0 D_j\}\Big)_{j\in J} .$$

Note that

$$\{S \subset V_j \mid \mathsf{card}(S) < 0 \ , \ S \subset D_j \ , \ \pi_0 S \xrightarrow{\cong} \pi_0 D_j\} \cong \coprod_{\alpha \in \pi_0 D_j} D_j^\alpha \ ,$$

which is contractible as a subspace of configuration space. Let $(S_j \subset V_j)_{j \in J} \in \prod_{j \in J} \coprod_{r \geq 0} \mathsf{Conf}_r(V_j)_{\Sigma_r}$. Then

$$\mathcal{U}_{(S_j \subset V_j)_{j \in J}} = \prod_{j \in J} \{D_j \in \mathcal{J}(V_j) \mid S_j \subset D_j \ , \ \pi_0 S_j \xrightarrow{\cong} \pi_0 D_j\} \ .$$

This poset is cofiltered (Definition A.0.3) since it is the product of cofiltered posets. Note that each factor is cofiltered by Example A.0.8. Namely, because open disks form a basis for the topology of a manifold. Therefore, we have verified the hypotheses of Theorem A.0.65 with respect to $\mathcal{U}$, so Theorem A.0.65 establishes an equivalence

$$\mathsf{B} \prod_{j \in J} \mathcal{J}(V_j) \simeq \prod_{j \in J} \coprod_{r \geq 0} \mathsf{Conf}_r(V_j)_{\Sigma_r} \ .$$

Earlier, we reduced the Segal condition to showing for each isotopy equivalence $c \to c'$ in $\mathcal{W}$, the induced map

$$\mathsf{B}(\mathcal{C}_{/c})^{\mathcal{W}} \to \mathsf{B}(\mathcal{C}_{/c'})^{\mathcal{W}} \tag{2.22}$$

is an equivalence. We have just shown that

$$\mathsf{B}(\mathcal{C}_{/c})^{\mathcal{W}} \simeq \prod_{j \in J} \coprod_{r \geq 0} \mathsf{Conf}_r(V_j)_{\Sigma_r} \times \prod_{j' \in J'} \coprod_{r \geq 0} \mathsf{Conf}_r(V_{j'})_{\Sigma_r}$$

and likewise for $\mathsf{B}(\mathcal{C}_{/c'})^{\mathcal{W}}$. Theorem A.0.66 states that the space of self-homeomorphisms of $\mathbb{R}^n$ is homotopy equivalent to the space of self-embeddings of $\mathbb{R}^n$. Note that we just functorially identified the domain and codomain of the functor in equation (2.22) with configuration spaces, which are homeomorphism invariant. Thus, the functor in equation

(2.22) is a weak homotopy equivalence.

We now prove that the Segal space

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}/(I_+,(U_i))}\Big([\bullet],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big) \tag{2.23}$$

is complete. That is, we will show that the map from the space of [0]-points

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}/(I_+,(U_i))}\Big([0],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)$$

into the space of [1]-points that are equivalences

$$\left(\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}/(I_+,(U_i))}\Big([\bullet],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)\right)^{\mathsf{equiv}}$$

is an equivalence of spaces. A key observation to proving this is that a morphism in

$$\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}$$

is in the isotopy equivalences

$$\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}$$

if and only if the underlying maps of finite sets are both bijections. Now, consider a [1]-point of the Segal space in equation (2.23) that is an equivalence. By definition, this is a point in the space

$$\mathsf{BFun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}/(I_+,(U_i))}\Big([1],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)\ .$$

Such a point is represented by an object in

$$\mathsf{Fun}^{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}/(I_+,(U_i))}\Big([0],\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\Big)\ ,$$

i.e. a functor

$$[1]\to {\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}}_{/(I_+,(U_i))}\ .$$

Let's say this functor selects out the morphism

$$\big((K_+,(V_k)),(L_+,(W_\ell)),K_+\wedge L_+\xrightarrow{f} I_+\big)\xrightarrow{\varphi,\psi}\big((K'_+,(V_{k'})),(L'_+,(W_{\ell'})),K'_+\wedge L'_+\xrightarrow{f'} I_+\big)\ . \tag{2.24}$$

Since there is a natural morphism of Segal spaces

$$\mathsf{BFun}^{{\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}}_{/(I_+,(U_i))}}\left([\bullet],{\mathsf{disk}(X)^{\otimes}_{\mathsf{c}}\times\mathsf{disk}(Y)^{\otimes}_{\mathsf{c}}}_{/(I_+,(U_i))}\right)\to\mathsf{Fin}_*\times\mathsf{Fin}_*\ ,$$

the equivalence in equation (2.24) gets carried to an equivalence in $\mathsf{Fin}_*\times\mathsf{Fin}_*$. This implies that φ and ψ are bijections, so by the key observation above, the equivalence in equation (2.24) lies in

$${\mathcal{J}(X)^{\otimes}_{\mathsf{c}}\times\mathcal{J}(Y)^{\otimes}_{\mathsf{c}}}_{/(I_+,(U_i))}\ .$$

Now, using the notation given in equation (2.16), observe the solid commutative diagram

$$\begin{tikzcd}
 & & {[1]} \arrow[dll, dashed, bend right] \arrow[dl, dashed] \arrow[ddl, bend left] \\
\mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] & \\
\mathsf{BFun}^{\mathcal{W}}([0],\mathcal{C}) \arrow[r] & \mathsf{BFun}^{\mathcal{W}}([\bullet],\mathcal{C}) &
\end{tikzcd}$$

where the solid arrow $[1]\to\mathsf{BFun}^{\mathcal{W}}([\bullet],\mathcal{C})$ is the assumed equivalence. We showed that this is represented by a dashed arrow $[1]\to\mathcal{C}$, and further that this dashed arrow actually factors through $\mathcal{W}$, hence the other dashed arrow $[1]\to\mathcal{W}$ in the diagram. This shows that the space of $[0]$-points is a deformation retract of the $[1]$-points that are equivalences, and is thus a homotopy equivalence as desired. □

Proposition 2.0.49. *An isotopy equivalence* $(I_+, (U_i)) \hookrightarrow (J_+, (V_j))$ *in* $\mathcal{J}(X \times Y)^{\otimes}_{\mathsf{c}}$*, induces an equivalence of* ∞*-categories*

$$(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(I_+,(U_i))}]$$
$$\xrightarrow{\simeq} (\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(J_+,(V_j))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(J_+,(V_j))}] \ .$$

Proof. We used Theorem A.0.28 to identify the localizations as complete Segal spaces in Lemma 2.0.48 above. Thus, to complete the proof, we establish an equivalence between [0]-points and [1]-points, which by Observation A.0.21, implies an equivalence of complete Segal spaces. First, we establish the equivalence of [0]-points. To do this, we first identify the [0]-points, that is the classifying space $\mathsf{B}(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}$, with the following space

$$\coprod_{[K_+,L_+,K_+\wedge L_+ \xrightarrow{f} I_+]} \left(\prod_{i \in I} \mathsf{Conf}_{f^{-1}(i) \subset K \times L}(U_i \subset X \times Y) \right)_{/\mathsf{Aut}(K_+,L_+,K_+\wedge L_+ \xrightarrow{f} I_+)} , \qquad (2.25)$$

where the coproduct is indexed by equivalence classes, and $\mathsf{Conf}_{f^{-1}(i) \subset K \times L}(U_i \subset X \times Y)$ is defined as the following pullback

$$\begin{tikzcd}
\mathsf{Conf}_{f^{-1}(i) \subset K \times L}(U_i \subset X \times Y) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & X^K \times Y^L \arrow[d] \\
 & (X \times Y)^{I \times J} \arrow[d] \\
\mathsf{Conf}_{f^{-1}(i)}(U_i) \arrow[r, hook] & (X \times Y)^{f^{-1}(i)}
\end{tikzcd} \ . \qquad (2.26)$$

Note that $\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i \subset X \times Y)$ carries a natural action by the pullback

$$\begin{tikzcd}
\mathsf{Aut}(f^{-1}(i) \subset K \times L) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathsf{Aut}(K) \times \mathsf{Aut}(L) \arrow[d] \\
 & \mathsf{Aut}(K \times L) \arrow[d] \\
\mathsf{Aut}(f^{-1}(i)) \arrow[r, hookrightarrow] & \mathsf{Inj}(S, K \times L)
\end{tikzcd} \quad . \tag{2.27}$$

Using Theorem A.0.63, we have the following pullback

$$\begin{tikzcd}
\mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|_{K_+\wedge L_+ \xrightarrow{f} I_+}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{B}\left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right) \arrow[d] \\
* \arrow[r, "\langle K_+\wedge L_+ \xrightarrow{f} I_+\rangle"] & \mathsf{B}\left(\mathsf{Obj}(\mathsf{Fin}_*) \times \mathsf{Obj}(\mathsf{Fin}_*)_{/I_+}\right)
\end{tikzcd} ,$$

with the bottom righthand corner equivalent to $\mathsf{BAut}(K_+, L_+, K_+ \wedge L_+ \xrightarrow{f} I_+)$. Also, the following diagram is a pullback

$$\begin{tikzcd}
\prod_{i\in I} \mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i \subset X \times Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_{[K_+,L_+,K_+\wedge L_+ \xrightarrow{f} I_+]} \left(\prod_{i\in I} \mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i \subset X \times Y)\right)_{/\mathsf{Aut}(K_+,L_+,K_+\wedge L_+ \xrightarrow{f} I_+)} \arrow[d] \\
* \arrow[r, "\langle K_+\wedge L_+ \xrightarrow{f} I_+\rangle"] & \mathsf{BAut}(K_+, L_+, K_+ \wedge L_+ \xrightarrow{f} I_+)
\end{tikzcd}$$

To identify $\mathsf{B}(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}$ with the space in equation (2.25), by Proposition A.0.10, it suffices to show an equivalence of the above fibers over $\mathsf{BAut}(K_+, L_+, K_+ \wedge L_+ \xrightarrow{f} I_+)$. We use Theorem A.0.65 to prove the equivalence of the fibers in the prior two diagrams. First, observe that

$$\mathcal{U} := \left(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}}{}_{/(I_+,(U_i))}\right)_{|_{K_+\wedge L_+ \xrightarrow{f} I_+}}$$

is a poset. The bottom horizontal morphism in the diagram in equation (2.27) is an open embedding, so by Proposition A.0.11 the top horizontal arrow is as well. Using this, we define a $\mathsf{Aut}(K_+, L_+, K_+ \wedge L_+ \xrightarrow{f} I_+)$ equivariant functor

$$\mathcal{U} \to \prod_{i\in I} \mathsf{open}\left(\mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i \subset X \times Y)\right) \subset \mathsf{open}\left(\prod_{i\in I} \mathsf{Conf}_{f^{-1}(i)\subset K\times L}(U_i \subset X\times Y)\right) \tag{2.28}$$

given by

$$(K_+,(V_k)),(L_+,(W_\ell)) \mapsto \{(v_k)_K \mid v_k \in V_k\} \times \{(w_\ell)_L \mid w_\ell \in W_L\} \ .$$

This is well-defined since it takes values in $\mathsf{open}(X^K \times Y^L)$ that evidently restrict to opens in $\mathsf{Conf}_{f^{-1}(i)}(U_i)$. To invoke Theorem A.0.65 we must verify that for $c \in \mathsf{Conf}_{f^{-1}(i)}(U_i \subset X \times Y)$, the classifying space $\mathsf{B}\mathcal{U}_c \simeq *$ is contractible. This is true since the category is cofiltered, which can be seen using the fact that disks form a basis for opens (Example A.0.8). Finally, observe that each value of the functor in equation (2.27) is a contractible subspace of configuration space, since it is a product of disks. Thus, by Theorem A.0.65, we identify the [0]-points of each space in the statement of this proposition in a functorial manner. Similar to the proof of Lemma 2.0.48, since the functor is induced by an isotopy equivalence, Theorem A.0.66 gives us a weak homotopy equivalence of [0]-points. It remains to verify an equivalence on [1]-points. Adopting the notation of Lemma 2.0.48, observe the following diagram

$$\begin{tikzcd}
 & & \mathsf{B}(\mathcal{C}'_{/c'})^{\mathcal{W}} \arrow[r] \arrow[d] & \mathsf{BFun}^{\mathcal{W}}([1],\mathcal{C}') \arrow[d, "\mathsf{ev}_1"] \\
\mathsf{B}(\mathcal{C}_{/c})^{\mathcal{W}} \arrow[r] \arrow[d] \arrow[urr, "\simeq"] & \mathsf{B}(\mathsf{Fun}^{\mathcal{W}}([1],\mathcal{C}) \arrow[d, "\mathsf{ev}_1"'] \arrow[urr, dashed] & * \arrow[r, "\langle c' \rangle"] & \mathsf{B}(\mathsf{Fun}^{\mathcal{W}}([0],\mathcal{C}') \\
* \arrow[r, "\langle c \rangle"'] \arrow[urr, "\simeq"'] & \mathsf{BFun}^{\mathcal{W}}([0],\mathcal{C}) \arrow[urr, "\simeq"'] & &
\end{tikzcd} .$$

The top left diagonal equivalence was establish in Lemma 2.0.48, and we just established the bottom right diagonal equivalence. Thus, the dashed arrow between [1]-points is manifestly an equivalence. $\square$

Corollary 2.0.50. *The adjunction*

$$\rho^* : \mathsf{Fun}_{\mathsf{opd}}^{\mathsf{m},J_\infty}(\mathsf{disk}(X \times Y)^\otimes, \mathcal{V}^\otimes) \rightleftarrows \mathsf{BiFun}^{\mathsf{m},J_\infty}(\mathsf{disk}(X)^\otimes, \mathsf{disk}(Y)^\otimes; \mathcal{V}^\otimes) : \rho_!$$

is an equivalence. Further, the adjunction restricts to an adjunction between locally constant subcategories

$$\rho^* : \mathsf{Fun}_{\mathsf{opd}}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X \times Y)^\otimes, \mathcal{V}^\otimes) \rightleftarrows \mathsf{BiFun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X)^\otimes, \mathsf{disk}(Y)^\otimes; \mathcal{V}^\otimes) : \rho_!.$$

Manifestly, the restricted adjunction is an equivalence.

Proof. The first statement is immediate from Theorem 2.0.19, Lemma 2.0.41, and Lemma 2.0.45. For the second statement, consider $\mathcal{F} \in \mathsf{BiFun}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X)^\otimes, \mathsf{disk}(Y)^\otimes; \mathcal{V}^\otimes)$. We wish to show that $\rho_!\mathcal{F} \in \mathsf{Fun}_{\mathsf{opd}}^{\mathsf{m},J_\infty,\mathsf{l.c.}}(\mathsf{disk}(X \times Y)^\otimes, \mathcal{V}^\otimes)$. To prove this, we restrict to the ∞-sub-operads of connected disks. That is, given an isotopy equivalence, $(I_+, (U_i)) \overset{\varphi}{\hookrightarrow} (J_+, (V_j))$ in $\mathsf{disk}(X \times Y)_{\mathsf{c}}^\otimes$, we show the morphism

$$\rho_!\mathcal{F}((I_+, (U_i))) \to \rho_!\mathcal{F}((J_+, (V_j)))$$

in $\mathcal{V}^\otimes$ is an equivalence. Recall that $\rho_!\mathcal{F}((I_+, (U_i)))$ is computed as the following colimit

$$\operatorname{colim}\left((\mathsf{disk}(X)_{\mathsf{c}}^\otimes \times \mathsf{disk}(Y)_{\mathsf{c}}^\otimes)_{/(I_+,(U_i))} \xrightarrow{\text{fgt}} (\mathsf{disk}(X)_{\mathsf{c}}^\otimes \times \mathsf{disk}(Y)_{\mathsf{c}}^\otimes)_{/I_+} \xrightarrow{\mathcal{F}} \mathcal{V}^\otimes_{/I_+} \xrightarrow{(-)_!} \mathcal{V}^\otimes_{|I_+}\right).$$

Note that the following diagram commutes

$$\begin{tikzcd}
(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))} \arrow[r, "\text{fgt}"] \arrow[d, "\text{loc}"] & \mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}} \arrow[r, "\mathcal{F}"] \arrow[d, "\text{loc}"] & \mathcal{V}^{\otimes} \\
(\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(I_+,(U_i))}] \arrow[r, "\text{fgt}"] & (\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}] \arrow[ur, dashed] &
\end{tikzcd}$$

The existence of the dashed arrow is given by the assumption that $\mathcal{F}$ is locally constant. By Proposition A.0.14 localizations are final, so

$$\rho_!\mathcal{F}((I_+,(U_i))) \simeq \operatorname{colim}\left((\mathsf{disk}(X)^{\otimes}_{\mathsf{c}} \times \mathsf{disk}(Y)^{\otimes}_{\mathsf{c}})_{/(I_+,(U_i))}[(\mathcal{J}(X)^{\otimes}_{\mathsf{c}} \times \mathcal{J}(Y)^{\otimes}_{\mathsf{c}})^{-1}_{/(I_+,(U_i))}] \to \mathcal{V}^{\otimes}|_{I_+}\right),$$

and likewise for $\rho_!\mathcal{F}((J_+,(V_j)))$. Thus, by Proposition 2.0.49, we have $\rho_!\mathcal{F}((I_+,(U_i))) \to \rho_!\mathcal{F}((J_+,(V_j)))$ is an equivalence. $\square$

THE COHOMOLOGY OF REAL GRASSMANNIANS

In this chapter we use the theory of stratified spaces, as recently developed in [5], to compute the additive R-cohomology of the real Grassmannian manifolds.

Notation

We will now fix some notational conventions that we will use throughout this chapter:

- For S a finite set, we let $\mathsf{card}(S)$ denote the cardinality of S.
- For $r \in \mathbb{Z}_{>0}$, we let $\underline{r}$ denote the set $\{1, \ldots, r\}$.
- For S a set, let $\mathsf{Sub}(S)$ denote the poset of subsets of S.
- We let $\left\{ {n \atop k} \right\}$ denote the set of all cardinality k subsets of the set $\{1, \ldots, n\}$.
- Let $\mathsf{Mat}_{n\times k}$ denote the collection of n-by-k matrices with real coefficients.
- Let $\mathbb{1}_{n\times n} \in \mathsf{Mat}_{n\times n}$ denote the n-by-n identity matrix.
- For each $1 \leq i \leq n$, $\mathsf{e}_i \in \mathbb{R}^n$ is the i-th standard basis vector of $\mathbb{R}^n$. That is, it is the vector consisting of a 1 in the i-th entry, and 0's in all other entries.
- For $S \in \left\{ {n \atop k} \right\}$, we define $\mathbb{R}^S$ to be the span

$$\mathbb{R}^S := \mathsf{span}\{\mathsf{e}_i \mid i \in S\} .$$

- For a X a topological space, we let X^+ denote the set, $X \amalg \{\infty\}$, where we attach a single disjoint point to X. This set is equipped with the following topology: $U \subset X^+$ is open iff U is an open subset of X, or $U = (X \setminus C) \amalg \{\infty\}$, for some closed and compact $C \subset X$.

- For $* \in X$ a pointed topological space, we let ΣX denote the reduced suspension,

$$\Sigma X := (X \times [0,1])/\sim \ ,$$

where $\sim$ is the equivalence relation defined by $(x,1) \sim (x',1), (x,0) \sim (x',0)$, and $(*,t) \sim (*,t')$, for all $x, x' \in X$, and $t, t' \in [0,1]$.

- We use homological grading conventions: $A[r]_n := A_{n-r}$.

Grassmannians

Grassmannians are the moduli space of subspaces of a fixed vector space. In this chapter, we restrict our attention to subspaces of Euclidean space. These are the *real* Grassmannians:

Definition 3.0.1. Let n, k be nonnegative integers such that $k \leq n$. We define the *Grassmannian*, $\mathsf{Gr}_k(n)$, to be the set consisting of all k-dimensional subspaces of $\mathbb{R}^n$.

The simplest case to understand is when $n = 1$, as $\mathsf{Gr}_1(1)$ is just a singleton. In general, $\mathsf{Gr}_n(n)$ is also a singleton. The first nontrivial example is $(n,k) = (2,1)$. One way to think about this set is to observe that, by taking its span, each choice of a unit vector in $\mathbb{R}^2$ defines a 1-dimensional subspace, hence a point in $\mathsf{Gr}_1(2)$. Further, by observing that antipodal vectors define the same subspace, we see that the following map of sets is a bijection

$$[0,\pi) \xrightarrow{\cong} \mathsf{Gr}_1(2), \qquad \theta \mapsto \mathsf{span}\{(\cos\theta, \sin\theta)\} \ .$$

Thinking of a Grassmannian as a set does not capture its entire being. In the above example of $\mathsf{Gr}_1(2) \cong [0,\pi)$, if we sweep through the angles θ, as we approach $\theta = \pi$, our subspace is getting closer to the x-axis, which is represented by the point $0 \in [0,\pi)$. In other words, thinking of $\mathsf{Gr}_1(2)$ as the set $[0, \pi)$, ignores the fact that points near π are actually near 0 as well. We will make this intuition precise below, and we will see that $\mathsf{Gr}_1(2)$ is actually diffeomorphic to S^1.

More generally, each $p \in \mathsf{Gr}_k(n)$ can be representated by a k-plane in $\mathbb{R}^n$, and we know how to wiggle planes in Euclidean space. This suggests two things. First, we know which planes are close to a given plane, so we might expect $\mathsf{Gr}_k(n)$ to be a topological space. Second, we know how to budge a plane into a nearby plane, so we might further expect $\mathsf{Gr}_k(n)$ to possess the structure of a manifold. This is indeed the case, as we briefly recall in Section 3. Even better than possessing a manifold structure, there is a natural way of decomposing $\mathsf{Gr}_k(n)$ into pieces called *Schubert cells.* These endow $\mathsf{Gr}_k(n)$ with the structure of a CW complex. We give a description of this Schubert CW structure and the corresponding CW chain complex in Section 3. We refer the reader to [24] for a more in depth treatment of this material.

Manifold structure

Let us fix positive integers n and k, with $k \leq n$. We define a manifold structure on $\mathsf{Gr}_k(n)$ by realizing it as the quotient of an open subset of Euclidean space called the Stiefel space.

Definition 3.0.2. We define the *Stiefel space* $V_k(n)$ to be the collection of all injective, linear maps from $\mathbb{R}^k$ to $\mathbb{R}^n$:

$$V_k(n) := \{\mathbb{R}^k \xrightarrow{\varphi} \mathbb{R}^n \mid \varphi \text{ is injective and linear}\} \subset \mathsf{Mat}_{n\times k} \ .$$

A point in $V_k(n)$ is a matrix $A \in \mathsf{Mat}_{n\times k}$ whose columns are linearly independent. Linear independence is an open condition. Indeed, observe the continuous map

$$F : \mathsf{Mat}_{n\times k} \to \mathbb{R} \ , \qquad A \mapsto \mathsf{det}(A^T A) \ .$$

Note that

$$V_k(n) = F^{-1}(\mathbb{R} \setminus \{0\}) \subset \mathsf{Mat}_{n\times k} \cong \mathbb{R}^{nk} \ .$$

Thus,t $V_k(n) \subset (\mathbb{R}^n)^{\times k}$ is an open subset of Euclidean space. As such we consider $V_k(n)$ as a topological space via the subspace topology. Each point $A \in V_k(n)$ defines a k-dimensional subspace of $\mathbb{R}^n$ by taking its column space. This defines a map

$$\mathsf{col} : V_k(n) \to \mathsf{Gr}_k(n) \ , \qquad A \mapsto \mathsf{col}(A) \ .$$

Observe that this map is surjective: given $V \in \mathsf{Gr}_k(n)$, choose a basis for it and define a matrix whose columns consist of those basis vectors. This defines a point in $V_k(n)$ whose column space is V. Thus, we endow $\mathsf{Gr}_k(n)$ with the quotient topology induced by the map col. Further, in Proposition 3.0.8 below, we show that $\mathsf{Gr}_k(n)$ is a compact topological space, and that it has a natural manifold structure. For this, it is convenient to introduce the orthogonal Stiefel space.

Definition 3.0.3. The *orthogonal Stiefel space* $V_k^o(n)$ is defined to be the set

$$V_k^o(n) := \{A \in \mathsf{Mat}_{n\times k} \mid A^T A = \mathbb{1}_{k\times k}\} \subset \mathsf{Mat}_{n\times k} \ ,$$

equipped with the subspace topology.

Remark 3.0.4. Similar to the case of the ordinary Stiefel space, there is a surjective continuous map

$$\mathsf{col} : V_k^o(n) \to \mathsf{Gr}_k(n) \ , \qquad A \mapsto \mathsf{col}(A) \ .$$

Remark 3.0.5. Consider the continuous map

$$G : \mathsf{Mat}_{n\times k} \to \mathsf{Mat}_{k\times k} \ , \qquad A \mapsto A^T A \ .$$

Then $V_k^o(n) = G^{-1}(\mathbb{1}_{k\times k})$. As such, $V_k^o(n)$ is closed and Hausdorff. Observe that the condition $A^TA = \mathbb{1}_{k\times k}$ implies that each entry of an element in $V_k^o(n)$ is bounded between -1 and 1. Since $V_k^o(n)$ is a closed and bounded subspace of Euclidean space, the Heine-Borel theorem states that $V_k^o(n)$ is in fact compact.

In the proof of Proposition 3.0.8 below, we introduce a smooth atlas for $\mathsf{Gr}_k(n)$. This atlas is composed of the following sets.

Notation 3.0.6. Let $S \in \genfrac\{\}{0pt}{}{n}{k}$. Denote the subset

$$U_S := \Big\{ V \subset \mathbb{R}^n \mid V \xhookrightarrow{\iota} \mathbb{R}^n \xrightarrow{\mathsf{pr}_S} \mathbb{R}^S \text{ is an isomorphism} \Big\} \subset \mathsf{Gr}_k(n) \ .$$

Lemma 3.0.7. *Let $V \in \mathsf{Gr}_k(n)$. Let $S = \{s_1 < \cdots < s_k\} \in \genfrac\{\}{0pt}{}{n}{k}$ where*

$$\begin{aligned}
s_k &:= \mathsf{max}\Big\{ 1 \leq s \leq n \mid V \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s\}} \textit{ is surjective} \Big\} \\
s_{k-1} &:= \mathsf{max}\Big\{ 1 \leq s < s_{k-1} \mid V \cap \mathbb{R}^{s_k - 1} \hookrightarrow \mathbb{R}^{s_k - 1} \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s\}} \textit{ is surjective} \Big\} \\
&\vdots \\
s_1 &:= \mathsf{max}\Big\{ 1 \leq s < s_2 \mid V \cap \mathbb{R}^{s_2} \hookrightarrow \mathbb{R}^{s_2} \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s\}} \textit{ is surjective} \Big\} \ .
\end{aligned}$$

Then such an S exists, and $V \in U_S$.

Proof. We proceed by induction on $n \geq k \geq 0$. The case $k = 0$ is trivial. Assume $k > 0$. Consider the subset

$$\Big\{ 1 \leq s \leq n \mid V \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s\}} \text{ is an isomorphism} \Big\} \subset \{1 < \cdots < n\} \ .$$

Since $\mathsf{dim}(V) = k > 0$, this subset is not empty. Since $\{1 < \cdots < n\}$ is a finite linearly ordered set,

$$s_k := \mathsf{max}\Big\{ 1 \leq s \leq n \mid V \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s\}} \text{ is surjective} \Big\}$$

exists and is unique. So $V \subset \mathbb{R}^{s_k} \subset \mathbb{R}^n$. By definition of s_k, we have

$$\mathsf{dim}(V \cap \mathbb{R}^{s_k - 1}) = k - 1 \ .$$

Indeed, either the dimension of this intersection is k or $k-1$ and it cannot be k for dimension reasons. Let $S' = S \setminus \{s_k\} \in \left\{ {s_k - 1 \atop k-1} \right\}$ By the induction hypothesis on k, $V \cap \mathbb{R}^{s_k-1} \subset U_{S'}$. In other words,

$$V \cap \mathbb{R}^{s_k - 1} \hookrightarrow \mathbb{R}^{s_k - 1} \xrightarrow{\mathsf{pr}} \mathbb{R}^{S'}$$

is an isomorphism. It remains for us to show that $V \in U_S$. To see this, consider the following diagram Each of these squares evidently commutes. To show that $V \in U_S$ is to show that

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d, "\mathsf{pr}"', "\cong"] & \mathbb{R}^{s_k} \arrow[r, hook] \arrow[d, "\mathsf{pr}"', "\cong"] & \mathbb{R}^n \arrow[r, "\mathsf{pr}"] \arrow[d, "\mathsf{pr}"', "\cong"] & \mathbb{R}^S \arrow[d, "\mathsf{pr}"', "\cong"] \\
(V \cap \mathbb{R}^{s_k-1}) \oplus (V \perp (V \cap \mathbb{R}^{s_k-1})) \arrow[r, hook] & \mathbb{R}^{s_k-1} \oplus (\mathbb{R}^{s_k} \perp \mathbb{R}^{s_k-1}) \arrow[r, hook] & \mathbb{R}^{s_k-1} \oplus (\mathbb{R}^n \perp \mathbb{R}^{s_k-1}) \arrow[r, "\mathsf{pr} \oplus \mathsf{pr}"] & \mathbb{R}^{S'} \oplus \mathbb{R}^{\{s_k\}} \ .
\end{tikzcd}$$

the top composite is surjective, and thus an isomorphism by dimension reasons. This follows because the bottom composite is surjective. Note that each bottom horizontal arrow is a direct sum of maps: the direct sum of inclusion for the first two maps, and the direct sum of projections for the last map. Indeed, the composite of the morphisms of the left factors in the direct sum is surjective by the induction assumption, and the composite of the right factors of the direct sum is surjective by definition of s_k. Therefore, $V \in U_S$, which completes the proof. □

Proposition 3.0.8. *The space $\mathsf{Gr}_k(n)$ is a compact, smooth manifold of dimension $k(n-k)$.*

Proof. First we show that $\mathsf{Gr}_k(n)$ is compact. Remark 3.0.4 says that $\mathsf{Gr}_k(n)$ is the continuous image of $V_k^o(n)$. Remark 3.0.5 shows that $V_k^o(n)$ is compact. It follows that $\mathsf{Gr}_k(n)$ is compact, since it is the continuous image of the compact space $V_k^o(n)$. Next, we show that $\mathsf{Gr}_k(n)$ is Hausdorff. To do this, we will show that each $V \neq W \in \mathsf{Gr}_k(n)$ can be separated by a continuous real valued function. Regard an element of $V_k^o(n)$ as a list of k orthonormal vectors of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, consider the function

$$f_x : V_k^o(n) \to \mathbb{R} \ , \qquad (u_1, \ldots, u_k) \mapsto x \cdot x - (x \cdot u_1)^2 - \cdots - (x \cdot u_k)^2 \ .$$

This function is evidently continuous. For $V \in \mathsf{Gr}_k(n)$, note that f_x evaluates the same on each element of $\mathsf{col}^{-1}(V)$. By the universal property of the quotient topology, f_x defines a continuous map $\widetilde{f_x} : \mathsf{Gr}_k(n) \to \mathbb{R}$. Now, for $V \neq W \in \mathsf{Gr}_k(n)$, choose a point $v \in V \setminus W$. By definition, $\widetilde{f_v}(V) = 0$ and $\widetilde{f_v}(W) \neq 0$. This shows that distinct points of $\mathsf{Gr}_k(n)$ can be separated by a continuous real valued function. It follows that $\mathsf{Gr}_k(n)$ is Hausdorff.

We will define a smooth atlas on $\mathsf{Gr}_k(n)$ as follows. Let $\mathsf{pr}_S : \mathbb{R}^n \to \mathbb{R}^S$ denote the orthogonal projection $x \mapsto \sum_{i \in S} \langle x, e_i \rangle e_i$. Given $S \in \left\{ {n \atop k} \right\}$, consider the subset from Notation 3.0.6

$$U_S = \left\{ V \subset \mathbb{R}^n \mid V \xhookrightarrow{\iota} \mathbb{R}^n \xrightarrow{\mathsf{pr}_S} \mathbb{R}^S \text{ is an isomorphism} \right\} \subset \mathsf{Gr}_k(n) \ .$$

Define a map

$$\mathsf{Graph}_S : \mathsf{Hom}^{\mathsf{lin}}(\mathbb{R}^S, \mathbb{R}^{\{1,\ldots,n\} \setminus S}) \to U_S$$

whose value on a linear map F is its graph

$$F \mapsto \mathsf{Graph}_S(F) := \mathsf{image}\left(\mathbb{R}^S \xhookrightarrow{(\mathsf{id}, F)} \mathbb{R}^S \oplus \mathbb{R}^{\underline{n} \setminus S} \xrightarrow[\cong]{Q_S} \mathbb{R}^n \right) \ .$$

Here, $Q_S : \mathbb{R}^S \oplus \mathbb{R}^{\underline{n} \setminus S} \to \mathbb{R}^n$ is the evident reordering of bases. In fact, Graph_S is a bijection with inverse given by the following. Given $V \in U_S$, define a linear map $F \in \mathsf{Hom}^{\mathsf{lin}}(\mathbb{R}^S, \mathbb{R}^{\{1,\ldots,n\} \setminus S})$ via the assignment

$$e_i \mapsto \mathsf{pr}_S^{-1}(e_i) \cap V - e_i \ ,$$

for all $i \in S$. Further, one can check that this in fact defines a homeomorphism. Thus, for each $S \in \left\{ {n \atop k} \right\}$, we have a homeomorphism

$$\mathsf{Graph}_S : \mathbb{R}^{k(n-k)} \cong \mathsf{Hom}^{\mathsf{lin}}(\mathbb{R}^S, \mathbb{R}^{\{1,\ldots,n\}\setminus S}) \xrightarrow{\cong} U_S \ .$$

By Lemma 3.0.7, the collection $\{\mathbb{R}^{k(n-k)} \xrightarrow{\mathsf{Graph}_S} U_S\}$ covers $\mathsf{Gr}_k(n)$. It remains to check that the transition functions for the cover $\{\mathbb{R}^{k(n-k)} \xrightarrow{\mathsf{Graph}_S} U_S\}$ are smooth. That is, for $S, T \in \left\{ {n \atop k} \right\}$ we need to show the composite

$$\mathsf{Graph}_S^{-1}(U_S \cap U_T) \xrightarrow{\mathsf{Graph}_S} U_S \cap U_T \xrightarrow{\mathsf{Graph}_S^{-1}} \mathsf{Graph}_S^{-1}(U_S \cap U_T) \tag{3.1}$$

is smooth. First, note that the composite

$$\mathbb{R}^S \oplus \mathbb{R}^{\underline{n}\setminus S} \xrightarrow[\cong]{Q_S} \mathbb{R}^n \xrightarrow[\cong]{Q_T^{-1}} \mathbb{R}^T \oplus \mathbb{R}^{\underline{n}\setminus T}$$

is some permutation matrix

$$\begin{bmatrix} P_{T\times S} & P_{T\times \underline{n}\setminus S} \\ P_{\underline{n}\setminus T\times S} & P_{\underline{n}\setminus T\times \underline{n}\setminus S} \end{bmatrix} ,$$

where the superscripts on the blocks denote the dimension of the block. Using this notation, the map (3.1) evaluates as

$$F \mapsto \left(P_{\underline{n}\setminus T\times S} + P_{\underline{n}\setminus T\times \underline{n}\setminus S}\right) \circ \left(P_{T\times S} + P_{T\times \underline{n}\setminus S}\right)^{-1} \ .$$

This map is evidently smooth since matrix multiplication, matrix addition, and taking inverses are smooth. □

CW structure

There is a natural CW structure on Grassmannians called the *Schubert* CW structure. The set of Schubert cells is in bijection with the set $\genfrac{\{}{\}}{0pt}{}{n}{k}$.

Definition 3.0.9. For positive integers $k \leq n$, let

$$\genfrac{\{}{\}}{0pt}{}{n}{k} := \{S \subset \{1, \ldots, n\} \mid \mathsf{card}(S) = k\} ,$$

denote the set consisting of all cardinality k subsets of the set $\{1, \ldots, n\}$. Define a partial order on the elements of $\genfrac{\{}{\}}{0pt}{}{n}{k}$ by declaring $S = \{s_1 < \cdots < s_k\} \leq T = \{t_1 < \cdots < t_k\}$ to mean $s_i \leq t_i$ for each $i \in \{1, \ldots, k\}$.

Example 3.0.10. The poset $\genfrac{\{}{\}}{0pt}{}{4}{2}$ can be depicted as

$$\begin{tikzcd}
 & & \{2,3\} \arrow[dr] & & \\
\{1,2\} \arrow[r] & \{1,3\} \arrow[ur] \arrow[dr] & & \{2,4\} \arrow[r] & \{3,4\} \\
 & & \{1,4\} \arrow[ur] & &
\end{tikzcd} .$$

For $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}$, the Schubert cell corresponding to S is the subspace

$$\mathsf{Gr}_k(n)_S := \{V \in \mathsf{Gr}_k(n) \mid S \text{ is the maximal element in } \genfrac{\{}{\}}{0pt}{}{n}{k} \text{ for which } V \in U_S\} .$$

Lemma 3.0.7 establishes that such an S exists for each $V \in \mathsf{Gr}_k(n)$. The following lemma shows that the resulting S from Lemma 3.0.7 is the unique maximal $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}$ for which $V \in U_S$.

Lemma 3.0.11. *Let $V \in \mathsf{Gr}_k(n)$. There exists a unique maximal $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}$ for which $V \in U_S$.*

Proof. Recall from Lemma 3.0.7 that there exists a maximal $S = \{s_1 < \cdots < s_k\} \in \left\{ {n \atop k} \right\}$ for which $V \in U_S$. The poset $\left\{ {n \atop k} \right\}$ is not linearly ordered, so it remains to show that the S constructed in Lemma 3.0.7 is in fact unique. We induct on $n \geq k \geq 0$. The case $k = 0$ is again trivial, so assume $k > 0$. Let $T \in \left\{ {n \atop k} \right\}$ be a maximal element for which $V \in U_T$. Recall from Lemma 3.0.7 that

$$s_k := \mathsf{max}\left\{1 \leq s \leq n \mid V \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s\}} \text{ is surjective}\right\} .$$

Thus, $t_k \leq s_k$. Since T is maximal by assumption, $t_k \not< s_k$. If this were the case, then $T < T' \cup \{s_k\}$ for $T' = T \setminus \{t_k\}$. So $t_k = s_k$. Now recall $\mathsf{dim}(V \cap \mathbb{R}^{s_k - 1}) = k - 1$. By assumption on T and S, we have that both

$$V \cap \mathbb{R}^{s_k-1} \hookrightarrow \mathbb{R}^{s_k-1} \xrightarrow{\mathsf{pr}} \mathbb{R}^{T'}$$

and

$$V \cap \mathbb{R}^{s_k-1} \hookrightarrow \mathbb{R}^{s_k-1} \xrightarrow{\mathsf{pr}} \mathbb{R}^{S'}$$

are isomorphisms. Thus by induction, $T' = S'$, and therefore $T = T' \cup \{t_k\} = S' \cup \{s_k\} = S$, which completes the proof. □

Observation 3.0.12. Let $S \in \left\{ {n \atop k} \right\}$. Lemma 3.0.7 and the proof of Lemma 3.0.11 above gives us the following characterization of the S-stratum of $\mathsf{Gr}_k(n)$:

$$\mathsf{Gr}_k(n)_S = \left\{V \in \mathsf{Gr}_k(n) \mid \forall\ 1 \leq i \leq k\ ,\ s_i - 1 = \mathsf{max}\{1 \leq s \leq n \mid \mathsf{dim}(V \cap \mathbb{R}^S) = i - 1\}\right\} .$$

Example 3.0.13. Consider $\left\{ {2 \atop 1} \right\} = \{\{1\} < \{2\}\}$, and take $S = \{1\}$. Then $\mathsf{Gr}_1(2)_{\{1\}}$ is a singleton because the only 1-dimensional subspace of $\mathbb{R}^2$ that does not project isomorphically onto the y-axis, $\mathbb{R}^{\{2\}} = \mathsf{span}\{e_2\}$, is the x-axis, $\mathbb{R}^{\{1\}} = \mathsf{span}\{e_1\}$. This also tells us that

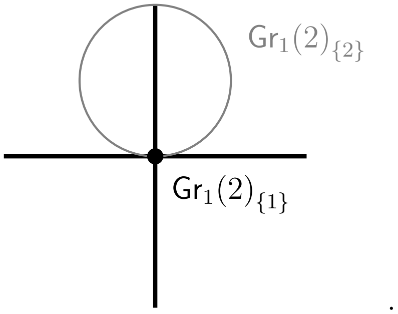
$\mathsf{Gr}_1(2)_{\{2\}}$ consists of all other points in $\mathsf{Gr}_1(2)$. Thus, $\mathsf{Gr}_1(2)$ is comprised of a single 0-dimensional cell,

$$\mathsf{Gr}_1(2)_{\{1\}} = \left\{ \mathsf{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} ,$$

and a single 1-dimensional cell,

$$\mathsf{Gr}_1(2)_{\{2\}} = \left\{ \mathsf{span} \begin{bmatrix} * \\ 1 \end{bmatrix} \;\middle|\; * \in \mathbb{R} \right\} .$$

We depict these cells in the following picture.



To name a CW structure, one must define attaching maps that specify how the higher dimensional cells attach to the lower dimensional cells. We will now explicitly describe this cellular decomposition of $\mathsf{Gr}_k(n)$.

For each $S \in \left\{ {n \atop k} \right\}$ we define a map Rot_S from a closed cube to $\mathsf{Gr}_k(n)$ whose interior is homeomorphic with $\mathsf{Gr}_k(n)_S$, and whose boundary maps to strictly lower dimensional cells. To define Rot_S we will use a product of rotation matrices, which are elements of $\mathsf{O}(n)$. Since $\mathsf{O}(n)$ is non-Abelian, the order in which we multiply these matrices is critical. To obtain the correct order of multiplication we use a particular linearly ordered index set Z_S

Definition 3.0.14. For $S = \{s_1 < \cdots < s_k\} \in \left\{ {n \atop k} \right\}$, define the set

$$Z_S := \{(i,j) \mid i \in \{1,\dots,k\}, j \in \{i,\dots,s_i - 1\}\} \ .$$

We equip this set with a total ordering by declaring $(i,j) \leq (i',j')$ to mean either $i < i'$, or $i = i'$ and $j \geq j'$. See Remark 3.0.16 for a further explanation of this ordering.

For $1 \leq j < n$ and $\theta \in \mathbb{R}$, let $\mathsf{R}_j(\theta)$ denote the $n \times n$ matrix implementing a rotation of $\mathbb{R}^n$ in the oriented $(j < j+1)$-plane by an angle of θ:

$$\mathsf{R}_j(\theta) := \begin{bmatrix} \mathbb{1}_{j-1 \times j-1} & & & \\ & \cos(\theta) & -\sin(\theta) & \\ & \sin(\theta) & \cos(\theta) & \\ & & & \mathbb{1}_{(n-j-1)\times(n-j-1)} \end{bmatrix} .$$

Here, the middle block lies in the j and $j+1$ rows and columns, and all unspecified entries are 0. For example, for $n = 3$,

$$\mathsf{R}_2(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} .$$

Note that for each j, this defines a continuous map

$$[0,\pi] \to \mathsf{O}(n) \ , \qquad \theta \mapsto \mathsf{R}_j(\theta) \ .$$

Given a matrix $A \in \mathsf{O}(n)$, the first k columns of A form an ordered set of k linearly independent vectors in $\mathbb{R}^n$, which is a point in $V_k(n)$. This assembles as a map

$$\mathsf{O}(n) \to V_k(n)\ , \qquad A \mapsto A\mathbb{1}_{n\times k}\ ,$$

where $\mathbb{1}_{n\times k}$ denotes the first k columns of the $n \times n$ identity matrix. This map is evidently continuous, since matrix multiplication is continuous.

Recall the linearly ordered set Z_S from Definition 3.0.14.

Definition 3.0.15. These maps, together with the group structure of $\mathsf{O}(n)$, and the linearly ordered index set Z_S allow us to define the composite continuous map

$$\widetilde{\mathsf{Rot}}_S : \prod_{(i,j)\in Z_S} [0,\pi] \xrightarrow{(\mathsf{R}_j(-))_{(i,j)\in Z_S}} \prod_{Z_S} \mathsf{O}(n) \xrightarrow{\text{mult}} \mathsf{O}(n) \xrightarrow{-\cdot\mathbb{1}_{n\times k}} V_k(n)\ ,$$

that sends

$$(\theta_{(i,j)})_{(i,j)\in Z_S} \mapsto \left(\prod_{(i,j)\in Z_S} \mathsf{R}_j(\theta_{(i,j)})\right) \mathbb{1}_{n\times k}\ .$$

Finally, postcomposing with $\mathsf{col} : V_k(n) \to \mathsf{Gr}_k(n)$, we obtain the continuous map

$$\mathsf{Rot}_S : \prod_{Z_S}[0,\pi] \xrightarrow{\widetilde{\mathsf{Rot}}_S} V_k(n) \xrightarrow{\mathsf{col}} \mathsf{Gr}_k(n)\ .$$

Remark 3.0.16. The reason for the above defined ordering on Z_S is to ensure that the matrix multiplication in the definition of $\widetilde{\mathsf{Rot}}_S$ occurs in the correct sequence. Namely, we have set this up so that if we evaluate $\widetilde{\mathsf{Rot}}_S$ at the point $(\pi/2)_{Z_S}$, then we obtain the plane $\mathbb{R}^S$. The defined ordering is such that $\widetilde{\mathsf{Rot}}_S$ first sends e_k to e_{s_k}, and ensures that the subsequent rotations will not move e_{s_k} further. Next, it sends e_{k-1} to $\mathsf{e}_{s_{k-1}}$, again ensuring that subsequent rotations will not move either e_{s_k} or $\mathsf{e}_{s_{k-1}}$. This process continues, until the final result produces $\mathbb{R}^S$.

Having defined the attaching maps, we now show that the collection of sets $\mathsf{Gr}_k(n)_S$ do form a CW structure on $\mathsf{Gr}_k(n)$. First, we establsih a homeomorphism of $\mathsf{Gr}_k(n)_S$ with the interior of the closed disk $\prod_{Z_S}[0,\pi]$. In particular, this will tell us that the dimension of the cell $\mathsf{Gr}_k(n)_S$ is the cardinality of Z_S. Let us denote this dimension by $d(S)$:

$$d(S) := \mathsf{card}(Z_S).$$

Observation 3.0.17. Recall that for $S = \{s_1 < \cdots < s_k\} \in \left\{ {n \atop k} \right\}$,

$$d(S) := \mathsf{card}(Z_S) = \sum_{i \in \{1,\ldots,k\}} \mathsf{card}(\{i, \ldots, s_i - 1\}) = \sum_{i \in \{1,\ldots,k\}} s_i - i \ .$$

Next, we establish that the boundaries of these cells attach to strictly lower dimensional cells. In what follows, denote

$$\partial \prod_{Z_S} [0,\pi] := \prod_{Z_S} [0,\pi] \setminus \prod_{Z_S} (0,\pi) \ .$$

Lemma 3.0.18. *For $S \in \left\{ {n \atop k} \right\}$, the restriction of Rot_S to the boundary factors through the union of the lower dimensional cells:*

$$\begin{tikzcd}
\partial \prod_{Z_S}[0,\pi] \arrow[r, hook] \arrow[dr, dashed] & \prod_{Z_S}[0,\pi] \arrow[r, "\mathsf{Rot}_S"] & \mathsf{Gr}_k(n) \\
& \bigcup_{T<S} \mathsf{Gr}_k(n)_T \arrow[ur, hook] &
\end{tikzcd} .$$

Proof. Let $\Theta = (\theta_{(i',j')}) \in \partial \prod_{Z_S} [0,\pi]$. We will show that for each $1 \leq i \leq k$, and each $j > s_i$, the (j,i) entry of $\widetilde{\mathsf{Rot}}_S(\Theta)$ is 0. Fix such a pair (j,i). Consider the partition of Z_S given by the sets $\{(i',j') \in Z_S \mid i' \leq i\}$ and $\{(i',j') \in Z_S \mid i' > i\}$. This gives a factorization

$\widetilde{\mathsf{Rot}}_S(\Theta) = BA$, where

$$B := \prod_{(i',j')\in Z_S,\ i'\leq i} R_{j'}(\theta_{(i',j')}) \ , \qquad A := \prod_{(i',j')\in Z_S,\ i'>i} R_{j'}(\theta_{(i',j')}) \ .$$

So, the (j,i)-entry of $\widetilde{\mathsf{Rot}}_S(\Theta)$ is $\mathsf{row}_j(B)\mathsf{col}_i(A)$. Since $j \geq s_i + 1$, and each $(i',j') \in Z_S$ with $i' \leq i$ has $j' < s_i$, the j-th row of B is e_j^T. Further, since each $(i',j') \in Z_S$ with $i' > i$ has $j' \geq i' > i$, the i-th column of A is e_i. Finally, since $i \leq s_i < j$, $\mathsf{e}_j \cdot \mathsf{e}_i = 0$. ☐

We will now show that, for each $S \in \left\{ n \atop k \right\}$, the subspace $\mathsf{Gr}_k(n)_S$ is homeomorphic with a Euclidean space. This is Lemma 3.0.24 below. The proof of Lemma 3.0.24 makes use of several intermediate results, which we now establish.

Notation 3.0.19. Let $0 \leq r \leq k \leq n$. Let $S \in \left\{ n \atop k \right\}$. Denote the element

$$S_{<r} \ := \ S \setminus \{s_r < \cdots < s_k\} := \{s_1 < \cdots < s_{r-1}\} \ \in \ \left\{ n \atop r-1 \right\} \ .$$

Consider the inclusion

$$Z_{S_{<r}} \ \hookrightarrow \ Z_S \ , \qquad (i,j) \mapsto (i,j) \ ,$$

whose image consists of those (i,j) for which $i < r$. Note that this inclusion is convex, and has the property that for $(i,j) \in Z_{S_{<r}}$, if $(i',j') \leq (i,j)$ in Z_S, then $(i',j') \in Z_{S_{<r}}$. This inclusion determines a projection between open cubes, whose values we denote as

$$(0,\pi)^{Z_S} \xrightarrow{\mathsf{proj}} (0,\pi)^{Z_{S_{<r}}} \ , \qquad \Theta_S \mapsto \Theta_{S_{<r}} \ .$$

Lemma 3.0.20. *Let $S \in \left\{ n \atop k \right\}$. Let $\Theta_S \in (0,\pi)^{Z_S}$.*

1. *The k-dimensional vector subspace $\mathsf{Rot}_S(\Theta_S) \subset \mathbb{R}^n$ in fact lies in $\mathbb{R}^{s_k} \subset \mathbb{R}^n$:*

$$\mathsf{Rot}_S(\Theta_S) \ \subset \ \mathbb{R}^{s_k} \ .$$

2. The $(k-1)$-dimensional vector subspace is

$$\mathit{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \;=\; \mathit{Rot}_S(\Theta_S) \cap \mathbb{R}^{s_k-1}$$

is the intersection of the k-dimensional vector subspace with $\mathbb{R}^{s_k-1}$.

Proof. Recall the Definition 3.0.15 of $\widetilde{\mathsf{Rot}}_S(\Theta_S)$, as a Z_S-fold (ordered) product of matrices, with the (i,j)-factor being a rotation matrix in the oriented $\{j < j+1\}$-coordinate plane. By Definition 3.0.14 of the linearly ordered set Z_S, each element $(i,j) \in Z_S$ has the property that $j < s_k$. Therefore, for each $s_k < t \leq n$, the value $\widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathsf{e}_t) = \mathsf{e}_t$. In particular, there is an equality between vector subspaces of $\mathbb{R}^n$:

$$\widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathbb{R}^{\{s_k+1<\cdots<n\}}) \;=\; \mathbb{R}^{\{s_k+1<\cdots<n\}} \;.$$

Because the $n \times n$ matrix $\widetilde{\mathsf{Rot}}_S(\Theta_S)$ is an orthogonal matrix, the fact that the vector subspace $\mathbb{R}^k$ is orthogonal with the vector subspace $\mathbb{R}^{\{s_k+1<\cdots<n\}}$ implies the vector subspace $\mathsf{Rot}_S(\Theta_S) := \widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathbb{R}^k)$ is orthogonal with the vector subspace $\widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathbb{R}^{\{s_k+1<\cdots<n\}}) = \mathbb{R}^{\{s_k+1<\cdots<n\}}$. We conclude an inclusion of one vector subspaces of $\mathbb{R}^n$ into the orthogonal complement of the other:

$$\mathsf{Rot}_S(\Theta_S) \;\subset\; \mathbb{R}^{s_k} \;=\; \mathbb{R}^n \perp \mathbb{R}^{\{s_k+1<\cdots<n\}} \;.$$

This proves the first statement of the lemma.

By inspection of the Definition 3.0.15 of $\widetilde{\mathsf{Rot}}_S$, for each $1 \leq i < k$, the values agree:

$$\widetilde{\mathsf{Rot}}_S(\mathsf{e}_i) \;=\; \widetilde{\mathsf{Rot}}_{S_{<k}}(\mathsf{e}_i) \;.$$

There follows an equality between $(k-1)$-dimensional vector subspaces of $\mathbb{R}^n$:

$$\begin{aligned} \widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathbb{R}^{\{1<\cdots<k-1\}}) &= \widetilde{\mathsf{Rot}}_{S_{<k}}(\Theta_{S_{<k}})(\mathbb{R}^{\{1<\cdots<k-1\}}) \\ &=: \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \ . \end{aligned} \tag{3.2}$$

Next, by careful inspection of the Definition 3.0.15 of $\widetilde{\mathsf{Rot}}_S$, the projection onto the e_{s_k}-coordinate is not the zero-vector:

$$\mathsf{Proj}_{\mathbb{R}^{\{s_k\}}}\big(\widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathsf{e}_k)\big) \ \neq \ \mathbf{0} \ .$$

In particular, the composite linear map

$$\widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathsf{e}_k) \ \hookrightarrow \ \mathbb{R}^n \xrightarrow{\mathsf{Proj}} \mathbb{R}^{\{s_k\}}$$

is surjective. Therefore, there is *not* a containment:

$$\widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathsf{e}_k) \ \not\subset \ \mathbb{R}^{s_k-1} \ . \tag{3.3}$$

Now, by Statement 1 of the lemma, applied to $S_{<k}$ and $\Theta_{S_{<k}}$, the $(k-1)$-dimensional vector subspace $\mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})$ is contained in the span of the first $s_k - 1$ coordinate vectors of $\mathbb{R}^n$:

$$\mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \underset{\text{State 1}}{\subset} \mathbb{R}^{s_{k-1}} \subset \mathbb{R}^{s_k-1} \subset \mathbb{R}^n \ . \tag{3.4}$$

We conclude a containment between vector subspaces of $\mathbb{R}^{s_k-1}$:

$$\begin{aligned} \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) &\underset{(3.2)\ \&\ (3.4)}{=} \widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathbb{R}^{\{1<\cdots<k-1\}}) \cap \mathbb{R}^{s_k-1} \\ &\subset \widetilde{\mathsf{Rot}}_S(\Theta_S)(\mathbb{R}^{\{1<\cdots<k\}}) \cap \mathbb{R}^{s_k-1} \\ &=: \mathsf{Rot}_S(\Theta_S) \cap \mathbb{R}^{s_k-1} \ . \end{aligned}$$

The domain of this containment has dimension $\mathsf{dim}\big(\mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})\big) = (k-1)$. The non-containment (3.25) implies the codomain of this containment has dimension $\mathsf{dim}\big(\mathsf{Rot}_S(\Theta_S)\cap\mathbb{R}^{s_k-1}\big) < \mathsf{dim}\big(\mathsf{Rot}_S(\Theta_S)\cap\mathbb{R}^{s_k-1}\big) = k$. It follows that this containment is, in fact, an equality, which proves the second statement of the lemma.

□

Lemma 3.0.20 yields the following, by induction on k.

Corollary 3.0.21. *Let $0 \leq i \leq k \leq n$. Let $S \in \left\{ {n \atop k} \right\}$. Let $\Theta_S \in (0,\pi)^{Z_S}$. There is an equality between $(i-1)$-dimensional vector subspaces of $\mathbb{R}^n$:*

$$\mathsf{Rot}_{S_{<i}}(\Theta_{S_{<i}}) \;=\; \mathsf{Rot}_S(\Theta_S)\cap\mathbb{R}^{s_i-1} \ .$$

We now establish two corollaries of Lemma 3.0.20 that will be used in the proof of Lemma 3.0.24. Consider the inclusion $(0,\pi)^{Z_S} \overset{0}{\hookrightarrow} [0,\pi]^{Z_M}$ given by $\Theta_S \mapsto \Theta_M$ where

$$(\Theta_M)_{(i,j)} = \begin{cases} (\Theta_S)_{(i,j)} \ , & \text{if } (i,j)\in Z_S \\ 0 \ , & \text{else} \ . \end{cases}$$

Corollary 3.0.22. *Let $S \in \left\{ {n \atop k} \right\}$. The restriction of $\mathsf{Rot}\colon [0,\pi]^{Z_M} \to \mathsf{Gr}_k(n)$ to $(0,\pi)^{Z_S}$ takes values in the S-stratum $\mathsf{Gr}_k(n)_S$:*

$$\begin{tikzcd}
(0,\pi)^{Z_S} \arrow[r, dashed, "\mathsf{Rot}_S"] \arrow[d, hook, "0"] & \mathsf{Gr}_k(n)_S \arrow[d, hook] \\
{[0,\pi]^{Z_M}} \arrow[r, "\mathsf{Rot}"] & \mathsf{Gr}_k(n) \ .
\end{tikzcd}$$

Proof. We proceed by induction on k. The base case in which $k=0$ is tautologically true. So assume $k > 0$. Let $\Theta_S \in (0,\pi)^{Z_S}$. We will show that

$$\mathsf{Rot}_S(\Theta_S) \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^S$$

is an isomorphism, and that S is the maximal such S for which this is true. Statement 1 of Lemma 3.0.20 implies, for $r > s_k$, the composite linear map

$$\mathsf{Rot}_S(\Theta_S) \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{r\}}$$

is the zero map. Statement 2 of Lemma 3.0.20 also implies the composite linear map

$$\mathsf{Rot}_S(\Theta_S) \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{\{s_k\}} \tag{3.5}$$

is surjective. So s_k is the maximal element in the finite linearly ordered set $\{1 < \cdots < n\}$ for which (3.5) is surjective. By induction on k, the element $S_{<k} \in \left\{ n \atop k-1 \right\}$ is maximal for which the composite linear map

$$\mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^{S_{<k}} \tag{3.6}$$

is an isomorphism. Statement 2 of Lemma 3.0.20 states that $\mathsf{Rot}_S(\Theta_S) \cap \mathbb{R}^{s_k - 1} = \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})$. Therefore we have an identification as a direct sum, with respect to which the inclusion into $\mathbb{R}^n$ respects this direct sum:

$$\begin{aligned}\mathsf{Rot}_S(\Theta_S) &= \big(\mathsf{Rot}_S(\Theta_S) \cap \mathbb{R}^{s_k-1}\big) \oplus \big(\mathsf{Rot}_S(\Theta_S) \perp (\mathsf{Rot}_S(\Theta_S) \cap \mathbb{R}^{s_k-1})\big) \\ &= \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \oplus \big(\mathsf{Rot}_S(\Theta_S) \perp \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})\big) \\ &\subset \mathbb{R}^{s_k-1} \oplus \mathbb{R}^{\{s_k\}} \subset \mathbb{R}^{s_k-1} \oplus \mathbb{R}^{\{s_k < \cdots < n\}} \ = \ \mathbb{R}^n \ .\end{aligned} \tag{3.7}$$

Note also that, with respect to the identification as a direct sum, $\mathbb{R}^S = \mathbb{R}^{S_{<k}} \oplus \mathbb{R}^{\{s_k\}}$,

the projection $\mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^S$ also respects this direct sum. These splittings together with the surjectivity of the linear maps (3.5) and (3.6) imply the linear map

$$\mathsf{Rot}_S(\Theta_S) \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{pr}} \mathbb{R}^S \tag{3.8}$$

is surjective. Because the domain and the codomain of this linear map both have dimension k, this map (3.8) is an isomorphism.

Lastly, recall from above that both s_k is maximal with the property that the linear map (3.5) is surjective, and $S_{<k}$ is maximal with the property that (3.6) is an isomorphism. It follows that $S = S_{<k} \cup \{s_k\}$ is maximal with the property that the linear map (3.8) is a surjective. Therefore, $\mathsf{Rot}_S(\Theta_S) \in \mathsf{Gr}_k(n)_S$, as desired. □

Corollary 3.0.23. *For $k > 0$, the map between sets*

$$\mathcal{I} : \mathsf{Gr}_k(n)_S \to \mathsf{Gr}_{k-1}(n)_{S_{<k}} , \qquad V \mapsto V \cap \mathbb{R}^{s_k - 1} , \tag{3.9}$$

is defined and continuous. With respect to this map, the following diagram commutes:

$$\begin{tikzcd}
(0,\pi)^{Z_S} \arrow[r, "\mathsf{Rot}_S"] \arrow[d, "\mathsf{pr}"] & \mathsf{Gr}_k(n)_S \arrow[d, "(3.9)"] \\
(0,\pi)^{Z_{S_{<k}}} \arrow[r, "\mathsf{Rot}_{S_{<k}}"'] & \mathsf{Gr}_{k-1}(n)_{S_{<k}} .
\end{tikzcd} \tag{3.10}$$

Proof. We first show that the map (3.9) is defined. Let $V \in \mathsf{Gr}_k(n)_S$ be an element in the S-stratum. By Observation 3.0.12, the element $s_k \in \{1 < \cdots < n\}$ is maximal among those elements $r \in \{1 < \cdots < n\}$ for which the composite linear map

$$V \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{Proj}} \mathbb{R}^{\{r\}}$$

is surjective. It follows that the vector subspace $V \cap \mathbb{R}^{s_k - 1} \subset \mathbb{R}^n$ is $(k-1)$-dimensional.

Observation 3.0.12 implies that this $(k-1)$-dimensional vector subspace indeed belongs to the $S_{<k}$-stratum of $\mathsf{Gr}_{k-1}(n)$:

$$V \cap \mathbb{R}^{s_k - 1} \ \in \ \mathsf{Gr}_{k-1}(n)_{S_{<k}} \ .$$

This shows that the map is defined.

We next show that the map (3.9) is continuous. Post-composition with projection onto $\mathbb{R}^{\{s_k\}}$ defines a continuous map

$$V_k(n) \xrightarrow{\mathsf{Proj}_{\mathbb{R}^{s_k}} \circ -} \mathsf{Mat}_{\{s_k\} \times k} \ , \qquad A \mapsto \mathbb{1}_{\{s_k\} \times s_k} \cdot A \ ,$$

which is given by left multiplication by a truncated identity matrix. Let $V_k(n)_{|S} \subset V_k(n)$ denote the subspace consisting of those $A \in V_k(n)$ for which $\mathsf{col}(A) \in \mathsf{Gr}_k(n)_S$. Observe the solid diagram among sets:

$$\begin{tikzcd}
V_k(n)_{|S} \arrow[d, "\mathsf{col}"] & V_k(s_k)_{|S} \arrow[l, hook', "="'] \arrow[r, hook] & \{A \in V_k(s_k) \mid \mathsf{rank}(\mathbb{1}_{\{s_k\} \times s_k} A) = 1\} \arrow[d, "\mathsf{null}(\mathbb{1}_{\{s_k\} \times s_k} \cdot -)"] \\
\mathsf{Gr}_k(n)_S \arrow[rr, "V \mapsto V \cap \mathbb{R}^{s_k - 1}"] & & \mathsf{Gr}_{k-1}(n)_{S_{<k}} \ .
\end{tikzcd}$$

Each map in this solid diagram is manifestly continuous. Since $\mathsf{Gr}_k(n)$ is a quotient of $V_k(n)$, the bottom horizontal map is continuous, as desired.

Lastly, the commutativity of the diagram in equation (3.14) follows directly from Statement 2 of Lemma 3.0.20. □

We are finally prepared to state and prove that each subspace $\mathsf{Gr}_k(n)_S$ is homeomorphic with a Euclidean space.

Lemma 3.0.24. *For $S \in \left\{ {n \atop k} \right\}$, the factorization of Corollary 3.0.22,*

$$\mathit{Rot}_S : (0, \pi)^{Z_S} \longrightarrow \mathit{Gr}_k(n)_S \ , \tag{3.11}$$

is a homeomorphism.

Proof. We first show that (3.11) is a bijection. We prove this by induction on $k \geq 0$. The $k = 0$ case is tautologically true.

We now consider the case in which $k = 1$. So $S = \{s\}$, for some $1 \leq s \leq n$. We proceed by induction on s. For $s = 1$, then $Z_{\{1\}} = \emptyset$ so that $(0,\pi)^{Z_{\{1\}}}$ is a singleton, as is the $\{1\}$-stratum $\mathsf{Gr}_1(n)_{\{1\}}$. So this case in which $s = 1$ is tautologically true. Now assume $s > 1$. Via the standard action of $\mathsf{O}(n)$ on $\mathsf{Gr}_1(n)$, the assignment,

$$(0,\pi) \times \mathsf{Gr}_1(n) \longrightarrow \mathsf{Gr}_1(n) \ , \qquad (\theta, L) \mapsto \mathsf{R}_{s-1}(\theta)(L) \ ,$$

is a continuous map. By inspection, the restriction of this map to the $\{s-1\}$-stratum takes values in the $\{s\}$-stratum:

$$(0,\pi) \times \mathsf{Gr}_1(n)_{\{s-1\}} \longrightarrow \mathsf{Gr}_1(n)_{\{s\}} \ , \qquad (\theta, L) \mapsto \mathsf{R}_{s-1}(\theta)(L) \ . \tag{3.12}$$

By the universal property of subspace topologies, this map is continuous. We now show that this continuous map (3.12) is a bijection. So let $V \in \mathsf{Gr}_1(n)_{\{s\}}$. By definition of the the $\{s\}$-stratum, the number $s \in \{1 < \cdots < n\}$ is maximal with respect to the property that the composite linear map $V \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{proj}} \mathbb{R}^{\{s\}}$ is an isomorphism. In particular, $V \subset \mathbb{R}^s \subset \mathbb{R}^n$, which is to say that, for $r > s$, the r-coordinate of a vector in V is zero. Furthermore, there is a unique unit vector $\mathbf{v} \in V$ for which the dot product $\mathbf{v} \cdot \mathsf{e}_s > 0$ is positive, which is to say there is a unique unit vector in V with positive s-coordinate. Now, there is a unique $\theta_{\mathbf{v}} \in (0,\pi)$ for which $\mathsf{R}_{s-1}(-\theta_{\mathbf{v}})(\mathbf{v}) \cdot \mathsf{e}_s = 0$. Then the element $(\theta_{\mathbf{v}}, R_{s-1}(-\theta_{\mathbf{v}})(V)) \in (0,\pi) \times \mathsf{Gr}_1(n)_{\{s-1\}}$ is the unique preimage of V under the map (3.12). So the map (3.12) is a bijection, as desired.

Next, notice the inclusion

$$Z_{\{s-1\}} \hookrightarrow Z_{\{s\}} \ , \qquad (1, j) \mapsto (1, j) \ .$$

This inclusion is convex, and has the property that for $(i,j) \in Z_{\{s-1\}}$, if $(i,j) \leq (i',j')$ in $Z_{\{s\}}$, then $(i',j') \in Z_{\{s-1\}}$. The complement of this inclusion is the singleton $\{(1,s-1)\} \subset Z_{\{s\}}$. There results a bijection:

$$(0,\pi)^{Z_{\{s\}}} \;\cong\; (0,\pi) \times (0,\pi)^{Z_{\{s-1\}}} \;. \tag{3.13}$$

Notice, now, from the Definition 3.0.15 of Rot, that the maps constructed just above fit into a commutative square among sets:

$$\begin{tikzcd}
(0,\pi)^{Z_{\{s\}}} \arrow[r, "\mathsf{Rot}_{\{s\}}"] \arrow[d, "\cong"', "(3.13)"] & \mathsf{Gr}_1(n)_{\{s\}} \\
(0,\pi) \times (0,\pi)^{Z_{\{s-1\}}} \arrow[r, "\mathsf{id}_{(0,\pi)} \times \mathsf{Rot}_{\{s-1\}}"] & (0,\pi) \times \mathsf{Gr}_1(n)_{\{s-1\}}. \arrow[u, "(3.12)", "\cong"']
\end{tikzcd}$$

By induction, the bottom horizontal map is a bijection. Because the vertical maps are bijections, it follows that the top horizontal map is a bijection, as desired. This completes the case in which $k = 1$.

Now assume $k > 1$. Corollary 3.0.23 fits the map Rot_S that we seek to show is a bijection, into a commutative diagram among topological spaces:

$$\begin{tikzcd}
(0,\pi)^{Z_S} \arrow[r, "\mathsf{Rot}_S"] \arrow[d, "\mathsf{pr}"] & \mathsf{Gr}_k(n)_S \arrow[d, "(3.9)"] \\
(0,\pi)^{Z_{S_{<k}}} \arrow[r, "\mathsf{Rot}_{S_{<k}}"'] & \mathsf{Gr}_{k-1}(n)_{S_{<k}} \;.
\end{tikzcd} \tag{3.14}$$

By induction, the bottom horizontal map is a homeomorphism, and in particular a bijection. Therefore, to show that the top horizontal map is a bijection, it is sufficient to show that, for each $\Theta_{S_{<k}} \in (0,\pi)^{Z_{S_{<k}}}$, the map between fibers,

$$\mathsf{pr}^{-1}(\Theta_{S_{<k}}) \xrightarrow{(\mathsf{Rot}_S)_|} \Big\{ V \in \mathsf{Gr}_k(n) \mid V \cap \mathbb{R}^{s_k - 1} = \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \Big\} \tag{3.15}$$

is a bijection. So let $\Theta_{S_{<k}} = (\theta_{(i,j)})_{(i,j)\in Z_{S_{<k}}} \in (0,\pi)^{Z_{S_{<k}}}$.

Let us enhance the notation $Z_S^n := Z_S$ to emphasize the implicit dependence on the ambient parameter n. Note the inclusion between linearly ordered sets:

$$Z^{s_k-k+1}_{\{s_k-k+1\}} \longrightarrow Z^n_S ~, \qquad (1,j) \mapsto (k, j+k-1) ~. \tag{3.16}$$

This inclusion is convex, and has the property that for $(i,j) \in Z^{s_k-k+1}_{\{s_k-k+1\}}$, if $(i,j) \leq (i',j')$ in $Z^n_{\{s\}}$, then $(i',j') \in Z^{s_k-k+1}_{\{s_k-k+1\}}$. Further, the image of this inclusion is precisely the complement $Z^n_S \setminus Z^n_{S_{<k}}$. Restriction along (3.16), which is simply projection off of the $Z^n_{S_{<k}}$-factor, thusly defines a homeomorphism:

$$\mathsf{pr}^{-1}(\Theta_{S_{<k}}) \xrightarrow{\text{projection}} (0,\pi)^{Z^{s_k-k+1}_{\{s_k-k+1\}}} ~.$$

The inverse of this homeomorphism is

$$(0,\pi)^{Z^{s_k-k+1}_{\{s_k-k+1\}}} \xrightarrow{\big(\{\Theta_{S_{<k}}\},\mathsf{id}\big)} \{\Theta_{S_{<k}}\} \times (0,\pi)^{Z^{s_k-k+1}_{\{s_k-k+1\}}} = \mathsf{pr}^{-1}(\Theta_{S_{<k}}) ~. \tag{3.17}$$

Now consider the continuous map

$$\{\mathbb{R}^{S_{<k}}\} \times \mathsf{Gr}_1(s_k-k+1) \longrightarrow \mathsf{Gr}_k(s_k) \subset \mathsf{Gr}_k(n) ~, \tag{3.18}$$

$$\big(\mathbb{R}^{S_{<k}}, L \subset \mathbb{R}^{\underline{s_k}\setminus S_{<k}}\big) \mapsto \mathbb{R}^{S_{<k}} \oplus L \subset \mathbb{R}^{s_{k-1}} \oplus \mathbb{R}^{\{s_{k-1}+1<\cdots<s_k\}} = \mathbb{R}^{s_k} \subset \mathbb{R}^n ~.$$

Recall the map

$$\mathcal{I} : \mathsf{Gr}_k(n)_S \to \mathsf{Gr}_{k-1}(n)_{S_{<k}} ~, \qquad V \mapsto V \cap \mathbb{R}^{s_k-1}$$

from Corollary 3.0.23. Note that the map (3.18) takes values in $\mathcal{I}^{-1}(\mathbb{R}^{S_{<k}}) \subset \mathsf{Gr}_k(n)_S$:

$$\{\mathbb{R}^{S_{<k}}\} \times \mathsf{Gr}_1(s_k - k + 1) \longrightarrow \mathcal{I}^{-1}(\mathbb{R}^{S_{<k}}) \ . \tag{3.19}$$

Furthermore, this continuous map (3.19) is a homeomorphism, with inverse given by

$$\mathcal{I}^{-1}(\mathbb{R}^{S_{<k}}) \longrightarrow \{\mathbb{R}^{S_{<k}}\} \times \mathsf{Gr}_1(s_k - k + 1) \ , \qquad V \mapsto (\mathbb{R}^{S_{<k}}, V \perp \mathbb{R}^{S_{<k}}) \ .$$

Consider the orthogonal $n \times n$ matrix

$$R \ := \ \widetilde{\mathsf{Rot}}_{S_{<k}}(\Theta_{S_{<k}})\widetilde{\mathsf{Rot}}_{S_{<k}}(\underline{\frac{\pi}{2}})^{-1} \ \in \ \mathsf{O}(n) \ .$$

This matrix R is just so that the two $(k-1)$-dimensional vector subspaces of $\mathbb{R}^n$,

$$R(\mathbb{R}^{S_{<k}}) \ = \ \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \ , \tag{3.20}$$

agree. Via the canonical action of $\mathsf{O}(n)$ on $\mathsf{Gr}_k(n)$, acting by this matrix R determines a homeomorphism

$$\mathcal{I}^{-1}(\mathbb{R}^{S_{<k}}) \xrightarrow{\cong} \mathcal{I}^{-1}(\mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})) \ , \qquad V \mapsto R(V) \ . \tag{3.21}$$

Concatenating the homeomorphisms (3.19) and (3.21) results in a homeomorphism

$$\mathsf{Gr}_1(s_k - k + 1) = \{\mathbb{R}^{S_{<k}}\} \times \mathsf{Gr}_1(s_k - k + 1) \xrightarrow{(3.19)} \mathcal{I}^{-1}(\mathbb{R}^{S_{<k}}) \xrightarrow{(3.21)} \mathcal{I}^{-1}(\mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})) \ , \tag{3.22}$$

$$L \mapsto R(\mathbb{R}^{S_{<k}} \oplus L) \underset{R \text{ orthog}}{=} R(\mathbb{R}^{S_{<k}}) \oplus R(L) \underset{(3.20)}{=} \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}}) \oplus R(L) \ .$$

Next, note that the map (3.15) fits into a commutative diagram among sets

$$\begin{tikzcd}
(0,\pi)^{Z^{s_k-k+1}_{\{s_k-k+1\}}} \arrow[rr, "\mathsf{Rot}_{\{s_k\}}"] \arrow[d, "\cong"', "(3.17)"] & & \mathsf{Gr}_1(s_k)_{\{s_k\}} \arrow[d, "\cong"', "(3.22)"] \\
\mathsf{pr}^{-1}(\Theta_{S_{<k}}) \arrow[r, "(\mathsf{Rot}_S)_|", "(3.15)"'] & \Big\{V \in \mathsf{Gr}_k(n) \mid V \cap \mathbb{R}^{s_k-1} = \mathsf{Rot}_{S_{<k}}(\Theta_{S_{<k}})\Big\}. &
\end{tikzcd}$$

The above case in which $k = 1$ implies the top horizontal map is a bijection. It follows that the bottom horizontal map is a bijection. This concludes the proof that the map Rot_S of the lemma is a bijection.

It remains to show that the map Rot_S is a homeomorphism. By the universal property of subspace topologies, the commutative diagram (3.14) reveals that Rot_S is a continuous map. The codomain of Rot_S is Hausdorff, because it is a subspace of a Hausdorff topological space. Being an open cube, the domain of Rot_S is compactly generated and Hausdorff. So to show that Rot_S is a homeomorphism, it only remains to show that Rot_S is a proper map. Recall the definition of Rot_S, supported by Corollary 3.0.22, as the restriction of Rot. The domain of Rot is compact and Hausdorff. So to show Rot_S is proper it is sufficient to show that the restriction of Rot to the point-set boundary $\partial[0,\pi]^{Z_S}$ of $(0,\pi)^{Z_S} \subset [0,\pi]^{Z_S}$, regarded as a subspace of the domain $[0,\pi]^{Z_M}$ of Rot, factors through the complement $\mathsf{Gr}_k(n)\setminus\mathsf{Gr}_k(n)_S$:

$$\begin{tikzcd}
\partial[0,\pi]^{Z_S} \arrow[r, dashed, "\mathsf{Rot}_|"] \arrow[d, hook] & \mathsf{Gr}_k(n) \setminus \mathsf{Gr}_k(n)_S \arrow[d, hook] \\
{[0,\pi]^{Z_M}} \arrow[r, "\mathsf{Rot}"] & \mathsf{Gr}_k(n)\ .
\end{tikzcd}$$

So let $(\theta_{(i,j)})_{(i,j)\in Z_S} \in \partial[0,\pi]^{Z_S}$. Let $(i_0, j_0) \in Z_S$ be maximal for which $\theta_{(i,j)} \notin (0,\pi)$. Then, for $r \leq i_0$,

$$\widetilde{\mathsf{Rot}}((\theta_{(i,j)})_{(i,j)\in Z_S})(\mathsf{e}_r) \in \mathbb{R}^{s_{i_0}-1}\ ,$$

yet, for $r > i_0$,

$$\widetilde{\mathsf{Rot}}((\theta_{(i,j)})_{(i,j)\in Z_S})(\mathsf{e}_r) \notin \mathbb{R}^{s_r - 1} \ .$$

It follows that the linear map from the k-dimensional vector subspace

$$\mathsf{Rot}((\theta_{(i,j)})_{(i,j)\in Z_S}) \hookrightarrow \mathbb{R}^n \xrightarrow{\mathsf{proj}} \mathbb{R}^S$$

is *not* an isomorphism. Therefore this k-dimensional vector subspace is *not* an element in the S-stratum:

$$\mathsf{Rot}((\theta_{(i,j)})_{(i,j)\in Z_S}) \ \notin \ \mathsf{Gr}_k(n)_S \ .$$

This completes this proof.

$\square$

Lemmas 3.0.24 and 3.0.18 give the following corollary.

Corollary 3.0.25. *The collection*

$$\left\{(\mathsf{Gr}_k(n)_S, \mathsf{Rot}_S) \mid S \in \left\{ {n \atop k} \right\}\right\}$$

defines a CW structure on $\mathsf{Gr}_k(n)$.

The Schubert CW chain complex

To give a general description of the CW chain complex for the Schubert CW structure on $\mathsf{Gr}_k(n)$, we need to know how many cells there are of each dimension. These can be counted by a partition function. For positive integers k, n with $k < n$, and $1 \leq d \leq k(n-k)$, define

$$p_k^n(d) := \left\{ \{d_1 \leq \cdots \leq d_\ell\} \mid \sum_{1\leq r\leq \ell} d_r = d, \ \ell \leq k, \ 0 < d_r \leq n-k \right\} \ .$$

This is the set of all partitions of d as the sum of at most k positive integers, each of which is less than or equal to $n-k$. For consistency, let us define $p^n_k(0) := \{0\}$.

Lemma 3.0.26. *There is a bijection between the set of d-dimensional cells of the Schubert decomposition of $\mathsf{Gr}_k(n)$ and the set $p^n_k(d)$.*

Proof. Consider the map of sets

$$p^n_k(d) \xrightarrow{\varphi} \left\{ {n \atop k} \right\} ,$$

via the assignment

$$\{d_1 \leq \cdots \leq d_\ell\} \mapsto \{1, 2, \ldots, k-\ell, k-(\ell-1)+i_1, k-(\ell-2)+i_2, \ldots, k+i_\ell\} \ .$$

First note that $\{1, 2, \ldots, k-\ell, k-(\ell-1)+i_1, k-(\ell-2)+i_2, \ldots, k+i_\ell\}$ indeed has cardinality k, and is thus an element of $\left\{ {n \atop k} \right\}$. Observe that this map is injective. We will now show that the image of φ is the collection of sets $S \in \left\{ {n \atop k} \right\}$ such that $d(S) = m$. Let $S = \{s_1 < \cdots < s_k\}$ be such that $d(S) = m$. By Observation 3.0.17,

$$\sum_{r=1}^{k} s_r - r = d \ .$$

Therefore there exists some set $L \subset \{1, \ldots, k\}$ where for each $r \in L$, $s_r - r \neq 0$ and

$$\sum_{r \in L} s_r - r = d \ .$$

Then $I := \{s_r - r \mid r \in L\}$ is an element of $p^n_k(d)$ for which $\varphi(I) = S$. Finally, we show that the only sets in the image of φ are those $S \in \left\{ {n \atop k} \right\}$ for which $d(S) = d$. Given $S = \{1, 2, \ldots, k-\ell, k-(\ell-1)+i_1, \ldots, k+i_\ell\} \in \left\{ {n \atop k} \right\}$ in the image of φ, by Observation

3.0.17, the dimension of the corresponding Schubert cell is

$$d(S) = \sum_{r=1}^{k} s_r - r = i_1 + \cdots + i_\ell = d\ .$$

□

Lemma 3.0.26 tells us that the CW chain complex for the Schubert CW structure on $\mathsf{Gr}_k(n)$ is of the form

$$0 \to \bigoplus_{p_k^n(k(n-k))} \mathbb{Z} \xrightarrow{\partial_{k(n-k)}} \cdots \xrightarrow{\partial_{d+1}} \bigoplus_{p_k^n(d)} \mathbb{Z} \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_1} \bigoplus_{p_k^n(0)} \mathbb{Z} \to 0$$

for some boundary maps ∂_i. In order to compute the homology of this chain complex, we need to have a description of these boundary maps, which is given in Proposition 3.0.46 below.

Homology of stratified spaces

As we saw in the preceeding section, the remaining task to compute the homology of $\mathsf{Gr}_k(n)$ is to determine the differentials. In order to do this, we exploit some natural extra regularity on the Schubert decomposition of $\mathsf{Gr}_k(n)$, namely, a stratification. For the purposes of computing homology, we need to impose some further regularity on this stratification, that of a *conically smooth structure*. In this section we give a brief introduction to the theory of stratified spaces à la [5], including the notion of conical smoothness. Then we discuss how this theory enables us to compute the homology of conically smooth stratified spaces.

Stratified spaces

In this section, we introduce the basic definitions of the theory of stratified spaces that we will use in the remainder of the chapter.

Definition 3.0.27. A *stratified topologcial space* is a triple $(X \overset{\psi}{\to} \mathcal{P})$ consisting of

- a paracompact, Hausdorff topological space, X,
- a poset $\mathcal{P}$, and
- a continuous map $\psi : X \to \mathcal{P}$, where $\mathcal{P}$ is equipped with the downward closed topology: $U \subset \mathcal{P}$ is closed if for each $p \in U$, if $q < p$, then $q \in U$.

Definition 3.0.28. For $X \overset{\varphi}{\to} \mathcal{P}$ a stratified topological space, and $p \in \mathcal{P}$, the *p-stratum* of X is the subspace

$$X_p := \varphi^{-1}(p) \subset X \ .$$

Example 3.0.29. Let M be a smooth manifold, and let $W \hookrightarrow M$ be a properly embedded submanifold. The map

$$M \to \{d < n\} \ , \qquad x \mapsto \begin{cases} 0, & \text{if } x \in W \\ 1, & \text{otherwise} \end{cases}$$

exhibits M as a stratified topological space with strata $M_0 = W$ and $M_1 = M \setminus W$.

Example 3.0.30. Let X be a topological space equipped with a CW structure. The skeleta of the CW structure give rise to a stratification of X. The stratifying poset is the nonnegative integers with their natural partial order, $(\mathbb{Z}_{\geq 0}, \leq)$. For $k \in \mathbb{Z}_{\geq 0}$, let X_k denote the k-skeleton of X. That is, X_k is the union of all cells, X_α, in the CW structure of X such that $\mathsf{dim}(X_\alpha) \leq k$

$$X_k := \bigcup_{\mathsf{dim}(X_\alpha) \leq k} X_\alpha \ .$$

Notice that each element $x \in X$ lies in $X_k \setminus X_{k-1}$ for some unique k. Define the map

$$X \to \mathbb{Z}_{\geq 0} \ , \qquad x \mapsto \ \text{the unique } k \geq 0 \text{ for which } x \in X_k \setminus X_{k-1} \ .$$

This map is continuous precisely because $X_k \subset X$ is closed.

Definition 3.0.31. Let $p \in \mathcal{P}$ be a minimal element of a poset. The *link of $\mathcal{P}$ along p* is defined to be the poset

$$\mathsf{Link}_p(\mathcal{P}) := \mathcal{P}_{p<} := \{q \in \mathcal{P} \mid p < q\} \ .$$

Definition 3.0.32. Let $\mathcal{P}$ and $\mathcal{Q}$ be posets. The *product poset* $\mathcal{P} \times \mathcal{Q}$ has underlying set $\mathcal{P} \times \mathcal{Q}$, and partial order given by declaring $(p, q) \leq (p', q')$ if $p \leq p'$ and $q \leq q'$.

Using the prior two defintions, we can define the blowup of a poset.

Definition 3.0.33. Given a diagram of posets

$$\begin{tikzcd} \mathcal{R} \arrow[r, "f"] \arrow[d, "g"] & \mathcal{Q} \\ \mathcal{P} & \end{tikzcd} ,$$

we can explicitly describe the pushout $\mathcal{P} \coprod_{\mathcal{R}} \mathcal{Q}$ as follows. As a set, $\mathcal{P} \coprod_{\mathcal{R}} \mathcal{Q}$ is the pushout of sets. That is, we quotient the product $\mathcal{P} \times \mathcal{Q}$ by the relation $f(r) \sim g(r)$ for all $r \in R$. The partial orders on $\mathcal{P} \setminus g(\mathcal{R})$ and $\mathcal{Q} \setminus f(\mathcal{R})$ are the given ones. On $[r]$, we declare $[r] \leq p \in \mathcal{P}$ if $g(r) \leq p$ in $\mathcal{P}$. SImilarly, $[r] \leq q \in \mathcal{Q}$ if $f(r) \leq q$ in $\mathcal{Q}$.

Definition 3.0.34. Let $p \in \mathcal{P}$ be a minimal element of a poset. The *blowup of $\mathcal{P}$ along p* is defined to be the poset

$$\mathsf{Bl}_p(\mathcal{P}) := \mathsf{Link}_p(\mathcal{P}) \times \{0 < 1\} \coprod_{\mathsf{Link}_p(\mathcal{P}) \times \{1\}} \mathcal{P} \setminus \{p\} \ .$$

Example 3.0.35. Consider the poset

$$\mathcal{P} = \begin{tikzcd} NW & N \arrow[l] \arrow[r] & NE \\ W \arrow[u] \arrow[d] & O \arrow[l] \arrow[r] \arrow[u] \arrow[d] & E \arrow[u] \arrow[d] \\ SW & S \arrow[l] \arrow[r] & SE \end{tikzcd} .$$

Here, the arrows indicate the partial order, with $a \to b$ meaning $a < b$. The link of $\mathcal{P}$ along O is

$$\mathsf{Link}_O(\mathcal{P}) = \begin{tikzcd} NW & N \arrow[l] \arrow[r] & NE \\ W \arrow[u] \arrow[d] & & E \arrow[u] \arrow[d] \\ SW & S \arrow[l] \arrow[r] & SE \end{tikzcd} ,$$

and the blowup of $\mathcal{P}$ along O is

$$\mathsf{Bl}_O(\mathcal{P}) = \begin{tikzcd} (NW,1) & & (N,1) \arrow[ll] \arrow[rr] & & (NE,1) \\ & (NW,0) \arrow[ul] & (N,0) \arrow[l] \arrow[r] \arrow[u] & (NE,0) \arrow[ur] & \\ (W,1) \arrow[uu] \arrow[dd] & (W,0) \arrow[l] \arrow[u] \arrow[d] & & (E,0) \arrow[r] \arrow[u] \arrow[d] & (E,1) \arrow[uu] \arrow[dd] \\ & (SW,0) \arrow[dl] & (S,0) \arrow[l] \arrow[r] \arrow[d] & (SE,0) \arrow[dr] & \\ (SW,1) & & (S,1) \arrow[ll] \arrow[rr] & & (SE,1) \end{tikzcd} .$$

Definition 3.0.36. Let $X \to \mathcal{P}$ be a smooth manifold equipped with the additional structure of a stratified topological space. Further, assume that each stratum, X_p, is a smooth submanifold of X. For $p \in \mathcal{P}$ a minimal element of the stratifying poset, we define the *link of* X *along* X_p to be the stratified topological space with underlying space the unit sphere bundle of the normal bundle

$$\mathsf{Link}_{X_p}(X) := S(N_{X_p \subset X}) \ .$$

The stratifying poset is given by $\mathsf{Link}_p(\mathcal{P})$.

Note that this definition in fact exhibits the link as a smooth fiber bundle

$$\pi : \mathsf{Link}_{X_p}(X) \to X_p \ .$$

A choice of a tubular neighborhood of $X_p \subset X$ gives us a smooth map $N_{X_p \subset X} \hookrightarrow X$, extending the inclusion $X_p \hookrightarrow X$. We then get a smooth map from the thickened link to X

$$\gamma : \mathsf{Link}_{X_p}(X) \times (0, \infty) \cong N_{X_p \subset X} \setminus X_p \hookrightarrow N_{X_p \subset X} \hookrightarrow X \ .$$

Definition 3.0.37. Let $X \to \mathcal{P}$ be a smooth manifold equipped with the additional structure of a stratified topological space, and let $p \in \mathcal{P}$ be a minimal element. We define the *blowup of X along X_p* to be the pushout

$$\mathsf{Bl}_{X_p}(X) := \big(\mathsf{Link}_{X_p}(X) \times [0, \infty)\big) \coprod_{\mathsf{Link}_{X_p}(X) \times (0,\infty)} X \setminus X_p \ .$$

This space is naturally stratified by the poset $\mathsf{Bl}_p(\mathcal{P})$. Indeed, the natural stratifications $\mathsf{Link}_{X_p}(X) \to \mathsf{Link}_p(\mathcal{P})$ and $[0, \infty) \to \{0 < 1\}$ define a continuous map $\mathsf{Bl}_{X_p}(X) \to \mathsf{Bl}_p(\mathcal{P})$.

Example 3.0.38. We can stratify $\mathbb{R}^2$ by the poset $\mathcal{P}$ from Example 3.0.35. We specify the structure map $\mathbb{R}^2 \to \mathcal{P}$ in Figure 3.1 by labeling the strata. There is a single 0-dimensional stratum $\mathbb{R}^2_O$ consisting of the origin. The resulting blowup and link of $(\mathbb{R}^2 \to \mathcal{P})$ along $\mathbb{R}^2_O$ are given in Figure 3.1.

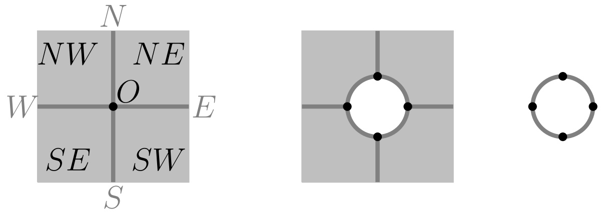


Figure 3.1: Left: $(\mathbb{R}^2 \to \mathcal{P})$ with indicated strata. Middle: the blowup $\mathsf{Bl}_{\mathbb{R}^2_O}(\mathbb{R}^2 \to \mathcal{P})$. Right: the link $\mathsf{Link}_{\mathbb{R}^2_O}(\mathbb{R}^2 \to \mathcal{P})$.

Example 3.0.39. Let $\mathcal{P}$ be a poset. We define the set of *subdivisions* of $\mathcal{P}$ to consist of all subsets $S \subset \mathcal{P}$ such that S is nonempty, finite, and the partial order on S induced by the partial order of $\mathcal{P}$ is a linear order. There is a natural partial order of the subdivisions of a poset given by inclusion of subsets. We denote this poset by $\mathsf{sd}(\mathcal{P})$. The closed interval $[0,1]$ admits a natural stratification by the poset, $\mathsf{sd}(\{0<1\})$. The map $[0,1] \to \mathsf{sd}(\{0<1\})$ is given by

$$x \mapsto \begin{cases} \{0\}, & \text{if } x = 0 \\ \{1\}, & \text{if } x = 1 \\ \{0,1\}, & \text{else} \end{cases} \cdot$$

More generally, let $k \in \mathbb{Z}_{>0}$. The following topological space

$$\mathsf{Cube}^k := [0,1]^{\times k} ,$$

admits a natural stratification by $\mathsf{sd}\left(\{0<1\}\right)^{\times k}$. This can be seen by taking the k-fold product of the above stratified space, $[0,1] \to \mathsf{sd}(\{0<1\})$.

Conical smoothness

By imposing some extra regularity on a stratified topological space, we can use the stratification to compute the homology of the underlying topological space. This is the notion of a *conically smooth structure*, which is analogous to a smooth structure on a topological space. We begin by briefly narrating key features of smooth structures on topological spaces (see [29] for an introduction).

Among paracompact Hausdorff topological spaces, C^0-manifolds are characterized by being locally Euclidean, that is, each point has a neighborhood that is homeomorphic to $\mathbb{R}^i$ for some $i \geq 0$. A *smooth structure* is a type of *regularity* on a C^0-manifold. A C^∞-manifold, or *smooth* manifold, is a C^0-manifold equipped with a smooth structure. There is a distinguished class of continuous maps between two smooth manifolds, called the *smooth* maps. This class of smooth maps consists precisely of those continuous maps that respect this smooth structure. Here, "respect" is just so that points (2) and (3) below are true. This regularity of a smooth manifold M is tailored precisely so that it has the following features:

1. For each point $x \in M$, there is a *tangent space*, T_xM, which is a vector space. This tangent space is a canonical local model of M about x, which is to say there is a basis for the topology about $x \in M$ comprised of images of smooth open embeddings $\varphi_x : T_xM \hookrightarrow M$ each that carries 0 to x

2. For $f : M \to N$ a smooth map, and for each $x \in M$ there is a linear map,

   $$D_xf : T_xM \longrightarrow T_{f(x)}N \ ,$$

   called the *derivative* of f at x. Through a choice of smooth open embeddings $T_xM \xhookrightarrow{\varphi_x} M$ and $T_{f(x)}N \xhookrightarrow{\psi_{f(x)}} N$ as in the above point for which $f(\mathsf{Image}(\varphi_x)) \subset \mathsf{Image}(\psi_{f(x)})$, this derivative can be identified as the limit

   $$T_xM \underset{\cong}{\xrightarrow{\varphi_x}} \mathsf{Image}(\varphi_x) \xrightarrow{f} \mathsf{Image}(\psi_{f(x)}) \underset{\cong}{\xrightarrow{\psi^{-1}_{f(x)}}} T_{f(x)}N \ , \qquad v \mapsto \lim_{t \to 0} \frac{\psi^{-1}_{f(x)} f \varphi_x(tv)}{t} \ . \tag{3.23}$$

3. The map (3.23) depends smoothly on $x \in M$, appropriately interpreted.

Example 3.0.40. Let $f = (f^1, \ldots, f^n)\colon \mathbb{R}^m \to \mathbb{R}^n$ be a sequence of n polynomial maps, each in m variables. Suppose, for each $x \in f^{-1}(0)$, that the total Jacobian matrix of f at x,

$$\left[\partial_i f^j{}_{|x}\right] ,$$

has rank n. The *Regular Value Theorem* states that the subspace $f^{-1}(0) \subset \mathbb{R}^m$ has the natural structure of a smooth manifold.

We now give a similarly brief narrative of features of conically smooth structures on stratified spaces (see [5] for an account). First, we recall a definition.

Definition 3.0.41. For X a topological space, we define the *open cone* of X as the pushout

$$\mathsf{C}(X) := * \coprod_{X\times\{0\}} \big(X \times [0,1)\big) .$$

Among paracompact Hausdorff stratified topological spaces, C^0-stratified spaces are characterized by being locally a product of a Euclidean space and an open cone: $\mathbb{R}^i \times \mathsf{C}(L)$ for some $i \geq 0$ and compact C^0-stratified space L. While this seems circular, because the topological dimension of $\mathsf{C}(L)$ is strictly greater than that of L, this notion can be grounded through induction (on dimension). A *conically smooth* structure is a type of *regularity* on a C^0-stratified space. A C^∞-stratified space, or *conically smooth* stratified space, is a C^0-stratified space equipped with a conically smooth structure. There is a special class of continuous maps between two conically smooth stratified spaces, called the *conically smooth* maps. This class of maps consists precisely of those continuous maps which respect this conically smooth structure. Here,“respect” is just so that points (2) and (3) below are true. This regularity of a conically smooth stratified space X is tailored precisely so that it has the following features:

1. For each point $x \in X$, there is a *tangent cone*, $T_xX \times \mathsf{C}(L_x)$, which is a product of a vector space and an open cone. Such a product has a scaling action by the group $\mathbb{R}_{>0}$, via $t \cdot (v, [s, \ell]) := (tv, [ts, \ell])$. This tangent cone is a canonical local model of X about x, which is to say there is a basis for the topology about $x \in X$ comprised of images of conically smooth open embeddings $\varphi_x \colon T_xX \times \mathsf{C}(L_x) \hookrightarrow X$ each that carries 0 to x.

2. For $f \colon X \to Y$ a conically smooth map, and for each $x \in X$ there is a $\mathbb{R}_{>0}$-equivariant map,
$$D_xf \colon T_xX \times \mathsf{C}(L_x) \longrightarrow T_{f(x)}Y \times \mathsf{C}(L_{f(x)}) ~,$$
called the *derivative* of f at x. Through a choice of smooth open embeddings $T_xX \times \mathsf{C}(L_x) \xrightarrow{\varphi_x} X$ and $T_{f(x)}Y \times \mathsf{C}(L_{f(x)}) \xrightarrow{\psi_{f(x)}} Y$ as in the above point for which $f(\mathsf{Image}(\varphi_x)) \subset \mathsf{Image}(\psi_{f(x)})$, this derivative can be identified as the limit
$$T_xX \times \mathsf{C}(L_x) \xrightarrow[\cong]{\varphi_x} \mathsf{Image}(\varphi_x) \xrightarrow{f} \mathsf{Image}(\psi_{f(x)}) \xrightarrow[\cong]{\psi^{-1}_{f(x)}} T_{f(x)}Y \times \mathsf{C}(L_{f(x)}) ~, \qquad (3.24)$$
$$(v, [s, l]) \mapsto \lim_{t \to 0} \frac{\psi^{-1}_{f(x)} f \varphi_x(tv, [ts, l])}{t} ~.$$

3. The map (3.24) depends conically smoothly on $x \in X$, appropriately interpreted.

Example 3.0.42. We follow up on Example 3.0.40. Let $f = (f^1, \dots, f^n) \colon \mathbb{R}^m \to \mathbb{R}^n$ be a sequence of n polynomial maps each in m variables. The Thom-Mather Theorem [22] states that, with no assumptions on the total Jacobian matrix, the subspace $f^{-1}(0) \subset \mathbb{R}^m$ has the natural structure of a conically smooth stratified space.

Example 3.0.43. We follow up on Example 3.0.29. Let $W \subset M$ be a properly embedded smooth d-submanifold of a smooth n-manifold. Assume $d < n$. Consider the stratified topological space of Example 3.0.29

$$(W \subset M) := \left(M \xrightarrow{x \mapsto d \text{ iff } x \in W} \{d < n\}\right) .$$

So the d-stratum is $W = (W \subset M)_d$, and the n-stratum is $M \setminus W = (W \subset M)_n$. For each $x \in W$, consider the vector space $N_x := \frac{T_x M}{T_x W}$. The unit sphere of the vector space $L_x := S(N_x) := (N_x \setminus 0)_{/\mathbb{R}_{>0}}$ is diffeomorphic with a $(n-d-1)$-sphere. Its open cone $\mathsf{C}(L_x) \cong N_x$ is conically homeomorphic with the normal space, which is linearly isomorphic with $(n-d)$-Euclidean space. A choice of tubular neighborhood of $W \subset M$ determines, for each $x \in W$, an open embedding

$$\varphi_x \colon T_x W \times \mathsf{C}(L_x) \cong T_x W \times N_x W \cong T_x M \hookrightarrow M$$

that carries 0 to x. These open embeddings determine a conically smooth structure on the stratified topological space $(W \subset M)$.

The following is a catalogue of the key features of a conically smooth stratified space $X = (X \to P)$:

1. For each $p \in P$, the stratum X_p is equipped with the structure of a connected smooth manifold.

2. For each strictly related pair $p < q$ in P, there is a smooth manifold $\mathsf{Bl}_{X_p}(X)_q$ with boundary $\mathsf{Link}_{X_p}(X)_q$ as well as a proper quotient map

   $$\overline{\pi}_{p<q} \colon \mathsf{Bl}_{X_p}(X)_q \longrightarrow X_p \cup X_q$$

   to the union in X of the p- and the q-strata. This continuous map $\overline{\pi}_{p<q}$ has the following features.

(a) The preimage of the p-stratum is precisely the boundary of $\mathsf{Bl}_{X_p}(X)_q$; equivalently, the preimage of the q-stratum is precisely the interior of $\mathsf{Bl}_{X_p}(X)_q$:

$$\overline{\pi}_{p<q}{}^{-1}(X_p) \ = \ \mathsf{Link}_{X_p}(X)_q \qquad \text{and} \qquad \overline{\pi}_{p<q}{}^{-1}(X_q) \ = \ \mathsf{Interior}\big(\mathsf{Bl}_{X_p}(X)_q\big) \ .$$

(b) The restriction of $\overline{\pi}_{p<q}$ to the interior of $\mathsf{Bl}_{X_p}(X)_q$ is a diffeomorphism onto the q-stratum:

$$(\overline{\pi}_{p<q})_{|} \colon \mathsf{Interior}\big(\mathsf{Bl}_{X_p}(X)_q\big) \xrightarrow{\cong} X_q \ .$$

(c) The restriction of $\overline{\pi}_{p<q}$ to the boundary of $\mathsf{Bl}_{X_p}(X)_q$ is a proper smooth fiber bundle:

$$\pi_{p<q} \colon \mathsf{Link}_{X_p}(X)_q \longrightarrow X_q \ .$$

Points (a)-(c) can be summarized as a commutative diagram

$$\begin{tikzcd}
\mathsf{Link}_{X_p}(X)_q \arrow[r, "\text{inclusion}"] \arrow[d, "\pi_{p<q}"] & \mathsf{Bl}_{X_p}(X)_q \arrow[d, "\overline{\pi}_{p<q}"] & \mathsf{Interior}\big(\mathsf{Bl}_{X_p}(X)_q\big) \arrow[l, "\text{inclusion}"'] \arrow[d, "\cong"] \\
X_p \arrow[r, "\text{inclusion}"] & X_p \cup X_q & X_q \arrow[l, "\text{inclusion}"']
\end{tikzcd}$$

in which each square is a pullback and the left square is a pushout. In particular, a choice of collaring of the boundary,

$$\overline{\gamma}_{p<q} \colon \mathsf{Link}_{X_p}(X)_q \times [0,1) \hookrightarrow \mathsf{Bl}_{X_p}(X)_q \ ,$$

restricts as a smooth open embedding

$$\gamma_{p<q} \colon \mathsf{Link}_{X_p}(X)_q \times (0,1) \hookrightarrow X_q \ . \tag{3.25}$$

3. For strictly related triples $p < q < r$ in P, there is a smooth manifold with $\langle 2 \rangle$-corners

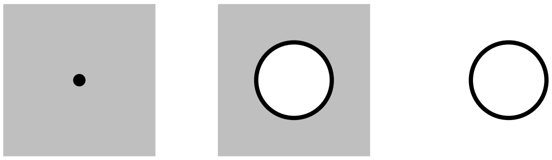


Figure 3.2: Left: the stratified space $(\{0\} \subset \mathbb{R}^2)$. Middle: the blowup $\mathsf{Bl}_{\{0\}}\big((\{0\} \subset \mathbb{R}^2)\big)$. Right: the link $\mathsf{Link}_{\{0\}}\big((\{0\} \subset \mathbb{R}^2)\big)$.

$\mathsf{Bl}_{X_p \cup X_q}(X)_r$, together with a proper quotient map $\overline{\pi}_{p<q<r} \colon \mathsf{Bl}_{X_p \cup X_q}(X)_r \to X_p \cup X_q \cup X_r$ with similar features to (a)-(c) above.

4. Etcetera, for finite strictly monotonic sequences $p_1 < \cdots < p_\ell$ in P.

Example 3.0.44. We follow up on Example 3.0.43. Namely, the smooth structures on the strata $W = (W \subset M)_d$ and on $M \setminus W = (W \subset M)_n$ are the given ones inherited from the given smooth structure on M. The link

$$\Big(\mathsf{Link}_W\big((W \subset M)\big) \xrightarrow{\pi_{d<n}} W\Big) \;=\; \Big(S(N_{W \subset M}) \xrightarrow{\mathsf{pr}} W\Big)$$

is identical with the unit sphere bundle of the normal bundle of $W \subset M$, which is a smooth manifold. The smooth n-manifold with boundary $\mathsf{Bl}_W\big((W \subset M)\big)$ is the *real* blow-up of M along W. The interior of this real blow-up is the complement $M \setminus W$. See Figure 3.2 in the case that $(W \subset M) = (\{0\} \subset \mathbb{R}^2)$.

Notation 3.0.45. Let $X \to \mathcal{P}$ be a conically smooth stratified space, and let $p < q$ be a pair of strictly related elements of $\mathcal{P}$. We will often denote the link simply by

$$\mathsf{L}_{p<q}(X) := \mathsf{Link}_{X_p}(X)_q \ .$$

Links and homology

We now explain how the previously defined links allow one to compute the homology of a conically smooth stratified space. In this section, we restrict attention to topological spaces, X, that possess the following structure:

- X is a smooth manifold;
- X is equipped with a conically smooth structure $X \to \mathcal{P}$ that is compatible with the smooth structure;
- Each stratum, X_p, is diffeomorphic to Euclidean space, and we further fix such a diffeomorphism
$$\alpha_p : X_p \xrightarrow{\cong} \mathbb{R}^{\mathsf{dim}(X_p)} ~,$$
for each $p \in \mathcal{P}$.

This structure gives rise to a well-defined map of posets

$$\mathsf{d} : \mathcal{P} \to \mathbb{Z}_{\geq 0} ~, \qquad p \mapsto \mathsf{dim}(X_p) ~.$$

We let $X_{(i)}$ denote the fiber of the composite $X \to \mathcal{P} \to \mathbb{Z}_{\geq 0}$

$$\begin{tikzcd}
X_{(i)} \arrow[r, hook] \arrow[dd] \arrow[rd, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& P \arrow[d] \\
\{i\} \arrow[r, hook] & \mathbb{Z}_{\geq 0}
\end{tikzcd} ~,$$

and let $X_{(\leq i)}$ denote the pullback

$$\begin{tikzcd}
X_{(\leq i)} \arrow[r, hook] \arrow[dd] \arrow[rdd, phantom, "\lrcorner", very near start] & X \arrow[d] \\
 & P \arrow[d] \\
\mathbb{Z}_{\leq i} \arrow[r, hook] & \mathbb{Z}_{\geq 0}
\end{tikzcd} .$$

The diffeomorphisms, α_p for each $p \in \mathcal{P}$, endows each stratum with an orientation, and thus a quasi-isomorphism

$$C_*^{\mathsf{red}}(X_p^+) \xleftarrow[\cong]{\alpha_p} \mathbb{Z}[\mathsf{dim}(X_p)]$$

between the reduced chain complex of X_p^+ and $\mathbb{Z}$ in dimension $\mathsf{dim}(X_p)$. To each such conically smooth stratified space, we obtain a sequence of graded abelian groups

$$\mathbb{Z}\langle\pi_0(X_{(0)})\rangle[0] \ , \qquad \mathbb{Z}\langle\pi_0(X_{(1)})\rangle[1] \ , \qquad \cdots \qquad \mathbb{Z}\langle\pi_0(X_{(d)})\rangle[d] \ , \qquad 0 \ , \qquad \cdots \ .$$

Here, we use the notation $\mathbb{Z}\langle A\rangle$ to denote the free abelian group generated by the set A.

For $i > 0$, we now name a homomorphism

$$\mathbb{Z}\langle\pi_0(X_{(i)})\rangle \xrightarrow{\partial_i} \mathbb{Z}\langle\pi_0(X_{(i-1)})\rangle \ .$$

To do this is to specify a $\pi_0(X_{(i-1)}) \times \pi_0(X_{(i)})$-matrix. Because each stratum $X_p \subset X$ is connected, there is a canonical bijection

$$\pi_0(X_{(i-1)}) \cong \{p \in \mathcal{P} \mid \mathsf{dim}(X_p) = i - 1\} \subset \mathcal{P} \ .$$

First, for each $p < q$ in $\mathcal{P}$, such that $\mathsf{dim}(X_p) + 1 = \mathsf{dim}(X_q)$, we define a map

$$\sigma_p^q : \pi_0(\mathsf{L}_{p<q}(X)) \to \{\pm 1\} \ , \qquad [\ell] \mapsto \mathsf{det}\left(D_\ell \gamma_{p<q} \circ \left((D_\ell \pi_{p<q})^{-1} \oplus \mathsf{id}\right)\right) \ ,$$

by sending $[\ell]$ to the determinant of the composite linear map

$$\mathbb{R}^{i-1} \oplus \mathbb{R} \cong T_{\pi(\ell)} X_p \oplus \mathbb{R} \xrightarrow{(D_\ell \pi)^{-1} \oplus \mathrm{id}} T_\ell \mathsf{L}_{p<q}(X) \oplus \mathbb{R} \cong T_{(\ell, 1/2)}(\mathsf{L}_{p<q}(X) \times (0,1)) \xrightarrow{D_\ell \gamma} T_{\gamma(\ell)} X_q \cong \mathbb{R}^i \ .$$

Now, we define the homomorphism

$$\partial_i : \mathbb{Z}\langle \pi_0(X_{(i)}) \rangle \to \mathbb{Z}\langle \pi_0(X_{(i-1)}) \rangle$$

by declaring, for $q \in \mathcal{P}_{(i)}$ and $p \in \mathcal{P}_{(i-1)}$, its (q,p)-entry to be

$$(\partial_i)_p^q := \sum_{[\ell] \in \pi_0(\mathsf{L}_{p<q}(X))} \sigma_p^q([\ell]) \ . \tag{3.26}$$

In other words,

$$\partial_i(q) := \sum_{p \in \mathcal{P}_{(i-1)}} \sum_{[\ell]} \sigma_p^q([\ell]) = \sum_{p \in \mathcal{P}_{(i-1)}} \sum_{[\ell]} \mathsf{det}\left(D_\ell \gamma_{p<q} \circ \left((D_\ell \pi_{p<q})^{-1} \oplus \mathsf{id}\right)\right) \ .$$

Proposition 3.0.46. *Let $X \to \mathcal{P}$ be a compact, connected, conically smooth stratified manifold such that each stratum is diffeomorphic to a Euclidean space. The previously defined sequence of abelian groups, and homomorphisms between them*

$$\partial_{i+1} : \mathbb{Z}\langle \pi_0(X_{(i+1)}) \rangle \to \mathbb{Z}\langle \pi_0(X_{(i)}) \rangle \ ,$$

is a chain complex. Furthermore, the homology of this chain complex is isomorphic with $H_*(X;\mathbb{Z})$, the singular homology of X with $\mathbb{Z}$-coefficients.

Proof. The filtration

$$X_{(\leq 0)} \hookrightarrow X_{(\leq 1)} \hookrightarrow \cdots \hookrightarrow X_{(\leq \ell)} \hookrightarrow \cdots \hookrightarrow X \ ,$$

of X induces a filtration of $C_*(X)$,

$$C_*(X_{(\leq 0)}) \to C_*(X_{(\leq 1)}) \to \cdots \to C_*(X_{(\leq \ell)}) \to \cdots \to C_*(X) \ .$$

The E^0 page of the spectral sequence associated to this filtration is

$$E^0_{i,j} = C_{i+j}(X_{(\leq i)})/C_{i+j}(X_{(\leq i-1)}) \ .$$

Hence the E^1-page is

$$E^1_{i,j} = H^{\mathrm{red}}_{i+j}\left(C_{i+j}(X_{(\leq i)})/C_{i+j}(X_{(\leq i-1)}\right) \cong H^{\mathrm{red}}_{i+j}\left(X_{(\leq i)}/X_{(\leq i-1)}\right) \ ,$$

where the homology is taken with respect to the d^1-differential on the E_0-page. Since X is compact and connected,

$$X_{(\leq i)}/X_{(\leq i-1)} \cong (X_{(i)})^+ \ ,$$

where $(X_{(i)})^+$ denotes the one-point compactification of $X_{(i)}$.Further, since each stratum is equipped with a diffeomorphism $\alpha_p : X_p \xrightarrow{\cong} \mathbb{R}^{\mathsf{dim}(X_p)}$, we have the based homeomorphism,

$$X^+_{(i)} \cong_{\alpha^+_p} \left(\coprod_{\pi_0(X_{(i)})} \mathbb{R}^i \right)^+ \cong \bigvee_{\pi_0(X_{(i)})} S^i \ . \tag{3.27}$$

Therefore,

$$E^1_{i,j} \cong H^{\mathrm{red}}_{i+j}\left(\bigvee_{\pi_0(X_{(i)})} S^i\right) \cong \begin{cases} \mathbb{Z}\langle \pi_0(X_{(i)})\rangle, & \text{if } j=0 \\ 0, & \text{otherwise} \end{cases} . \tag{3.28}$$

The d^1-differential is then for each $i \geq 0$, simply a homomorphism

$$d^1_i : \mathbb{Z}\langle \pi_0(X_{(i)})\rangle \to \mathbb{Z}\langle \pi_0(X_{(i-1)})\rangle \ .$$

We will now identify d^1 with the homomorphism (3.26). This will complete the proof of the first statement, since $d^1 \circ d^1 = 0$. For dimension reasons, this spectral sequence collapses at the E^2-page. Indeed, (3.28) reveals that $E^1_{i,\bullet}$ is concentrated in degree $\bullet = 0$. It follows that the d^2-differential is 0. This implies the spectral sequence collapses at the E^2-page. Further, because this filtration is finite, this spectral sequence converges to $H_*(X;\mathbb{Z})$.

The d^1-differential is induced by applying H^{red}_* to the following composite morphism of based spaces

$$\frac{X_{(\leq i)}}{X_{(\leq i-1)}} \xleftarrow{\simeq} \mathsf{Cone}\left(X_{(\leq i-1)} \hookrightarrow X_{(\leq i)}\right) \to \Sigma X^+_{(\leq i-1)} \to \Sigma \frac{X_{(\leq i-1)}}{X_{(\leq i-2)}} \ ,$$

where the first morphism is given by collapsing $X_{(\leq i)}$ in the mapping cone, and the second morphism is the quotient. Consider the continuous map

$$\gamma^!_{(\leq i-1)} : X^+_{(i)} \to \Sigma \mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)})^+ \ ,$$

which is the following composite

$$\begin{tikzcd}
X_{(i)}^{+} \arrow[r, dashed, "\gamma^{!}_{(\leq i-1)}"] \arrow[d, "\cong"'] & \Sigma\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)})^{+} \\
\frac{X_{(\leq i)}}{X_{(\leq i-1)}} & (\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)}) \times (0,1))^{+} \arrow[u, "\cong"'] \\
\frac{\mathsf{Bl}_{X_{(\leq i-1)}}(X_{(\leq i)})}{\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)})} \arrow[u, "\cong"] \arrow[r, "\text{collapse}"] & \frac{(\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)}) \times [0,1))^{+}}{\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)}) \times \{0\}} \arrow[u, "\cong"']
\end{tikzcd} .$$

The top right vertical homeomorphism is using the fact that for Y a locally compact and Hausdorff space, there is a based homeomorphism

$$\Sigma Y^{+} \cong (Y \times (0,1))^{+} \ . \tag{3.29}$$

Further, the collapse map uses

$$\gamma_{\leq i-1} : \mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)}) \times [0,1) \to \mathsf{Bl}_{X_{\leq i-1}}(X_{\leq i})$$

to collapse $\mathsf{Bl}_{X_{\leq i-1}}(X_{\leq i}) \setminus \Big(\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)}) \times [0,1)\Big)$ to the basepoint. So, $\gamma^{!}_{(\leq i-1)}$ evaluates as

$$\gamma^{!}_{(\leq i-1)}(x) = \begin{cases} \gamma^{-1}(x) \ , & \text{if } x \in \mathsf{image}(\gamma) \\ + \ , & \text{else.} \end{cases}$$

Consider the following solid commutative diagram

$$\begin{tikzcd}
X_{(i)}^+ \arrow[r, "\gamma^!_{(\leq i-1)}"] \arrow[rd] \arrow[rr, dashed, bend left, "\gamma^!_{(i-1)}"] & \Sigma\mathsf{Link}_{X_{(\leq i-1)}}(X_{(\leq i)})^+ \arrow[r, "\text{collapse}"] \arrow[d, "\Sigma\pi_{(\leq i-1)}"] & \Sigma\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)})^+ \arrow[d, "\Sigma\pi_{(i-1)}"] \\
& \Sigma X^+_{(\leq i-1)} \arrow[r] & \Sigma X^+_{(i-1)}
\end{tikzcd},$$

Denote the indicated horizontal composite as $\gamma^!_{(i-1)}$. Since the bottom composite induces the d^1 differential upon applying reduced homology, we have identified the d^1 differential with the map induced on homology by the composite $\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}$. Recall from the definition of links that $\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}) \to X_{(i-1)}$ is finite sheeted cover. Each connected component of $X_{(i-1)}$ is homeomorphic to $\mathbb{R}^{i-1}$, so

$$\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}) \cong \left(\coprod_{\pi_0(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}))} \mathbb{R}^{i-1} \right) . \tag{3.30}$$

Thus, there is a based homeomorphism

$$\begin{aligned}
\Sigma\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)})^+ &\overset{(3.29)}{\cong} \Big(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}) \times (0,1)\Big)^+ \\
&\overset{(3.30)}{\cong} \left(\left(\coprod_{\pi_0(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}))} \mathbb{R}^{i-1} \right) \times (0,1) \right)^+ \\
&\cong \bigvee_{\pi_0(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}))} S^i .
\end{aligned} \tag{3.31}$$

Using these identifications together with (3.27), we identify $\gamma^!_{(i-1)}$ as a map between wedges of spheres

$$\gamma^!_{(i-1)} : \bigvee_{\pi_0(X_{(i)})} S^i \cong_{\alpha_i^+} X^+_{(i)} \xrightarrow{\gamma^!_{(i-1)}} \Sigma\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)})^+ \overset{\cong}{\to} \bigvee_{\pi_0(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}))} S^i \ .$$

Here, the first homeomorphism is from (3.27), and the last homeomorphism is from (3.31). We identify $\Sigma\pi_{(i-1)}$ as a map between wedges of spheres,

$$\Sigma\pi_{(i-1)} : \bigvee_{\pi_0(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)}))} S^i \cong \Sigma\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)})^+ \xrightarrow{\Sigma\pi_{(i-1)}} \Sigma X^+_{(i-1)} \cong \bigvee_{\pi_0(X_{(i-1)})} S^i \ .$$

Here, the first homeomorphism is from (3.31) and the last homeomorphism is the suspension of (3.27). Thus, upon applying H^{red}_*, the d^1-differential is a composite homomorphism

$$\gamma^!_{(i-1)} : \mathbb{Z}\langle \pi_0(X_{(i)}) \rangle \to \mathbb{Z}\langle \pi_0(\mathsf{Link}_{X_{(i-1)}}(X_{(i-1)} \cup X_{(i)})^+) \rangle \to \mathbb{Z}\langle \pi_0(X_{(i-1)}) \rangle \ .$$

So, for $q \in \mathcal{P}_{(i+1)}$ and $p \in \mathcal{P}_{(i)}$, the (p, q)-entry of the $\mathcal{P}_{(i)} \times \mathcal{P}_{(i+1)}$-matrix associated to this homomorphism is

$$\left(\gamma^!_{(i-1)}\right)^q_p = \mathsf{deg}\left(S^{i+1} \overset{q}{\hookrightarrow} (S^{i+1})^{\vee \mathcal{P}_{(i+1)}} \xrightarrow{\Sigma\pi_{(i)} \circ \gamma^!_{(i)}} (S^i)^{\vee \mathcal{P}_{(i)}} \overset{p}{\to} S^i \right) \ .$$

The arrow labeled q signifies the inclusions of sphere associated to q, and the arrow labeled by p signifies the projection onto the sphere associated to p. To identify the d^1-differential with (3.26), and thus complete the proof, we now identify these degrees.

The degree of a smooth map $f : X \to Y$ between compact, oriented smooth manifolds can be computed by choosing a regular value $y \in Y$, and then taking a count of preimages of y, signed according to whether f is orientation preserving or reversing at that point:

$$\mathsf{deg}(f) = \sum_{x \in f^{-1}(y)} \mathsf{sgn}\left(\mathsf{det}(D_x f)\right) \ ,$$

(see [14], for instance). Note that $\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}$ is smooth away from the basepoint. Further, each element of $\pi_0(X_{(i)})$ is a stratum X_q, for some $q \in \mathcal{P}$, such that $d(X_q) = i$. Similarly, each element of $\pi_0(X_{(i-1)})$ is a stratum X_p, for some $p \in \mathcal{P}$, such that $d(X_p) = i-1$. Thus, we can compute the degree by computing the degree of the induced map

$$\left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)_{p<q} : S^i_q := X^+_q \to (X_p \times (0,1))^+ =: S^i_p \ ,$$

for each $p < q \in \mathcal{P}$ with $d(p) + 1 = d(q)$. Choosing a regular value, $y \in S^i_p$, we have

$$\mathsf{deg}\left(\left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)_{p<q}\right) = \sum_{x \in \left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)^{-1}_{p<q}(y)} \mathsf{sgn}\left(\mathsf{det}\left(D_x \left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)_{p<q}\right)\right) \ .$$

Observe that

$$\mathsf{card}\left(\left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)^{-1}_{p<q}(y)\right) = \mathsf{card}(\pi_0(\mathsf{L}_{p<q}(X))) \ .$$

It just remains to establish that

$$\mathsf{sgn}\left(\mathsf{det}\left(D_x \left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)_{p<q}\right)\right) = \mathsf{sgn}(\mathsf{det}(D_\ell\gamma \circ ((D_\ell\pi)^{-1} \oplus \mathsf{id}_{(0,\varepsilon)}))) \ ,$$

as in (3.26). Since $\left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)_{p<q}$ is given as the composite, we have

$$D_x \left(\Sigma\pi_{(i-1)} \circ \gamma^!_{(i-1)}\right)_{p<q} = (D_\ell\pi \oplus \mathsf{id}_{(0,\varepsilon)}) \circ (D_x\gamma)^{-1} \ ,$$

where $\ell = \gamma^{-1}_{p<q}(x)$. Hence the sign of this determinant matches that of Equation (3.26),

which completes the proof. $\square$

The Schubert stratification

In Section 3, we showed that the cells of the Grassmannian $\mathsf{Gr}_k(n)$ are parameterized by the set

$$\genfrac\{\}{0pt}{}{n}{k} := \{S \subset \{1, \dots, n\} \mid \mathsf{card}(S) = k\} \ .$$

Since each element in $\genfrac\{\}{0pt}{}{n}{k}$ corresponds to a cell in the CW structure on $\mathsf{Gr}_k(n)$, there is a canonical map

$$\mathsf{Gr}_k(n) \to \genfrac\{\}{0pt}{}{n}{k}$$

given by sending a k-plane to the set indexing its cell. This realizes $\mathsf{Gr}_k(n)$ as a stratfied topological space with S-stratum the S-cell:

$$\mathsf{Gr}_k(n)_S := \{V \in \mathsf{Gr}_k(n) \mid S \text{ is the maximal element in } \genfrac\{\}{0pt}{}{n}{k} \text{ for which } V \in U_S\} \ .$$

In particular, recall from Lemma 3.0.24 that each stratum of $\mathsf{Gr}_k(n)$ is diffeomorphic to Euclidean space. In fact, the following theorem is proven in a forthcoming paper.

Theorem 3.0.47. *For $0 \leq k \leq n$, the Schubert stratification of the Grassmannian $\mathsf{Gr}_k(n)$ can be naturally upgraded to the structure of a conically smooth stratified space. With respect to this stratification, for each pair $S < T$ in $\genfrac\{\}{0pt}{}{n}{k}$ with $d(S){+}1 = d(T)$, there are identifications*

$$\begin{tikzcd}
\mathsf{Gr}_k(n)_S & \mathsf{L}_{S<T}(\mathsf{Gr}_k(n)) \arrow[l, "\pi_{S<T}"'] & & \mathsf{L}_{S<T}(\mathsf{Gr}_k(n)) \times (0,\pi) \arrow[r, "\gamma_{S<T}"] & \mathsf{Gr}_k(n)_T \\
(0,\pi)^{Z_S} \arrow[u, "\cong", "\mathsf{Rot}_S"'] & \mathsf{L}_{S<T} \arrow[l, "\mathsf{proj}"'] \arrow[u, "\cong"] & \text{and} & \mathsf{L}_{S<T} \times (0,\pi) \arrow[u, "\cong"] \arrow[r, "\mathsf{swap} \amalg \mathsf{Rswap}"] & (0,\pi)^{Z_T} \arrow[u, "\cong", "\mathsf{Rot}_T"']
\end{tikzcd} \ .$$

Here,

$$\mathsf{L}_{S<T} := \left\{ \left(\theta_{(i,j)}\right) \in [0,\pi]^{Z_T} \middle|\ \textit{for all } (i,j) \in Z_T,\ \textit{if } (i,j) \in Z_S, \theta_{(i,j)} \in (0,\pi),\ \textit{else } \theta_{(i,j)} \in \{0,\pi\} \right\}$$
$$\cong (0,\pi)^{Z_S} \times \{0,\pi\}\ .$$

We now give a description of the map

$$\mathsf{swap} \amalg \mathsf{Rswap} : \mathsf{L}_{S<T} \times (0,1) \cong (0,\pi)^{Z_S} \times \{0,\pi\} \times (0,1) \to (0,\pi)^{Z_T} \tag{3.32}$$

that appears in the statement of Theorem 3.0.47. Let $S < T \in \left\{ {n \atop k} \right\}$ with $d(S) + 1 = d(T)$. Recall that $d(S) = \mathsf{card}(Z_S)$, so we have $\mathsf{card}(Z_T \setminus Z_S) = 1$. Let $\Theta_S = (\theta^S_{(i,j)})_{Z_S} \in (0,\pi)^{Z_S}$. The map (3.32) is given by the following two maps. The first map

$$\mathsf{swap} : (0,\pi)^{Z_S} \times \{0\} \times (0,1) \to (0,\pi)^{Z_T}$$

maps $(\Theta_S, 0, \theta) \mapsto \Theta_T = (\theta^T_{(i,j)})_{Z_T}$ where

$$\theta^T_{(i,j)} = \begin{cases} \theta^S_{(i,j)}\ , & \text{if } (i,j) \in Z_S \\ \theta\ , & \text{if } (i,j) \in Z_T \setminus Z_S \end{cases}\ .$$

That is, swap takes the extra coordinate in the domain and swaps it into the $Z_T \setminus Z_S$-coordinate in the codomain Z_T. The second map

$$\mathsf{Rswap} : (0,\pi)^{Z_S} \times \{\pi\} \times (0,1) \to (0,\pi)^{Z_T}$$

maps $(\Theta_S, 0, \theta) \mapsto \Theta_T = (\theta^T_{(i,j)})_{Z_T}$ where

$$\theta^T_{(i,j)} = \begin{cases} \theta^S_{(i,j)} \ , & \text{if } (i,j) \in Z_S \\ 1 - \theta \ , & \text{if } (i,j) \in Z_T \setminus Z_S \end{cases} \ .$$

That is, swap takes the extra coordinate in the domain and swaps it into the $Z_T \setminus Z_S$-coordinate of the codomain Z_T, except with reversed orientation.

As stated by Proposition 3.0.46, to compute the homology of $\mathsf{Gr}_k(n)$, we need to understand the differential given in (3.26). In particular, for $S < T \in \left\{ {n \atop k} \right\}$ with $d(S) + 1 = d(T)$, we must understand the map

$$\sigma^T_S : \pi_0(\mathsf{L}_{S<T}) \to \{\pm 1\} \ , \qquad [\ell] \mapsto \mathsf{sgn}\,\mathsf{det}\left(D_\ell \gamma^T_S \circ ((D_\ell \pi^T_S)^{-1} \oplus \mathsf{id})\right)$$

whose value on $[\ell]$ is the sign of the determinant of the composite map

$$\begin{tikzcd}
\mathbb{R}^i \oplus \mathbb{R} & & \mathbb{R}^{i+1} \\
T_{\pi^T_S(\ell)}\mathsf{Gr}_k(n)_S \oplus \mathbb{R} \arrow[u, phantom, sloped, "\cong"] \arrow[r, "(D_\ell \pi^T_S)^{-1} \oplus \mathrm{id}"] & T_\ell \mathsf{L}_{S<T} \oplus \mathbb{R} \cong T_{(\ell, \varepsilon/2)}(\mathsf{L}_{S<T} \times (0, \varepsilon)) \arrow[r, "D_\ell \gamma^T_S"] & T_{\gamma^T_S(\ell)}\mathsf{Gr}_k(n)_T \arrow[u, phantom, sloped, "\cong"] \ .
\end{tikzcd}$$

By the chain rule, each such value can be computed as the product of signs

$$\mathsf{sgn}\,\mathsf{det}\left((D_\ell \pi^T_S)^{-1}\right) \cdot \mathsf{sgn}\,\mathsf{det}\left(D_\ell \gamma^T_S\right) \ .$$

So, let us fix $S < T \in \left\{ {n \atop k} \right\}$ with $d(S) + 1 = d(T)$.

The following immediate consequence of Theorem 3.0.47 tells us that we only need to compute σ^T_S for two values.

Corollary 3.0.48. *For* $S < T \in \left\{ {n \atop k} \right\}$ *with* $d(S) + 1 = d(T)$, $\pi_0(\mathsf{L}_{S<T}) \cong \{0, \pi\}$.

Let us fix two such values, one in each connected component of $\mathsf{L}_{S<T}$:

$$\Theta^0_{S<T} = (\theta^0_{(i,j)})_{(i,j)\in Z_T} \ , \qquad \theta^0_{(i,j)} := \begin{cases} \pi/2, & \text{if } (i,j) \in Z_S \\ 0, & \text{if } (i,j) \in Z_T \setminus Z_S \end{cases} \ ,$$

and

$$\Theta^\pi_{S<T} = (\theta^\pi_{(i,j)})_{(i,j)\in Z_T} \ , \qquad \theta^\pi_{(i,j)} := \begin{cases} \pi/2, & \text{if } (i,j) \in Z_S \\ \pi, & \text{if } (i,j) \in Z_T \setminus Z_S \end{cases} \ .$$

We will now unpack the maps $D_\ell \pi^T_S$ and $D_\ell \gamma^T_S$. Let us now fix $S < T \in \left\{ {n \atop k} \right\}$ with $d(S) + 1 = d(T)$. Let $M = \{n-k+1 < \cdots < n\} \in \left\{ {n \atop k} \right\}$ be the maximal element.

By Theorem 3.0.47, we have the following commutative diagrams of spaces:

$$\begin{tikzcd}
\mathsf{L}_{S<T} \arrow[rr, "\pi_{S<T}"] \arrow[d, "\mathsf{pr}"] & & \mathsf{Gr}_k(n)_S \arrow[d, hook, "\mathsf{inc}_S"] \\
(0,\pi)^{Z_S} \arrow[rr, "\mathsf{Rot}_S"] \arrow[dr, "\widetilde{\mathsf{Rot}_S}"'] & & \mathsf{Gr}_k(n) \\
& \mathsf{V}^o_k(n) \arrow[ur, "\mathsf{col}"'] &
\end{tikzcd} \ , \qquad
\begin{tikzcd}
\mathsf{L}_{S<T} \times (0,\pi) \arrow[rr, "\gamma_{S<T}"] \arrow[d, "\mathsf{swap} \amalg \mathsf{Rswap}"] & & \mathsf{Gr}_k(n)_T \arrow[d, hook, "\mathsf{inc}_S"] \\
(0,\pi)^{Z_T} \arrow[rr, "\mathsf{Rot}_T"] \arrow[dr, "\widetilde{\mathsf{Rot}_T}"'] & & \mathsf{Gr}_k(n) \\
& \mathsf{V}^o_k(n) \arrow[ur, "\mathsf{col}"'] &
\end{tikzcd} \ ,$$

from which we compute $D_\ell \pi_{S<T}$ and $D_\ell \gamma_{S<T}$ as their respective composites.

Remark 3.0.49. Let $S = \{s_1 < \cdots < s_k\} < T = \{t_1 < \cdots < t_k\} \in \left\{ {n \atop k} \right\}$ with $d(S) + 1 = d(T)$. We claim there exists a unique integer $1 \leq i_{S<T} \leq k$ for which $s_{i_{S<T}} + 1 = t_{i_{S<T}}$. Indeed, the condition $S < T$ implies that for each $1 \leq i \leq k$, we have $s_i \leq t_i$. Now, recall that

$$d(S) := \sum_{i=1}^{k} s_i - i \ , \qquad d(T) := \sum_{i=1}^{k} t_i - i \ .$$

Thus, $d(S) + 1 = d(T)$ if and only if there exists precisely one integer $1 \leq i_{S<T} \leq k$ for which $s_{i_{S<T}} + 1 = t_{i_{S<T}}$.

Lemma 3.0.50. *Through the linear isomorphism*

$$\mathsf{Mat}_{\underline{n}\setminus S\times S}\xrightarrow{\cong}\mathsf{Hom}(\mathbb{R}^S,\mathbb{R}^{\underline{n}\setminus S})=T_{\mathbf{0}}\mathsf{Hom}(\mathbb{R}^S,\mathbb{R}^{\underline{n}\setminus S})\xrightarrow[\cong]{D_0\mathsf{Graph}_S}T_{\mathbb{R}^S}\mathsf{Gr}_k(n)\ ,$$

the vector subspace $T_{\mathbb{R}^S}\mathsf{Gr}_k(n)_S\subset T_{\mathbb{R}^S}\mathsf{Gr}_k(n)$ is identified as the vector subspace of $\mathsf{Mat}_{\underline{n}\setminus S\times S}$ consisting of those F such that for all $1\leq s<r\leq n$ with $s\in S$ and $r\in\underline{n}\setminus S$, the (r,s)-entry of F is 0.

Proof. Recall from Proposition 3.0.8 the open embedding

$$\mathsf{Mat}_{\underline{n}\setminus S\times S}\xrightarrow{\mathsf{Graph}_S}\mathsf{Gr}_k(n)\ ,\qquad F\mapsto\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S}\right)\ .$$

Here, $S\star(\underline{n}\setminus S)$ denotes the linearly ordered set in which all elements of S are less than those of $\underline{n}\setminus S$. Thus, we have a nested sequence of subspaces

$$\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_1}}\right)\subset\cdots\subset\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_{k-1}}}\right)\subset\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_k}}\right)\ .$$

By Lemma 3.0.7, $\mathsf{Graph}_S(F)\in\mathsf{Gr}_k(n)_S$ if and only if for each $1\leq i\leq k$, the element $s_i\in S$ is the maximal element for which

$$\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_i}}\right)\subset\mathbb{R}^{s_i}\ ,\qquad\text{yet}\qquad\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_i}}\right)\not\subset\mathbb{R}^{s_i-1}\ .$$

For $1\leq i\leq k$, the condition on F from the statement of the lemma directly implies the composite

$$\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S)_{\geq s_i})\times S_{\leq s_i}}\right)\hookrightarrow \mathbb{R}^{S\star(\underline{n}\setminus S)}\xrightarrow{\mathsf{pr}}\mathbb{R}^{\{r\}} \tag{3.33}$$

is the zero map if $r > s_i$ and is not the zero map for $r = s_i$. In particular, this implies

$$\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S)_{\geq s_i})\times S_{< s_i}}\right)\subset \mathsf{ker}\left(\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S)_{\geq s_i})\times S_{\leq s_i}}\right)\hookrightarrow \mathbb{R}^{S\star(\underline{n}\setminus S)}\xrightarrow{\mathsf{pr}}\mathbb{R}^{\{s_i<\cdots<n\}}\right). \tag{3.34}$$

The left hand side of (3.34) has dimension $i-1$ by inspection. Equation (3.33) implies that the right hand side of (3.34) also has dimension $i-1$. Therefore, for each $1 \leq i \leq k$, the element s_i is the maximal element of S for which

$$\mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_i}}\right)\subset \mathbb{R}^{s_i}, \qquad \text{yet} \qquad \mathsf{col}\left(\begin{bmatrix}\mathbb{1}\\ F\end{bmatrix}_{(S\star(\underline{n}\setminus S))\times S_{\leq s_i}}\right)\not\subset \mathbb{R}^{s_i-1}.$$

Thus, $\mathsf{Graph}_S(F) \in \mathsf{Gr}_k(n)_S$, as desired. □

Lemma 3.0.51. *For $S < T \in \left\{ {n \atop k} \right\}$ with $d(S)+1 = d(T)$,*

$$\sigma_S^T([\ell]) = \begin{cases} (-1)^{1+k+i_{S<T}+\sum\limits_{i=i_{S<T}+1}^{k} s_i-i}, & \text{if } \ell \in [0] \in \pi_0(\mathsf{L}_{S<T}) \\ (-1)^{\sum\limits_{i=i_{S<T}}^{k} s_i-i}, & \text{if } \ell \in [\pi] \in \pi_0(\mathsf{L}_{S<T}) \end{cases},$$

where $1 \leq i_{S<T} \leq k$ is the index for which $s_{i_{S<T}} + 1 = t_{i_{S<T}}$.

Proof. Let us first consider the derivative of col. For $A \in V_k^o(n)$, $T_A V_k^o(n) = \{V \in \mathsf{Mat}_{n\times k} \mid V^T A + A^T V = 0\}$, and $T_{\mathsf{col}(A)}(\mathsf{Gr}_k(n)) \cong \mathsf{Hom}_{\mathsf{lin}}(\mathsf{col}(A), \mathsf{col}(A)^{\perp}) \cong \mathsf{Mat}_{(n-k)\times k}$. Thus, a basis for $T_{\mathsf{col}(A)}(\mathsf{Gr}_k(n))$ is given by the set $\mathcal{M} := \{\mathsf{M}_{(i,j)} \mid 1 \leq i \leq n \setminus k,\ 1 \leq j \leq k\}$, where $\mathsf{M}_{(i,j)}$ has a 1 in the (i,j) entry and all other entries are 0. For $\mathsf{col}(A) \in \mathsf{Gr}_k(n)_S$, Lemma 3.0.50 says $T_{\mathsf{col}(A)}(\mathsf{Gr}_k(n)_S)$ has a basis given by

$$\mathcal{M}_S := \{\mathsf{M}_{(i,j)} \mid 1 \leq j \leq k,\ 1 \leq i < s_j\} \subset \mathcal{M}\ .$$

Note that we will refer to elements of $\mathcal{M}$ or $\mathcal{M}_S$ as the indexing pairs, (i,j). Recall that $\mathsf{col} : V_k^o(n) \to \mathsf{Gr}_k(n)$ takes an $n \times k$ matrix to its column space. Let $A \in \{\widetilde{\mathsf{Rot}}_S(\Theta^0_{S<T}), \widetilde{\mathsf{Rot}}_S(\Theta^\pi_{S<T})\}$. The surjective submersion $\mathsf{O}(n) \to V_k^o(n)$ tells us that for each $V \in T_A V_k^o(n)$, there exists $\widetilde{V} \in \mathsf{Skew}(n)$ for which $e^{t\widetilde{V}}A$ represents the equivalence class of the tangent vector V. Let $\mathbb{1}_{(\underline{n}\setminus S)\times n}$ denote the $n \times n$ identity matrix with the rows labeled by S removed. We compute the derivative of col at $A \in V_k^o(n)$ via the derivative of the following path

$$\mathbb{R} \to V_k^o(n)\ , \qquad t \mapsto e^{t\widetilde{V}A}\ .$$

Restricting to an open chart $\mathsf{Mat}_{(\underline{n}\setminus S)\times k} \hookrightarrow \mathsf{Gr}_k(n)$, we have the following diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, dashed] \arrow[d] & \mathsf{Mat}_{(\underline{n}\setminus S)\times k} \arrow[d, hook] \\
V_k^o(n) \arrow[r] & \mathsf{Gr}_k(n)\ .
\end{tikzcd}$$

The dashed arrow is given by

$$t \mapsto \mathbb{1}_{(\underline{n}\setminus S)\times n} e^{t\widetilde{V}} A (A^T e^{t\widetilde{V}} A)^{-1}\ .$$

We now use this path to compute $D_A\mathsf{col}$ for $A \in \{\widetilde{\mathsf{Rot}}_S(\Theta^0_{S<T}), \widetilde{\mathsf{Rot}}_S(\Theta^\pi_{S<T})\}$. Namely, the derivative of this path at $t = 0$ is

$$\left.\frac{d}{dt}\right|_{t=0} \left(\mathbb{1}_{(\underline{n}\setminus S)\times n} e^{t\widetilde{V}} A (A^T e^{t\widetilde{V}} A)^{-1}\right) = \mathbb{1}_{(\underline{n}\setminus S)\times n} \widetilde{V} A\ .$$

So, when $A = \widetilde{\mathsf{Rot}}_S(\Theta^0_{S<T})$, we have an equaltiy of $n \times k$-matrices $\widetilde{V}A = V$. Thus,

$$D_{\Theta^0_{S<T}}\mathsf{col} : V_k^o(n) \to \mathsf{Gr}_k(n)\ , \qquad V \mapsto \mathbb{1}_{\underline{n}\setminus S \times k}V$$

sends V to the $\underline{n} \setminus S \times k$ matrix that consists of the $\underline{n} \setminus S$ rows of V. When $A = \widetilde{\mathsf{Rot}}_S(\Theta^\pi_{S<T})$, the $n \times k$-matrix $\widetilde{V}A$ is V except with the $i_{S<T}$-column negated. Let $\mathbb{1}^-_{\underline{n}\setminus S\times k}$ denote $\mathbb{1}_{\underline{n}\setminus S\times k}$ except with the $i_{S<T}$-column negated. Then,

$$D_{\Theta^\pi_{S<T}}\mathsf{col} : V_k^o(n) \to \mathsf{Gr}_k(n)\ , \qquad V \mapsto \mathbb{1}^-_{\underline{n}\setminus S \times k}V$$

sends V to the matrix that consists of the $\underline{n} \setminus S$ rows of V, but with the $i_{S<T}$-column negated. Therefore, $T_A V_k^o(n) = \{V \in \mathsf{Mat}_{n\times k} \mid V^T A + A^T V = 0\}$ has a natural basis consisting of two types of matrices. The first type consists of $n \times k$-matrices whose only nonzero entry is the (i, j) entry, for $(i, j) \notin S \times S$, one such matrix for each such (i, j). The second type of matrices will have their nonzero entries concentrated in the rows and columns labeled by S. Note that our description of the derivative shows that for both 0 and π, the derivative sends all matrices of the second type to 0. Note that $T_{\mathsf{col}(A)}\mathsf{Gr}_k(n) \cong \mathsf{Mat}_{(n-k)\times k}$ has a natural basis consisting of the matrices of the first type described above. Thus, deleting the second type of basis elements of $T_A V_k^o(n)$, $D_{\widetilde{\mathsf{Rot}}_S(\Theta^0_{S<T})}\mathsf{col}$ is simply the identity matrix. Since we are only interested in those basis elements for which $(i, j) \in \mathcal{M}_S$, as discussed above, we conclude that

$$\mathsf{det}(D_{\widetilde{\mathsf{Rot}}_S(\Theta^0_{S<T})}\mathsf{col}) = 1\ .$$

Also, we see $D_{\widetilde{\mathsf{Rot}}_S(\Theta^\pi_{S<T})}\mathsf{col}$ is the identity matrix except where all columns labeled by (i, j) for which $j = i_{S<T}$ are negated. Thus, this will introduce a negative sign in the determinant

$$\mathsf{det}(D_{\widetilde{\mathsf{Rot}}_S(\Theta^\pi_{S<T})}\mathsf{col})$$

for each basis element of the first type. Namely, one negative for each $(i,j) \in \underline{n} \times \underline{k}$ for which $(i,j) \notin S \times S$ and $j - i_{S<T}$. Therefore.

$$\mathsf{det}(D_{\widetilde{\mathsf{Rot}}_S(\Theta^{\pi}_{S<T})}\mathsf{col}) = (-1)^{\mathsf{card}\{(i,j)\in\underline{n}\times\underline{k} \ \mid \ (i,j)\notin S\times S \ , \ j=i_{S<T}\}} = (-1)^{s_{i_{S<T}}-i_{S<T}} \ .$$

Note that $\widetilde{\mathsf{Rot}}_S(\Theta^0_{S<T}) = \widetilde{\mathsf{Rot}}_S(\Theta^{\pi}_{S<T}) \in V^o_k(n)$ is the $n \times k$ matrix whose ith column is e_{s_i}. We will first consider

$$D_{\Theta^0_{S<T}}\widetilde{\mathsf{Rot}}_S = \left[\left(\frac{\partial\widetilde{\mathsf{Rot}}_S}{\partial\theta_{(i,j)}}\right)_{xy}\right] \ .$$

That is, for each pair, $(i,j) \in Z_S$ and $(x,y) \in \mathcal{M}_S$, we will identify the (x,y)-entry in the matrix $\frac{\partial\widetilde{\mathsf{Rot}}_S}{\partial\theta_{(i,j)}}$. Note that we ignore those entries indexed by elements $(x,y) \in \mathcal{M} \setminus \mathcal{M}_S$, as $D\mathsf{col}$ will map those to 0, as discussed above. Let us now fix $(i,j) \in Z_S$. Then

$$\frac{\partial\widetilde{\mathsf{Rot}}_S}{\partial\theta_{(i,j)}}(\Theta^0_{S<T}) = \left(\prod_{(i',j')<(i,j)} \mathsf{R}_{j'}(\theta_{(i',j')})\right) \frac{\partial\mathsf{R}_j}{\partial\theta_{(i,j)}}\bigg|_0 \left(\prod_{(i'',j'')>(i,j)} \mathsf{R}_{j''}(\theta_{(i'',j'')})\right) \ .$$

For ease of notation, let us denote

$$\mathsf{Rot}_S^{<(i,j)} := \prod_{(i',j')<(i,j)} \mathsf{R}_{j'}(\theta_{(i',j')}) \ ,$$

and

$$\mathsf{Rot}_S^{>(i,j)} := \prod_{(i'',j'')>(i,j)} \mathsf{R}_{j''}(\theta_{(i'',j'')}) \ .$$

Recall that $\theta_{(i',j')} = \pi/2$ for each $(i',j') \in Z_S$, so each entry in $\frac{\partial\mathsf{R}_j}{\partial\theta_{(i,j)}}\Big|_0$ is zero, except for the (j,j) and $(j+1,j+1)$ entries which are both -1. Thus, we only need to know which e_r gets sent to e_j and e_{j+1} under the map $\mathsf{Rot}_S^{>(i,j)}$. By definition, $\mathsf{Rot}_S^{>(i,j)}$ is a product of cyclic permutation matrices

$$[\mathsf{e}_i \mapsto \mathsf{e}_j][\mathsf{e}_{i+1} \mapsto \mathsf{e}_{s_{i+1}}]\cdots[\mathsf{e}_k \mapsto \mathsf{e}_{s_k}] \ .$$

The first block $[\mathsf{e}_i \mapsto \mathsf{e}_j]$ sends

$$\mathsf{e}_r \mapsto \begin{cases} \mathsf{e}_j, & \text{if } r = i \\ \mathsf{e}_{j+1}, & \text{if } r = j+1 \end{cases} .$$

For the next block, $[\mathsf{e}_{i+1} \mapsto \mathsf{e}_{s_{i+1}}]$, we care about what gets sent to e_i and e_{j+1}. We see that $[\mathsf{e}_{i+1} \mapsto \mathsf{e}_{s_{i+1}}]$ sends

$$\mathsf{e}_r \mapsto \begin{cases} \mathsf{e}_i, & \text{if } r = i \\ -\mathsf{e}_{j+1}, & \text{if } r = j+2 \end{cases} .$$

Similarly, the next block will fix e_i, and send e_{j+3} to $-\mathsf{e}_{j+2}$. Since there are $k-i$ blocks after $[\mathsf{e}_i \mapsto \mathsf{e}_j]$, we see that

$$\left.\frac{\partial \mathsf{R}_j}{\partial \theta_{(i,j)}}\right|_0 \circ \mathsf{Rot}_S^{>(i,j)} : \mathsf{e}_r \mapsto \begin{cases} -\mathsf{e}_j, & \text{if } r = i \\ (-1)^{k-i+1}\mathsf{e}_{j+1}, & \text{if } r = j+k-i+1 \\ 0, & \text{else} \end{cases} .$$

Next, we will determine where $\mathsf{Rot}_S^{<(i,j)}$ sends e_j and e_{j+1}. Note that $\widetilde{\mathsf{Rot}}_S$ is the following product of cyclic permutation matrices

$$[\mathsf{e}_1 \mapsto \mathsf{e}_{s_1}] \cdots [\mathsf{e}_{i-1} \mapsto \mathsf{e}_{s_{i-1}}][\mathsf{e}_{j+1} \mapsto \mathsf{e}_{s_i}] \ . \tag{3.35}$$

The $[\mathsf{e}_{j+1} \mapsto \mathsf{e}_{s_i}]$ block fixes e_j and sends $\mathsf{e}_{j+1} \mapsto \mathsf{e}_{s_i}$. The next block from the right, $[\mathsf{e}_{i-1} \mapsto \mathsf{e}_{s_{i-1}}]$ sends

$$\mathsf{e}_j \mapsto \begin{cases} \mathsf{e}_j, & \text{if } j > s_{i-1} \\ -\mathsf{e}_{j-1}, & \text{else} \end{cases} .$$

Similarly, the next block sends

$$\mathsf{e}_{j-1} \mapsto \begin{cases} \mathsf{e}_{j-1}, & \text{if } j-1 > s_{i-2} \\ -\mathsf{e}_{j-2}, & \text{else} \end{cases} .$$

Thus we see that the composite in (3.35) will move e_j some number of times. We let $\beta_S(i,j) := \mathsf{card}\{1 \leq \ell < i \mid j+1-\ell \leq s_{i-\ell}\}$ denote this number. As indicated in the formulas above, each movement of e_j also introduces a negative. Thus, we see that $\widetilde{\mathsf{Rot}}_S$ sends $\mathsf{e}_j \mapsto (-1)^\beta \mathsf{e}_{j-\beta}$ and $\mathsf{e}_{j+1} \mapsto \mathsf{e}_{s_i}$. Therefore, the composite $\frac{\partial \widetilde{\mathsf{Rot}}_S}{\partial \theta_{(i,j)}}(\Theta^0_{S<T})$ maps $\mathsf{e}_i \mapsto (-1)^{\beta+1}\mathsf{e}_{j-\beta}$ and $\mathsf{e}_{j+k-i+1} \mapsto (-1)^{k-i+1}\mathsf{e}_{s_i}$. As discussed above, these are the only values that matter. Namely, $\left.\frac{\partial \mathsf{R}_j}{\partial \theta_{(i,j)}}\right|_0$ only has two non-zero entries, so all other basis elements get sent to 0.. The (i,j) column of

$$D_{\Theta^0_{S<T}} \widetilde{\mathsf{Rot}}_S = \left[\left(\frac{\partial \widetilde{\mathsf{Rot}}_S}{\partial \theta_{(i,j)}} \right)^y_x \right]$$

has a single nonzero entry of $(-1)^{\beta+1}$ in the $(j-\beta, i) \in \mathcal{M}_S$ row. To compute the determinant of $D_{\Theta^0_{S<T}} \widetilde{\mathsf{Rot}}_S$, we first observe that for $(i,j) < (i, j-1) \in Z_S$, there is an inequality, $j-1-\beta_S(i,j-1) < j - \beta_S(i,j)$. Therefore, ignoring signs, $D_{\Theta^0_{S<T}} \widetilde{\mathsf{Rot}}_S$, is a block sum of antidiagonal matrices. There will be a block of size $s_i - i$ for each $1 \leq i \leq k$, so ignoring signs, the determinant of $D_{\Theta^0_{S<T}} \widetilde{\mathsf{Rot}}_S$ is

$$(-1)^{\sum\limits_{i=1}^{k} \sum\limits_{r=1}^{s_i-i-1} r} .$$

The number of negative entries of $D_{\Theta^0_{S<T}}$ is given by

$$\sum_{(i,j) \in Z_S} (\beta_S(i,j) + 1) = d(S) + \sum_{(i,j) \in Z_S} \beta_S(i,j) .$$

All told,

$$\mathsf{det}(D_{\Theta^0_{S<T}}\widetilde{\mathsf{Rot}}_S) = (-1)^{\sum\limits_{i=1}^{k}\sum\limits_{r=1}^{s_i-i-1} r}(-1)^{d(S)+\sum\limits_{(i,j)\in Z_S}\beta_S(i,j)} .$$

The only difference in computing $\mathsf{det}(D_{\Theta^\pi_{S<T}}\widetilde{\mathsf{Rot}}_S)$, as opposed to $\mathsf{det}(D_{\Theta^0_{S<T}}\widetilde{\mathsf{Rot}}_S)$, is that the matrix in the $(i_{S<T}, S_{i_{S<T}})$-entry is no longer the identity matrix, but rather the diagonal matrix all of whose diagronal entries are 1, except the $(i_{S<T}, i_{S<T})$- and $(i_{S<T}+1, i_{S<T}+1)$-entries are both -1. For $(i,j) \in Z_S$, if $(i,j) < (i_{S<T}, s_{i_{S<T}})$, then $i < i_{S<T}$, and thus $\widetilde{\mathsf{Rot}}_S$ still sends $e_i \mapsto e_j$ as in the $\Theta^0_{S<T}$ case. Let us now consider the case $(i,j) > (i_{S<T}, s_{i_{S<T}})$ in Z_S. If $i = i_{S<T}$, then the effect on $\widetilde{\mathsf{Rot}}_S$ is that the $i_{S<T}$ block now sends $\mathsf{e}_{j+1} \mapsto -\mathsf{e}_{s_{i_{S<T}}}$ and $\mathsf{e}_{i_{S<T}+1} \mapsto -\mathsf{e}_{i_{S<T}+1}$. Thus, this block sends $\mathsf{e}_j \mapsto \mathsf{e}_j$ as in the $\Theta^0_{S<T}$ case. The last case to consider is that $i > i_{S<T}$. The effect on $\widetilde{\mathsf{Rot}}_S$ is that the $i_{S<T}$ block now sends $\mathsf{e}_{i_{S<T}} \mapsto -\mathsf{e}_{i_{S<T}}$ and $\mathsf{e}_{i_{S<T}+1} \mapsto -\mathsf{e}_{i_{S<T}+1}$. This will introduce an extra factor of -1 on the image e_j precisely if $j+1-(i-i_{S<T}) = s_{i_{S<T}}+1$, or more simply if $j-i = s_{i_{S<T}} - i_{S<T}$. Thus, the determinant of $D_{\Theta^\pi_{S<T}}\widetilde{\mathsf{Rot}}_S$ will have one extra factor (compared to $D_{\Theta^0_{S<T}}\widetilde{\mathsf{Rot}}_S$) of -1 for each pair $(i,j) \in Z_S$ for which $j-i = s_{i_{S<T}} - i_{S<T}$. Note that

$$\mathsf{card}\{(i,j) \in Z_S \mid j-i = s_{i_{S<T}} - i_{S<T}\} = k - i_{S<T} .$$

Therefore,

$$\mathsf{det}(D_{\Theta^\pi_{S<T}}\widetilde{\mathsf{Rot}}_S) = (-1)^{\sum\limits_{i=1}^{k}\sum\limits_{r=1}^{s_i-i-1} r}(-1)^{d(S)+\sum\limits_{(i,j)\in Z_S}\beta_S(i,j)}(-1)^{k-i_{S<T}} .$$

Recall that $\mathsf{swap} : (0,\pi)^{Z_S} \times \{0\} \times (0,\pi) \to Z_T$ simply moves the last coordinate to the $(i_{S<T}, s_{i_{S<T}})$-coordinate in Z_T. Thus, the derivative of swap is a matrix consisting of 1's along the diagonal until the $(i_{S<T}, s_{i_{S<T}})$-row. The $(i_{S<T}, s_{i_{S<T}})$-row will consist of all zeros, except the final column will be a 1. There will be 1's along the subdiagonal, and all other entries are 0. Likewise, the derivative of Rswap will be the same as the derivative of swap,

except the last column of the $(i_{S<T}, s_{i_{S<T}})$-row will be a -1, since Rswap is orientation reversing in that factor. Thus,

$$\mathsf{det}(D_{\Theta^0_{S<T}}\mathsf{swap}) = (-1)^{\mathsf{card}\{(i,j)\in Z_T \ | \ (i,j)>(I_{S<T}, s_{I_{S<T}})\}} \ ,$$

and

$$\mathsf{det}(D_{\Theta^0_{S<T}}\mathsf{Rswap}) = (-1)^{\mathsf{card}\{(i,j)\in Z_T \ | \ (i,j)>(I_{S<T}, s_{I_{S<T}})\}+1} \ .$$

All told, we see that $\sigma^T_S([\Theta^0_{S<T}])$ is equal to the number

$$\begin{aligned}&(-1)^{\sum_{i=1}^{k}\sum_{r=1}^{s_i-i-1} r+d(S)+\sum_{(i,j)\in Z_S}\beta_S(i,j)+\sum_{i=1}^{k}\sum_{r=1}^{t_i-i-1} r+d(T)+\sum_{(i,j)\in Z_T}\beta_T(i,j)+\mathsf{card}\{(i,j)\in Z_T \ | \ (i,j)>(i_{S<T}, s_{i_{S<T}})\}}\\ &\quad= (-1)^{1+\sum_{i=1}^{k}\sum_{r=1}^{s_i-i-1} r+\sum_{(i,j)\in Z_S}\beta_S(i,j)+\sum_{i=1}^{k}\sum_{r=1}^{t_i-i-1} r+\sum_{(i,j)\in Z_T}\beta_T(i,j)+\mathsf{card}\{(i,j)\in Z_T \ | \ (i,j)>(i_{S<T}, s_{i_{S<T}})\}} \ ,\end{aligned}$$

since $d(S) + d(T) = 2d(S) + 1$. Notice that for $i \neq i_{S<T}$,

$$\sum_{r=1}^{t_i-i-1} r = \sum_{r=1}^{s_i-i-1} r \ .$$

Further, $t_{i_{S<T}} = s_{i_{S<T}} + 1$. Thus,

$$(-1)^{\sum_{i=1}^{k}\sum_{r=1}^{s_i-i-1} r+\sum_{i=1}^{k}\sum_{r=1}^{t_i-i-1} r} = (-1)^{s_{i_{S<T}}-i_{S<T}} \ .$$

Next, note that

$$\mathsf{card}\{(i,j)\in Z_T \ | \ (i,j) > (i_{S<T}, s_{i_{S<T}})\} = \sum_{i=i_{S<T}}^{k} (s_i - i) \ .$$

To assess when $\beta_S(i,j) \neq \beta_T(i,j)$, we must figure out if there exists $1 \leq \ell < i$ for which $j + 1 - \ell \leq t_{i-\ell}$, yet $j + 1 - \ell > s_{i-\ell}$. Since $s_i = t_i$ for all i except $i = i_{S<T}$, the only ℓ for

which this could possibly hold is $\ell = i - i_{S<T}$. Then, we are seeking $(i,j) \in Z_S$ for which $j - i + i_{S<T} + 1 \leq t_{i_{S<T}} = s_{i_{S<T}} + 1$ and $j - i + i_{S<T} + 1 > s_{i_{S<T}}$. Both of these inequalities hold precisely if $j - i = s_{i_{S<T}} - i_{S<T}$. Therefore,

$$(-1)^{\sum\limits_{(i,j)\in Z_S} \beta_S(i,j) + \sum\limits_{(i,j)\in Z_T} \beta_T(i,j)} = (-1)^{\beta_T(i_{S<T}, s_{i_{S<T}}) + \mathsf{card}\{(i,j)\in Z_S \mid j-i = s_{i_{S<T}} - i_{S<T}\}} = (-1)^{k - i_{S<T}},$$

since $\beta_T(i_{S<T}, s_{i_{S<T}}) = 0$, and $\mathsf{card}\{(i,j) \in Z_S \mid j - i = s_{i_{S<T}} - i_{S<T}\} = k - i_{S<T}$. Hence, the formula for $\sigma_S^T([\Theta^0_{S<T}])$ is proven. The above reductions also yield the stated formula for $\sigma_S^T([\Theta^\pi_{S<T}])$. □

Observation 3.0.52. We can further simplify the boundary formula

$$\begin{aligned}
\sigma_S^T([\Theta^0_{S<T}]) + \sigma_S^T([\Theta^\pi_{S<T}]) &= \sum (-1)^{1+k+i_{S<T}+\sum\limits_{i=i_{S<T}+1}^{k} s_i - i} + (-1)^{\sum\limits_{i=i_{S<T}}^{k} s_i - i} \\
&= (-1)^{i_{S<T} + \sum_{i=i_{S<T}+1}^{k}(s_i - i)} \left((-1)^{(s_{i_{S<T}})} - (-1)^k \right) \\
&= (-1)^{\sum_{i=i_{S<T}}^{k}(s_i - i)} \left(1 - (-1)^{k - s_{i_{S<T}}} \right) \\
&= \begin{cases} (-1)^{\sum_{i=i_{S<T}}^{k}(s_i - i)} 2, & \text{if } S < T \text{ and } k \not\equiv s_{i_{S<T}} \mod 2 \\ 0, & \text{otherwise} \end{cases} .
\end{aligned}$$

Lemma 3.0.51 gives a simple, easy to compute formula for the boundary maps in $C_*^{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z})$, the integral chain complex of the Schubert CW structure on $\mathsf{Gr}_k(n)$. Note that this gives us a description of $C^*_{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z})$, the integral cochain complex. Namely, $C^i_{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z}) = C_i^{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z})$. The differential $C^i_{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z}) \to C^{i+1}_{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z})$ is simply the transpose of the differential $C_{i+1}^{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z}) \to C_i^{\mathsf{Sch}}(\mathsf{Gr}_k(n); \mathbb{Z})$. In other words, given $S < T \in \left\{ {n \atop k} \right\}$ with $d(T) = d(S) + 1$, the coefficient $\sigma_T^S = \sigma_S^T$. This is codified in Theorem 3.0.53 below.

For each $0 \leq r \leq k$, consider the map

$$d_r : \left\{ {n \atop k} \right\} \xrightarrow{\{s_1<\cdots<s_k\}\mapsto\sum_{r\leq i\leq k} s_i - i} \mathbb{Z}_{\geq 0} \ .$$

Theorem 3.0.53. *The Schubert CW cochain complex of $\mathsf{Gr}_k(n)$ with $\mathbb{Z}$-coefficients $(C^*_{\mathsf{Sch}}(\mathsf{Gr}_k(n);\mathbb{Z}),\partial)$, has underlying graded abelian group given by the free graded abelian group on the graded set $\left\{ {n \atop k} \right\} \xrightarrow{-d_1} \mathbb{Z}$. The differential $\partial^i : C^i(\mathsf{Gr}_k(n);\mathbb{Z}) \to C^{i+1}(\mathsf{Gr}_k(n);\mathbb{Z})$ is given by*

$$\partial^i : S = \{s_1 < \cdots < s_k\} \ \mapsto \sum_{r\in\{1\leq r\leq k \ \mid \ s_{r+1}-s_r>1 \text{ and } k-s_r \text{ is odd}\}} (-1)^{d_{r-1}(S)} 2 \cdot S_r \ , \tag{3.36}$$

where $S_r := \{s_1 < \cdots < s_{r-1} < s_r + 1 < s_{r+1} < \cdots < s_k\} \in \left\{ {n \atop k} \right\}$.

Observation 3.0.54 (Arbitrary coefficients)**.** Thus, for the Schubert CW chain complex with coefficients in a commutative ring R, we see that $C^*_{\mathsf{Sch}}(\mathsf{Gr}_k(n);R)$ has the same underlying chain groups, and differential specified by

$$\sigma^S_T([\Theta^0_{S<T}]) + \sigma^S_T([\Theta^\pi_{S<T}]) = \begin{cases} (-1)^{\sum_{i=i_{S<T}}^{k}(s_i-i)} 2, & \text{if } S < T \text{ and } k \not\equiv s_{i_{S<T}} \mod 2 \\ 0, & \text{otherwise} \end{cases} .$$

Thus, if $0 = 2$ in R, the differentials are all 0, which in particular, recovers the case of $\mathbb{Z}/2\mathbb{Z}$ coefficients, e.g. [24]. These computations should also specialize to agree with those of [7] and [28], though we do not verify this.

The cohomology of $\mathsf{Gr}_k(n)$

The visualization of $C^*_{\mathsf{Sch}}(\mathsf{Gr}_6(12);\mathbb{Z})$ from Figures 3.4 and 3.6 suggests the chain complex can be written as a finite direct sum of cubes. We will now prove this remarkable

$$\begin{tikzcd}
 & & 134 \arrow[r, "0"] \arrow[dr, "-2"] & 234 \arrow[r, "-2"] & 235 \arrow[r, "0"] \arrow[dr, "0"] & 245 \arrow[r, "-2"] \arrow[dr, "0"] & 345 \arrow[r, "0"] & 346 \arrow[dr, "-2"] & & \\
123 \arrow[r, "0"] & 124 \arrow[ur, "-2"] \arrow[dr, "-2"] & & 135 \arrow[r, "0"] \arrow[ur, "0"] \arrow[dr, "0"] & 145 \arrow[ur, "0"] \arrow[dr, "0"] & 236 \arrow[r, "0"] & 246 \arrow[ur, "2"] \arrow[dr, "-2"] & & 356 \arrow[r, "0"] & 456 \\
 & & 125 \arrow[r, "0"] \arrow[ur, "2"] & 126 \arrow[r, "-2"] & 136 \arrow[r, "0"] \arrow[ur, "0"] & 146 \arrow[r, "-2"] \arrow[ur, "0"] & 156 \arrow[r, "0"] & 256 \arrow[ur, "-2"] & &
\end{tikzcd}$$

Figure 3.3: A visual of the chain complex for $\mathsf{Gr}_3(6)$. Underlying this figure is the poset $\left\{ {6 \atop 3} \right\}$, with each relation arrow labeled by the coefficient of the differential.

observation in Theorem 3.0.56. A consequence of this theorem is a closed formula for the R-cohomology of $\mathsf{Gr}_k(n)$, which is proven in Corollary 3.0.61.

For $S \in \left\{ {n \atop k} \right\}$, define

$$\mathsf{In}(S) := \{ i \in \underline{k} \mid s_i \equiv k \pmod 2 \text{ and } s_{i-1} < s_i - 1 \} \subset \underline{k} \ ,$$

and

$$\mathsf{Out}(S) := \{ i \in \underline{k} \mid s_i \not\equiv k \pmod 2 \text{ and } s_i + 1 < s_{i+1} \} \subset \underline{k} \ ,$$

where we set $s_0 := 0$, and $s_{k+1} := n+1$, for notational purposes. Further, define

$$\left\{ {n \atop k} \right\}_{\mathsf{Out}} := \left\{ S \in \left\{ {n \atop k} \right\} \mid \mathsf{Out}(S) = \emptyset \right\} \ , \qquad \left\{ {n \atop k} \right\}_{\mathsf{In}} := \left\{ S \in \left\{ {n \atop k} \right\} \mid \mathsf{In}(S) = \emptyset \right\} \ ,$$

$$\left\{ {n \atop k} \right\}_{\mathsf{Min}} := \left\{ S \in \left\{ {n \atop k} \right\} \mid \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{In}(S) \right\} \ .$$

Consider the subposet $\left\{ {n \atop k} \right\}^{\mathsf{adm}} \subset \left\{ {n \atop k} \right\}$ consisting of the same objects as $\left\{ {n \atop k} \right\}$, yet only those relations $S < T$ that factor as a sequence of relations $S = U_0 < U_1 < \cdots < U_\ell = T$ in which for all $0 < r \leq \ell$, we have $d(U_r) - d(U_{r-1}) = 1$ and $i_{U_{r-1} < U_r} \not\equiv k \bmod 2$. This subposet is generated by the relations $S < T$ for which $d(T) - d(S) = 1$ and $\sigma^S_T([\Theta^0_{S<T}]) + \sigma^S_T([\Theta^\pi_{S<T}]) \neq 0$.

Observation 3.0.55. Let $S < T$ be a relation in $\left\{ {n \atop k} \right\}$. This relation belongs to $\left\{ {n \atop k} \right\}^{\mathsf{adm}}$ if and only if $s_i = t_i - 1$ for all $i \in \mathsf{Out}(S) \cap \mathsf{In}(T)$ and $s_i = t_i$ for all $i \notin \mathsf{Out}(S) \cap \mathsf{In}(T)$.

We are now prepared to state our main result, whose proof relies on lemmas that follow.

Theorem 3.0.56. *Let R be a commutative ring. There is an isomorphism of chain complexes*

$$C_*^{\mathit{Sch}}(\mathit{Gr}_k(n);\mathbb{Z}) \cong \bigoplus_{S\in \left\{{n \atop k}\right\}_{\mathit{Out}}} \mathit{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\otimes \mathit{In}(S)}[d(S) - \mathit{card}(\mathit{In}(S))]$$

$$\cong \left(\bigoplus_{S\in \left\{{n \atop k}\right\}_{\mathit{Out}} \setminus \left\{{n \atop k}\right\}_{\mathit{In}}} \mathbb{Z}[d(S)] \right) \oplus \left(\bigoplus_{S\in \left\{{n \atop k}\right\}_{\mathit{Min}}} \mathit{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[d(S)-1] \right) .$$

Proof. The first isomorphism is immediate from Lemmas 3.0.57 and 3.0.58. The first summand of the second isomorphism comes from the $S \in \left\{{n \atop k}\right\}$ such that $\mathsf{In}(S) = \emptyset$. The second factor follows from Lemma 3.0.59 below. □

See Figure 3.4 for a depiction of $C^*_{\mathsf{Sch}}(\mathsf{Gr}_5(10); R)$ and Figure 3.5 for a depiction of the isomorphic complex

$$\left(\bigoplus_{S\in \left\{{10 \atop 5}\right\}_{\mathsf{Out}} \setminus \left\{{n \atop k}\right\}_{\mathsf{In}}} \mathbb{Z}[d(S)] \right) \oplus \left(\bigoplus_{S\in \left\{{10 \atop 5}\right\}_{\mathsf{Min}}} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[d(S)-1] \right) .$$

We now prove the lemmas that yield the proof of Theorem 3.0.56.

Lemma 3.0.57. *There is an isomorphism of chain complexes*

$$C_*^{\mathit{Sch}}(\mathit{Gr}_k(n);\mathbb{Z}) \cong \bigoplus_{S\in \left\{{n \atop k}\right\}_{\mathit{Out}}} C_*(\mathit{Sub}(\mathit{In}(S))^{\mathit{op}};\mathbb{Z}) .$$

Proof. We can regard $\left\{{n \atop k}\right\}^{\mathsf{adm}}$ as a weighted level graph with weights given by the coefficients $\sigma_T^S([\Theta^0_{S<T}]) + \sigma_T^S([\Theta^\pi_{S<T}])$, and the level given by the dimension of the corresponding Schubert cells. Recall there is a bijection between weighted level graphs whose adjacency matrices square to the zero matrix and chain complexes. Since $C_*^{\mathsf{Sch}}(\mathsf{Gr}_k(n))$ is

a chain complex, and $\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}$ precisely selects out the relations $S < T$ with $\sigma_T^S([\Theta^0_{S<T}]) + \sigma_T^S([\Theta^\pi_{S<T}]) \neq 0$, the adjacency matrix of $(\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}})^{\mathsf{op}}$ squares to the zero matrix. Let us denote the chain complex associated to $(\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}})^{\mathsf{op}}$ by $C_*((\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}})^{\mathsf{op}})$. Thus, by construction, $C_*^{\mathsf{Sch}}(\mathsf{Gr}_k(n)) \cong C_*((\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}})^{\mathsf{op}})$. Next, we claim the canonical functor between posets

$$\coprod_{S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}} \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}_{\leq S} \to \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}$$

is an isomorphism. To prove this, note that for each $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}$, the canonical inclusion $\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}_{\leq S} \hookrightarrow \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}$ is fully faithful. Thus, to prove the claim, it is enough to show that for each $T \in \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}$, there exists a unique $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}$ such that $T \in \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}_{\leq S}$. So, for $T \in \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}$, define a set $S^T \in \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}$ by defining $S_i^T = T_i$ in $i \notin \mathsf{Out}(T)$, and $S_i^T = T_i + 1$ otherwise. By construction, $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}$. Observation 3.0.55 implies that $T \in \genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}_{\leq S^T}$, and further that S^T is the unique element of $\genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}$ for which this is true. Now, observe that for $S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}$, there is an isomorphism of posets $\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}_{\leq S} \xrightarrow{\cong} \mathsf{Sub}(\mathsf{In}(S))$ given by sending $T \mapsto \mathsf{Out}(T)$. Therefore,

$$\begin{aligned} C_*^{\mathsf{Sch}}(\mathsf{Gr}_k(n)) &\cong C_*\left((\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}})^{\mathsf{op}}\right) \\ &\cong C_*\left(\coprod_{S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}} (\genfrac{\{}{\}}{0pt}{}{n}{k}^{\mathsf{adm}}_{\leq S})^{\mathsf{op}}\right) \\ &\cong C_*\left(\coprod_{S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}} \mathsf{Sub}(\mathsf{In}(S))^{\mathsf{op}}\right) \\ &\cong \bigoplus_{S \in \genfrac{\{}{\}}{0pt}{}{n}{k}_{\mathsf{Out}}} C_*\left(\mathsf{Sub}(\mathsf{In}(S))^{\mathsf{op}}\right) . \end{aligned}$$

□

Lemma 3.0.58. *There is an isomorphism of chain complexes*

$$C_*(\mathit{Sub}(\mathit{In}(S)^{op})) \cong \mathit{Cone}\left(\mathbb{Z} \xrightarrow{2} \mathbb{Z}\right)^{\otimes \mathit{In}(S)} .$$

Proof. In terms of their associated weighted level graphs, note that both underlying digraphs are isomorphic to the poset $\mathsf{Sub}(\{1 < \cdots < \mathsf{card}(\mathsf{In}(S))\})$. Further, the weights in both weighted level graphs are all ± 2. We will now prove that for any cardinality $r \in \mathbb{Z}_{\geq 0}$, any two choices of weights drawn from the set $\{\pm 2\}$ on the digraph $\mathsf{Sub}(\underline{r})$, such that their adjacency matrices square to 0, are isomorphic. In fact, proceeding by induction on r, we only need to prove the case $r = 2$, as the cases $r = 0, 1$ are clear. Namely, we will show that given any solid diagrams with weights $u_*^*, v_*^* \in \{\pm 2\}$,

$$\begin{tikzcd}
& \{1,2\} \arrow[rrr, dashed, bend left, "f_{12}"] & & & \{1,2\} & \\
\{1\} \arrow[ur, "u_1^{12}"] \arrow[rrr, dashed, "f_1"] & & \{2\} \arrow[ul, "u_2^{12}"'] \arrow[rrr, dashed, "f_2"'] & \{1\} \arrow[ur, "v_1^{12}"] & & \{2\} \arrow[ul, "v_2^{12}"'] \\
& \emptyset \arrow[ul, "u_\emptyset^1"] \arrow[ur, "u_\emptyset^2"'] \arrow[rrr, dashed, bend right, "f_\emptyset"'] & & & \emptyset \arrow[ul, "v_\emptyset^1"] \arrow[ur, "v_\emptyset^2"'] &
\end{tikzcd}$$

there exists filler arrows, $f_\emptyset, f_1, f_2, f_{12} \in \{\pm 1\}$. Further, f_1, f_2, and f_{12} are uniquely determined by $f_\emptyset$. The commutivity conditions give the follwing four equations

$$f_1 u_\emptyset^1 = v_\emptyset^1 f_\emptyset \tag{3.37a}$$

$$f_2 u_\emptyset^2 = v_\emptyset^2 f_\emptyset \tag{3.37b}$$

$$f_{12}u_1^{12} = v_1^{12}f_1 \qquad (3.38a) \qquad\qquad f_{12}u_2^{12} = v_2^{12}f_2\ . \qquad (3.38b)$$

The fact that each solid diagram defines a chain complex results in the following two equations

$$u_1^{12}u_\emptyset^1 = u_2^{12}u_\emptyset^2 \qquad (3.39a) \qquad\qquad v_1^{12}v_\emptyset^1 = v_2^{12}v_\emptyset^2\ . \qquad (3.39b)$$

Equations (3.37a) and (3.37b) uniquely determine f_1 and f_2 in terms of $f_\emptyset$. Multplying (3.37a) on the left by v_1^{12} and substituting in (3.38a) and (3.39b) yields $f_{12}u_1^{12}u_\emptyset^1 = v_2^{12}v_\emptyset^2 f_\emptyset$, which uniquely determines f_{12} in terms of $f_\emptyset$. It remains to show that this value of f_{12} is compatible with the value determined by (3.38b). Multiplying (3.37b) on the left by v_2^{12} and substituting in (3.38b) and (3.39a) also yields the equation $f_{12}u_1^{12}u_\emptyset^1 = v_2^{12}v_\emptyset^2 f_\emptyset$. Thus, there is a consistent value of f_{12} uniquely determined by $f_\emptyset$. ☐

Lemma 3.0.59. *For $r > 0$, there is an isomorphism of chain complexes*

$$\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\otimes r} \cong \bigoplus_{0 \leq a \leq r-1} \left(\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[a]\right)^{\oplus \binom{r-1}{a}}\ .$$

Proof. We will prove this by induction on r. As $r = 1$ is clear, consider the case $r = 2$.

The following diagram exhibits such an isomorphism

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\begin{bmatrix} 2 \\ 2 \end{bmatrix}"] \arrow[d, "1"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\begin{bmatrix} 2 & -2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}"] & \mathbb{Z} \arrow[d, "-1"] \\
\mathbb{Z} \arrow[r, "\begin{bmatrix} 2 \\ 0 \end{bmatrix}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\begin{bmatrix} 0 & 2 \end{bmatrix}"'] & \mathbb{Z}
\end{tikzcd} .$$

Assume $r > 1$, then

$$\begin{aligned}
&\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\otimes r} \\
&\qquad = \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\otimes r-1} \\
&\qquad \cong \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes \left(\bigoplus_{0 \le a \le r-2} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[a]^{\oplus \binom{r-2}{a}} \right) \\
&\qquad \cong \bigoplus_{0 \le a \le r-2} \Big(\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \Big) [a]^{\oplus \binom{r-2}{a}} \\
&\qquad \cong \bigoplus_{0 \le a \le r-2} \Big(\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \oplus \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \Big) [a]^{\oplus \binom{r-2}{a}} \\
&\qquad \cong \left(\bigoplus_{0 \le a \le r-2} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[a]^{\oplus \binom{r-2}{a}} \right) \oplus \left(\bigoplus_{0 \le a \le r-2} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[a+1]^{\oplus \binom{r-2}{a}} \right) \\
&\qquad \cong \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \\
&\qquad\quad \oplus \left(\bigoplus_{0 < a \le r-2} \Big(\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\oplus \binom{r-2}{a}} \oplus \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})^{\oplus \binom{r-2}{a-1}} \Big) [a] \right) \oplus \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[r-1] \\
&\qquad \cong \bigoplus_{0 \le a \le r-1} \mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z})[a]^{\oplus \binom{r-1}{a}} .
\end{aligned}$$

□

We now prove several corollaries of Theorem 3.0.56. In particular, these will provide a closed formula for the R-cohomology of $\mathsf{Gr}_k(n)$.

Corollary 3.0.60. *Let R be a commutative ring. There is an isomorphism of chain complexes*

$$C^*_{\mathsf{Sch}}(\mathsf{Gr}_k(n);R) \cong \left(\bigoplus_{S \in \left\{ n \atop k \right\}_{\mathsf{Out}} \cap \left\{ n \atop k \right\}_{\mathsf{In}}} R[-d(s)] \right) \oplus \left(\bigoplus_{S \in \left\{ n \atop k \right\}_{\mathsf{Out}} \setminus \left\{ n \atop k \right\}_{\mathsf{In}}} \bigoplus_{0 \le a < \mathsf{card}(\mathsf{In}(S))} \mathsf{Cone}(R \xrightarrow{2} R)^{\oplus \binom{\mathsf{card}(\mathsf{In}(S))-1}{a}}[-d(S)+a] \right) .$$

Proof. This follows from the prior lemmas, and observing that

$$\mathsf{Hom}(\mathsf{Cone}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}), R) \cong \mathsf{Cone}(R \xrightarrow{2} R)[-1] .$$

□

Corollary 3.0.61. *There is an isomorphism of graded R-modules*

$$H^*(\mathsf{Gr}_k(n);R) \cong \bigoplus_{S \in \left\{ n \atop k \right\}} V_S[d(S)] ,$$

where

$$V_S := \begin{cases} R, & \text{if } \mathsf{In}(S) = \emptyset = \mathsf{Out}(S) \\ \mathsf{ker}(R \xrightarrow{2} R), & \text{if } \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{Out}(S) \\ \mathsf{coker}(R \xrightarrow{2} R), & \text{if } \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{In}(S) \end{cases} .$$

Proof. Taking the cohomology of the chain complex in Corollary 3.0.60 yields an isomorphism of $H^*(\mathsf{Gr}_k(n);R)$ with

$$\left(\bigoplus_{S\in \left\{ n \atop k\right\}_{\mathsf{Out}} \cap \left\{ n \atop k\right\}_{\mathsf{In}}} R[d(S)]\right)$$

$$\oplus \bigoplus_{S\in \left\{ n \atop k\right\}_{\mathsf{Out}} \setminus \left\{ n \atop k\right\}_{\mathsf{In}}} \bigoplus_{0\leq a<\mathsf{card}(\mathsf{In}(S))} H^*\left(0\to R[a]\xrightarrow{2} R[a-1]\to 0\right)[d(S)-\mathsf{card}(\mathsf{In}(S))]^{\oplus \binom{\mathsf{card}(\mathsf{In}(S))-1}{a}} .$$

Note that

$$\begin{aligned} &H^*\left(0\to R[a]\xrightarrow{2} R[a-1]\to 0\right)[d(S)-\mathsf{card}(\mathsf{In}(S))] \\ &\quad\cong \mathsf{ker}(R\xrightarrow{2} R)[d(S)-\mathsf{card}(\mathsf{In}(S))+a]\oplus \mathsf{coker}(R\xrightarrow{2} R)[d(S)-\mathsf{card}(\mathsf{In}(S))+a+1] .\end{aligned}$$

Thus, the number of summands in

$$\bigoplus_{0\leq a<\mathsf{card}(\mathsf{In}(S))} H^*\left(0\to R[a]\xrightarrow{2} R[a+1]\to 0\right)[d(S)-\mathsf{card}(\mathsf{In}(S))]^{\oplus \binom{\mathsf{card}(\mathsf{In}(S))-1}{a}}$$

is equal to $2^{\mathsf{card}(\mathsf{In}(S))}$. Hence there is one summand for each set in $\left\{ n \atop k\right\}^{\mathsf{adm}}_{\leq S}$. Let us choose the following convention. Assign to each $T\in \left\{ n \atop k\right\}^{\mathsf{adm}}_{\leq S}$, the R-module

$$T\mapsto V_T := \begin{cases} \mathsf{ker}(R\xrightarrow{2} R), & \text{if } \mathsf{Min}(\mathsf{In}(T)\cup \mathsf{Out}(T))\in \mathsf{Out}(T) \\ \mathsf{coker}(R\xrightarrow{2} R), & \text{if } \mathsf{Min}(\mathsf{In}(T)\cup \mathsf{Out}(T))\in \mathsf{In}(T) \end{cases} .$$

This choice amounts to the following: Consider the weighted level graph assoicatied to the complex in the description of $C^*_{\mathsf{Sch}}(\mathsf{Gr}_k(n);R)$ from Corollary 3.0.60. See Figure 3.5 for the example of $\mathsf{Gr}_5(10)$. We assign $\mathsf{ker}(R\xrightarrow{2} R)$ to the start node of each edge, and $\mathsf{coker}(R\xrightarrow{2} R)$ to the end node of each edge. Recall from the proof of Proposition 3.0.57 that for each T, there is a unique $S \in \left\{ {n \atop k} \right\}$ such that $T \in \left\{ {n \atop k} \right\}^{\mathsf{adm}}_{\leq S}$. Therefore,

$$\begin{aligned} \bigoplus_{S \in \left\{ {n \atop k} \right\}_{\mathsf{Out}} \setminus \left\{ {n \atop k} \right\}_{\mathsf{In}}} \bigoplus_{0 \leq a < \mathsf{card}(\mathsf{In}(S))} & H^* \left(0 \to R[a] \xrightarrow{2} R[a+1] \to 0 \right) [d(S) - \mathsf{card}(\mathsf{In}(S))]^{\oplus \binom{\mathsf{card}(\mathsf{In}(S))-1}{a}} \\ \cong & \bigoplus_{S \in \left\{ {n \atop k} \right\}_{\mathsf{Out}} \setminus \left\{ {n \atop k} \right\}_{\mathsf{In}}} \bigoplus_{T \in \left\{ {n \atop k} \right\}^{\mathsf{adm}}_{\leq S}} V_T[d(T)] \\ \cong & \bigoplus_{S \in \left\{ {n \atop k} \right\} \setminus \left(\left\{ {n \atop k} \right\}_{\mathsf{Out}} \setminus \left\{ {n \atop k} \right\}_{\mathsf{In}} \right)} V_S[d(S)] \ . \end{aligned}$$

Extending the assignment $S \mapsto V_S$ by defining $V_S := R$ if $\mathsf{In}(S) = \emptyset = \mathsf{Out}(S)$, yields the desired result: $H^*(\mathsf{Gr}_k(n); R) \cong \bigoplus_{S \in \left\{ {n \atop k} \right\}} V_S[d(S)]$. □

Computations

We present some computations for various n and k. Figure 3.3 provides a graphical depiction of the chain complex of $\mathsf{Gr}_3(6)$. The vertical columns list the generators of $C^i_{\mathsf{Sch}}(\mathsf{Gr}_3(6))$ in terms of increasing dimension, with the generator of $C^0_{\mathsf{Sch}}(\mathsf{Gr}_3(6))$ on the left, and the generator of $C^6_{\mathsf{Sch}}(\mathsf{Gr}_3(6))$ on the right. The arrows specify the poset structure of the poset, $\left\{ {6 \atop 3} \right\}$, which stratifies the space $\mathsf{Gr}_3(6)$. Each arrow, $S \to T$, is labeled by the coefficient of its differential: $\sigma^S_T([\Theta^0_{S<T}]) + \sigma^S_T([\Theta^\pi_{S<T}])$. Figures 3.4 and 3.6 give a graphical depiction of $C^*_{\mathsf{Sch}}(\mathsf{Gr}_5(10); \mathbb{Z})$ and $C^*_{\mathsf{Sch}}(\mathsf{Gr}_6(12); \mathbb{Z})$, respectively. Each node in the images represents a set $S \in \left\{ {n \atop k} \right\}$. To make the images more legible, in Figure 3.4, the nodes have been labeled using base 11 notation, where $a = 10$, and we left the nodes unlabeled in Figure 3.6. The blue dotted arrows indicate the coefficient of the differential is -2 and the solid red lines indicate the coefficient of the differential is $+2$. Figure 3.5 is a depiction of the chain complex from Theorem 3.0.56 that is isomorphic to $C^*_{\mathsf{Sch}}(\mathsf{Gr}_5(10); R)$. Table 3.1 displays the number of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ summands in each integral cohomology group of $\mathsf{Gr}_{12}(24)$.

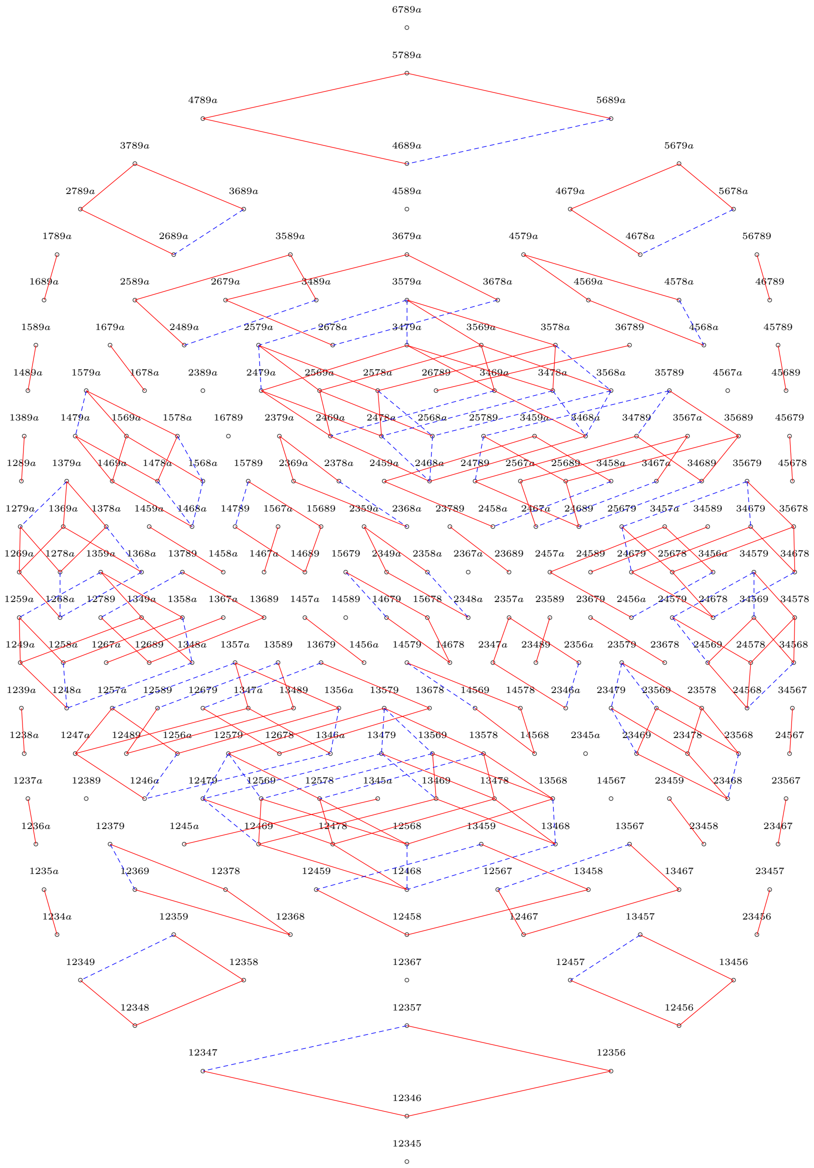


Figure 3.4: The chain complex for $\mathsf{Gr}_5(10)$. The dotted blue lines indicate the differential is $+2$, the solid red lines indicate the coefficient of the differential is -2. The sets are labeled using base 11: $a = 10$.

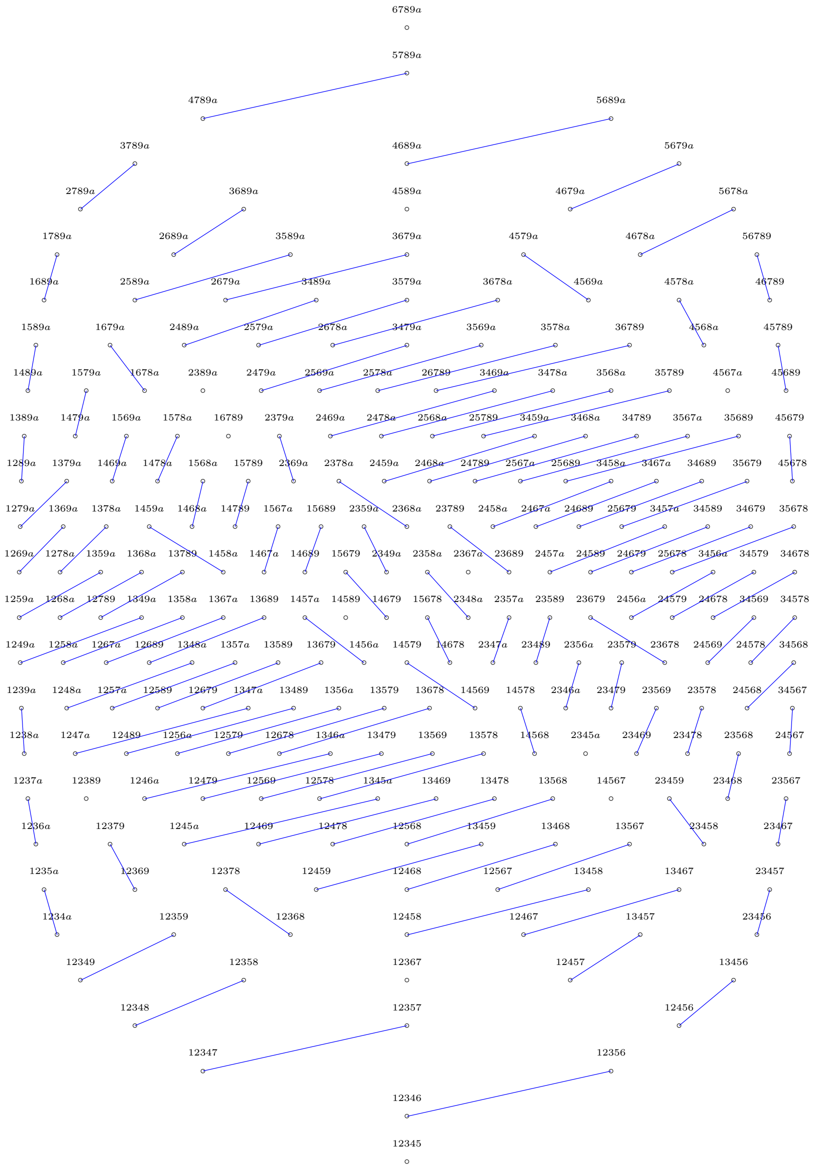


Figure 3.5: The chain complex from Theorem 3.0.56 that is isomorphic to $C^*_{\mathsf{Sch}}(\mathsf{Gr}_5(10); R)$. Each blue line corresponds to one of the copies of $\mathsf{Cone}(R \xrightarrow{2} R)$. The sets are labeled using base 11: $a = 10$.

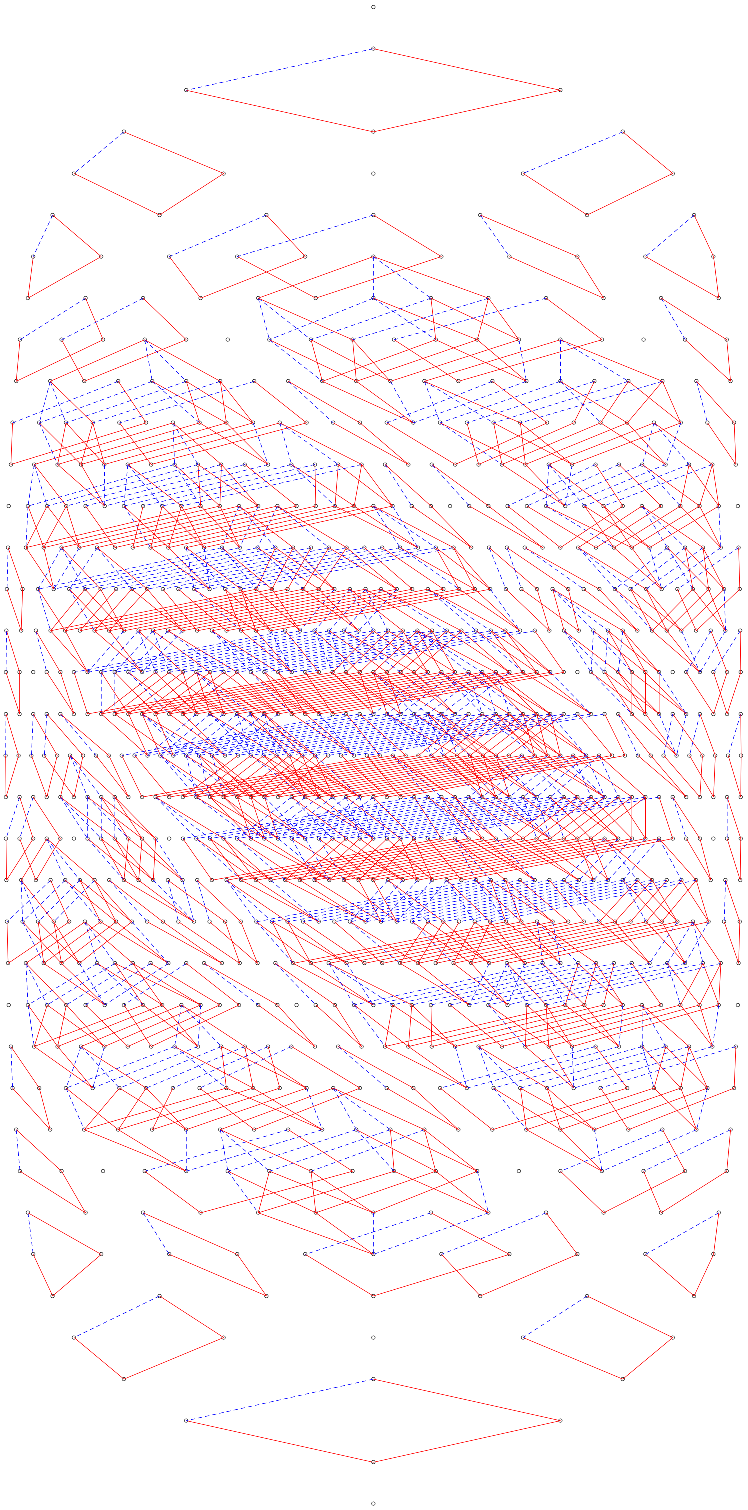


Figure 3.6: The chain complex for $\mathsf{Gr}_6(12)$. The dotted blue lines indicate the differential is $+2$, the solid red lines indicate the coefficient of the differential is -2. The labels of the nodes were omitted for legibility reasons.

$Gr_{12}(24)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
H^0	1	0
H^1	0	0
H^2	0	1
H^3	0	1
H^4	1	2
H^5	0	2
H^6	0	5
H^7	0	6
H^8	2	9
H^9	0	11
H^{10}	0	19
H^{11}	0	23
H^{12}	3	33
H^{13}	0	41
H^{14}	0	58
H^{15}	0	73
H^{16}	5	95
H^{17}	0	117
H^{18}	0	156
H^{19}	0	191
H^{20}	7	239
H^{21}	0	291
H^{22}	0	367
H^{23}	0	441
H^{24}	11	536
H^{25}	0	638
H^{26}	0	777
H^{27}	0	916
H^{28}	13	1087
H^{29}	0	1268
H^{30}	0	1503
H^{31}	0	1740
H^{32}	18	2017
H^{33}	0	2316
H^{34}	0	2680
H^{35}	0	3048
H^{36}	22	3470
H^{37}	0	3917
H^{38}	0	4440
H^{39}	0	4974
H^{40}	28	5562
H^{41}	0	6179
H^{42}	0	6886
H^{43}	0	7594
H^{44}	32	8356
H^{45}	0	9151
H^{46}	0	10029
H^{47}	0	10901
H^{48}	39	11821
H^{49}	0	12760
H^{50}	0	13776
H^{51}	0	14770
H^{52}	42	15791
H^{53}	0	16814
H^{54}	0	17898
H^{55}	0	18936
H^{56}	48	19967
H^{57}	0	20990
H^{58}	0	22040
H^{59}	0	23015
H^{60}	51	23959
H^{61}	0	24864
H^{62}	0	25764
H^{63}	0	26573
H^{64}	55	27307
H^{65}	0	27984
H^{66}	0	28635
H^{67}	0	29166
H^{68}	55	29596
H^{69}	0	29963
H^{70}	0	30272
H^{71}	0	30456
H^{72}	58	30525
H^{73}	0	30525
H^{74}	0	30456
H^{75}	0	30272
H^{76}	55	29963
H^{77}	0	29596
H^{78}	0	29166
H^{79}	0	28635
H^{80}	55	27984
H^{81}	0	27307
H^{82}	0	26573
H^{83}	0	25764
H^{84}	51	24864
H^{85}	0	23959
H^{86}	0	23015
H^{87}	0	22040
H^{88}	48	20990
H^{89}	0	19967
H^{90}	0	18936
H^{91}	0	17898
H^{92}	42	16814
H^{93}	0	15791
H^{94}	0	14770
H^{95}	0	13776
H^{96}	39	12760
H^{97}	0	11821
H^{98}	0	10901
H^{99}	0	10029
H^{100}	32	9151
H^{101}	0	8356
H^{102}	0	7594
H^{103}	0	6886
H^{104}	28	6179
H^{105}	0	5562
H^{106}	0	4974
H^{107}	0	4440
H^{108}	22	3917
H^{109}	0	3470
H^{110}	0	3048
H^{111}	0	2680
H^{112}	18	2316
H^{113}	0	2017
H^{114}	0	1740
H^{115}	0	1503
H^{116}	13	1268
H^{117}	0	1087
H^{118}	0	916
H^{119}	0	777
H^{120}	11	638
H^{121}	0	536
H^{122}	0	441
H^{123}	0	367
H^{124}	7	291
H^{125}	0	239
H^{126}	0	191
H^{127}	0	156
H^{128}	5	117
H^{129}	0	95
H^{130}	0	73
H^{131}	0	58
H^{132}	3	41
H^{133}	0	33
H^{134}	0	23
H^{135}	0	19
H^{136}	2	11
H^{137}	0	9
H^{138}	0	6
H^{139}	0	5
H^{140}	1	2
H^{141}	0	2
H^{142}	0	1
H^{143}	0	1
H^{144}	1	0

Table 3.1: This table displays the integral cohomology of $\mathsf{Gr}_{12}(24)$. We indicate the number of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ summands in each cohomology group.

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APPENDIX

BACKGROUND

In this chapter we briefly recall the necessary definitions and notations that underlie this dissertation.

Category theory

In this section we record some essential notation, definitions, and results from category theory that we use freely in this dissertation. For more details on the foundations of category theory, we refer the reader to [1] and [26].

Definition A.0.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let $d \in \mathcal{D}$. We define the *overcategory* as the pullback

$$\begin{tikzcd} \mathcal{C}_{/_d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}_{/_d} \arrow[d, "\mathsf{fgt}"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} , \end{tikzcd}$$

where the category $\mathcal{D}_{/_d}$ is defined as the pullback

$$\begin{tikzcd} \mathcal{D}_{/_d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}([1], \mathcal{D}) \arrow[d, "\mathsf{ev}_1"] \\ * \arrow[r, "\langle d \rangle"] & \mathcal{D} . \end{tikzcd}$$

Definition A.0.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let $d \in \mathcal{D}$. We define the *fiber of F over d* as the pullback

$$\begin{tikzcd} \mathcal{C}_{|_d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "F"] \\ * \arrow[r, "\langle d \rangle"] & \mathcal{D} . \end{tikzcd}$$

Definition A.0.3. An ∞-category $\mathcal{C}$ is called *cofiltered* if every functor $\mathcal{K} \xrightarrow{F} \mathcal{C}$ between ∞-categories extends to a functor out of the left cone

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] \arrow[d, hook] & \mathcal{C} \\ \mathcal{K}^{\triangleleft} . \arrow[ur, dashed] & \end{tikzcd}$$

Proposition A.0.4. *Let P be a poset and $\mathcal{K}$ be an ∞-category. The restriction*

$$\mathit{Fun}(\mathcal{K}, P) \to \mathit{Fun}(\mathit{Obj}(\mathcal{K}), P) \simeq \mathit{Fun}(\pi_0 \mathit{Obj}(\mathcal{K}), P)$$

is a monomorphism.

Corollary A.0.5. *Let P be a poset and $\mathcal{K}$ be an ∞-category. A functor $F : \mathcal{K} \to P$ admits an extension*

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "F"] \arrow[d, hook] & P \\
\mathcal{K}^{\triangleleft} \arrow[ur, dashed] &
\end{tikzcd}$$

if and only if there exists an extension

$$\begin{tikzcd}
\pi_0\mathcal{K} \arrow[r, "F"] \arrow[d, hook] & \mathcal{C} \\
\pi_0(\mathit{Obj}(\mathcal{K})^{\triangleleft}) \arrow[ur, dashed] &
\end{tikzcd}$$

Corollary A.0.6. *A poset P is cofiltered (as an ∞-category) if and only if for all finite subsets $S \subset P$, there exists $p_{-\infty} \in P$ such that for all $s \in S$, we have $p_{-\infty} \leq s$.*

Example A.0.7. Let X be a topological space. By Corollary A.0.6, the poset $\mathsf{open}(X)$ is cofiltered since the finited intersection of open sets is open.

Example A.0.8. Let X be a topological manifold. Using Corollary A.0.6, the subposet $\mathsf{disk}(X)$ is cofiltered. Namely, the finite intersection of disks is open, and since X is a manifold, $\mathsf{disk}(X)$ is a basis, so we can find an element of $\mathsf{disk}(X)$ in the finite intersection.

Proposition A.0.9. *If a diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\
\mathcal{B} \arrow[r, "F"] & \mathcal{B}'
\end{tikzcd}$$

is a pullback, then for all $b \in \mathcal{B}$ the functor

$$f_{|_b} : \mathcal{E}_{|_b} \to \mathcal{E}'_{|_{F(b)}}$$

is an equivalence.

Proposition A.0.10. *A diagram of spaces*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\
\mathcal{B} \arrow[r, "F"] & \mathcal{B}'
\end{tikzcd}$$

is a pullback, if and only if for all $[b] \in \pi_0\mathcal{B}$, there exists some $b' \in [b]$ for which the functor

$$F_{|_{b'}} : \mathcal{E}_{|_{b'}} \to \mathcal{E}'_{|_{F(b')}}$$

is an equivalence.

Proposition A.0.11. *Consider a pullback diagram in topological spaces*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & X' \arrow[d] \\
Y \arrow[r, "f"] & Y' .
\end{tikzcd}$$

If f is an open embedding, then so is g.

There is a technical condition that we need to assume on our target symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ in Chapter 2 to allow us to compute colimits.

Definition A.0.12 ([3] Definition 3.4)**.** A symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ is $\otimes$-*presentable* if it is presentable, and if for each $V \in \mathcal{V}^{\otimes}$, the functor $V \otimes - : \mathcal{V} \to \mathcal{V}$ takes colimit diagrams to colimit diagrams.

In particular, $\mathcal{V}^{\otimes}$ being presentable means that all colimits exist. We also use that $V \otimes -$ distributes over colimits to compute certain colimits in some of our proofs.

Definition A.0.13. A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is *final* if for each functor $\mathcal{D} \to \mathcal{E}$ to another ∞-category the canonical morphism

$$\operatorname{colim}(\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathcal{E}) \to \operatorname{colim}(\mathcal{D} \to \mathcal{E})$$

is an equivalence, provided the colimits exist.

Note that if $\mathcal{D}$ has a final object, d, then the inclusion $* \xrightarrow{\langle d \rangle} \mathcal{D}$ is a final functor.

Proposition A.0.14 ([4] Proposition 5.13)**.** *A localization of ∞-categories is both final and initial.*

Complete Segal spaces and localization

Complete Segal spaces as developed by Rezk in [25] are one model for the theory of ∞-categories. Though we work model independently in this paper, we explicitly use complete Segal spaces to use a theorem of Mazel-Gee [23] to identify localizations of ∞-categories. Here we recall the basics of complete Segal spaces.

Complete Segal spaces are simplicial presheaves of spaces satisfying two conditions. To describe simplicial objects, we recall the simplex category.

Definition A.0.15. The *simplex category* Δ is the category of finite nonempty linearly ordered sets and order preserving maps between them.

We denote objects in Δ by $[p] := \{0 < \cdots < p\}$ for $p \in \mathbb{Z}_{>0}$.

Now, we define what we mean by a space.

Definition A.0.16. The *∞-category of spaces* Spaces is the category of topological spaces that admit a CW structure localized on the weak homotopy equivalences.

Definition A.0.17. A *simplicial space* is a functor $\Delta^{\mathsf{op}} \to \mathsf{Spaces}$.

There is a special class of simplicial spaces called the Segal spaces

Definition A.0.18. A simplicial space, $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ is a *Segal space* if for every integer $p > 1$ the diagram

$$\begin{tikzcd}
F[p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F\{p-1 < p\} \arrow[d] \\
F\{0 < \cdots < p-1\} \arrow[r] & F\{p-1\}
\end{tikzcd}$$

is a pullback of spaces.

Given a Segal space, there is a subspace of [1]-points that have both left and right inverses. We call these [1]-points equivalences.

Definition A.0.19. Let $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ be a Segal space. An *equivalence* in F is a [1]-point

$$[1] \xrightarrow{\langle x \xrightarrow{f} y \rangle} F$$

such that the dashed arrows in the following two diagrams exist

$$\begin{tikzcd}
\{0 < 1\} \arrow[d, hook] \arrow[dr, bend left, "f"] & \\
{[2]} \arrow[r, dashed] & F \\
\{0 < 2\} \arrow[u, hook] \arrow[r] & * \arrow[u, "\langle x \rangle"]
\end{tikzcd} \tag{A.1}$$

$$\begin{tikzcd}
\{1 < 2\} \arrow[d, hook] \arrow[dr, bend left, "f"] & \\
{[2]} \arrow[r, dashed] & F \\
\{0 < 2\} \arrow[u, hook] \arrow[r] & * \arrow[u, "\langle y \rangle"]
\end{tikzcd} \tag{A.2}$$

We denote the subspace of equivalences by $F^{\mathsf{equiv}}[1] \subset F[1]$.

The diagram in equation (A.1) asserts that f has a left inverse, and the diagram in equation (A.2) asserts that f has a right inverse.

Let $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ be a Segal space. The unique map from $[1] \to [0]$ induces a map $F[0] \to F[1]$ that uniquely factors through the equivalences F^{equiv}. This maps the $[0]$-points to degenerate $[1]$-points.

Definition A.0.20. A *complete Segal space* is a Segal space $F : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ for which the map $F[0] \to F^{\mathsf{equiv}}[1]$ is an equivalence of spaces.

The Segal condition says that the $[0]$-points and $[1]$-points determine the $[p]$-points. This is useful for identifying two complete Segal spaces, as codified in the following observation.

Observation A.0.21. Two complete Segal spaces $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is an equivalence between $[0]$-points and $[1]$-points.

Central to the proof of the Theorem 2.0.36 is the identification of the localization of an ∞-category via Theorem A.0.28 below. We define localizations using classifying spaces, or ∞-groupoid completions. The idea of localization of an ∞-category $\mathcal{C}$ is to formally invert a specific class of morphisms in $\mathcal{C}$. If we simply invert only the isomorphisms, then we obtain the original ∞-category $\mathcal{C}$. On the other hand, if we invert every morphism in $\mathcal{C}$, then we obtain the classifying space, or ∞-groupoid completion of $\mathcal{C}$.

Definition A.0.22. As developed in [20], there exists a left adjoint to the inclusion

$$\begin{tikzcd} \mathsf{Cat}_{(\infty,1)} \arrow[r, bend left, "\mathsf{B}"] & \mathsf{Spaces} \arrow[l, hook'] \end{tikzcd}$$

of the ∞-category of spaces into the ∞-category of ∞-categories. For $\mathcal{C}$ an ∞-category, we call the value of the left adjoint $\mathsf{B}\mathcal{C}$ the *classifying space of* $\mathcal{C}$.

Remark A.0.23. If one takes complete Segal spaces as a model for ∞-categories, then the classifying space of a complete Segal space $\mathcal{C} : \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ is given by the colimit $\mathsf{B}\mathcal{C} := \operatorname{colim} \mathcal{C}$. If one takes quasicategories as a model for ∞-categories, then the classifying space of a quasicategory $\mathcal{C} : \Delta^{\mathsf{op}} \to \mathsf{Set}$ is given by the geometric realization $|\mathcal{C}|$.

Observation A.0.24. Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ possesses an initial object, then its classifying space $\mathsf{B}\mathcal{C}$ is contractible. Dually, if $\mathcal{C}$ possesses a final object, then its classifying space $\mathsf{B}\mathcal{C}$ is contractible.

Proposition A.0.25 ([23] Corollary 1.28)**.** *An adjunction between ∞-categories induces an equivalence between their classifying spaces.*

Using the notion of a classifying space, we can now define the localization of an ∞-category.

Definition A.0.26. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{W} \subset \mathcal{C}$ be an ∞-subcategory that contains all the equivalences in $\mathcal{C}$. The *localization of $\mathcal{C}$ at $\mathcal{W}$* is defined to be the pushout

$$\begin{tikzcd} \mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathsf{B}\mathcal{W} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{C}[\mathcal{W}^{-1}] \end{tikzcd} .$$

Example A.0.27. If we take $\mathcal{W} = \mathcal{C}^{\sim}$ to be the maximal ∞-subgroupoid of $\mathcal{C}$, then $\mathcal{C}[\mathcal{C}^{\sim}] \simeq \mathcal{C}$. At the other end of the spectrum, if we localize $\mathcal{C}$ on all morphisms, then we obtain the classifying space of $\mathcal{C}$. That is, $\mathcal{C}[\mathcal{C}^{\sim}] \simeq \mathsf{B}\mathcal{C}$.

Theorem A.0.28 ([23] Theorem 3.8)**.** *Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{W} \subset \mathcal{C}$ be an ∞-subcategory that contains the maximal ∞-subgroupoid of $\mathcal{C}$. If the classifying space $\mathsf{BFun}^{\mathcal{W}}([\bullet], \mathcal{C})$ is a complete Segal space, then there is an equivalence of ∞-categories*

$$\mathsf{BFun}^{\mathcal{W}}([\bullet], \mathcal{C}) \simeq \mathcal{C}[\mathcal{W}^{-1}] \ .$$

Here, $\mathsf{Fun}^{\mathcal{W}}([\bullet], \mathcal{C})$ denotes the simplicial category whose $[p]$-points are defined as the following pullback of ∞-categories

$$\begin{tikzcd}
\mathsf{Fun}^{\mathcal{W}}([p], \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}([p], \mathcal{C}) \arrow[d] \\
\mathsf{Fun}([p]^{\sim}, \mathcal{W}) \arrow[r] & \mathsf{Fun}([p]^{\sim}, \mathcal{C})
\end{tikzcd}$$

where $[p]^{\sim}$ denotes the maximal ∞-subgroupoid of $[p]$.

Observation A.0.29. Note the $[0]$-points are equivalent to $\mathcal{W}$. Also, the $[1]$-points are given by natural transformations whose morphisms are drawn from $\mathcal{W}$.

∞-operads

We use the theory of ∞-operads as developed by Lurie in [21]. The notion of ∞-operad is an ∞-categorical analog of a multicategory, or colored operad. We now recall the basic definitions and notation of this theory.

Colored operads can be thought of as symmetric monoidal categories where the symmetric monoidal product is not actually representable. The category of based finte sets is used to organize ∞-operads.

Definition A.0.30. Let Fin_* denote the category of based finite sets with based maps between them.

Typically, we will denote objects of Fin_* by I_+, where I is a finite set and $+$ is a disjoint basepoint. There are several special classes of morphisms in Fin_*.

Definition A.0.31. A morphism $I_+ \xrightarrow{f} J_+$ in Fin_* is called

- *inert* if $f^{-1}(j) \simeq *$ for all $j \in J$;
- *active* if $f^{-1}(+) = \{+\}$.

Observation A.0.32. The inert and active morphisms form a factorization system on Fin_*. By this we mean that every morphism $I_+ \to J_+$ in Fin_* can be uniquely factored as the composition $I_+ \to \widetilde{I}_+ \to J_+$ of an inert morphism followed by an active morphism.

Definition A.0.33. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. A morphism f in $\mathcal{D}$ is called *F-coCartesian* if there exists an initial filler for each solid diagram of categories

$$\begin{tikzcd}
* \arrow[r, "\langle C \rangle"] \arrow[d, "\langle s \rangle"'] & \mathcal{C} \arrow[d, "F"] \\
\{s < t\} \arrow[r, "\langle f \rangle"'] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd} .$$

We will denote the lift of f by $f_!$.

Definition A.0.34. Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an F-coCartesian morphism in $\mathcal{D}$, $f : D \to D'$, we can consider the *coCartesian monodromy functor of f*

$$f_! : \mathcal{C}_{|_D} \to \mathcal{C}_{|_{D'}}$$

that sends $C \in \mathcal{C}_{|_D}$ to $f_!(C)$, the coCartesian lift of f evaluated at C.

Definition A.0.35. A functor $F : \mathcal{C} \to \mathcal{D}$ is a *coCartesian fibration* if every morphism in $\mathcal{D}$ is F-coCartesian.

Proposition A.0.36. *Let $E \to B$ be a coCartesian fibration. For each $b \in B$, the canonical functor*

$$E_{|_b} \hookrightarrow E_{/_b}$$

is a right adjoint.

Definition A.0.37. Let $E \to B$ be a coCartesian fibration. For $b \xrightarrow{f} b'$ in B, the coCartesian monodromy functor is defined via the left adjoint to the above right adjoint:

$$\begin{tikzcd}
E_{|_b} \arrow[r, dashed, "f_!"] \arrow[d, hook] & E_{|_{b'}} \arrow[d, hook, shift right] \\
E_{/_b} \arrow[r, "f \circ -"] & E_{/'_b} \arrow[u, shift right]
\end{tikzcd} .$$

Definition A.0.38. A functor $F : \mathcal{C} \to \mathsf{Fin}_*$ is called *inert-coCartesian fibration* if each inert morphism in Fin_* is F-coCartesian.

Definition A.0.39. An ∞-*operad* is an ∞-category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathsf{Fin}_*$ such that

1. F is an inert-coCartesian fibration;
2. for all $I_+ \in \mathsf{Fin}_*$, the canonical functor

   $$\mathcal{C}|_{I_+} \xrightarrow{((c_i)_!)_{i \in I}} \prod_{i \in I} \mathcal{C}|_{\{i\}_+}$$

   is an equivalence of ∞-categoires;

3. for every $f : I_+ \to J_+$ in Fin_* and every $O \in \mathcal{C}|_{I_+}$ and $P \in \mathcal{C}|_{J_+}$, the canonical map between spaces
$$\mathrm{Map}_{\mathcal{C}}(O, P)|_f \xrightarrow{((c_j)_!(O)\circ-)_{j\in J}} \prod_{j\in J} \mathrm{Map}_{\mathcal{C}}(O, P_j)|_{c_j\circ f}$$
is an equivalence of ∞-categories.

We now make precise the idea that ∞-operads generalize ordinary colored operads, or multicategories.

Construction A.0.40. Let $\mathcal{O}$ be a multicategory. There exists a category $\mathcal{O}^{\otimes} \xrightarrow{\pi_{\mathcal{O}}} \mathsf{Fin}_*$ over the category of non-empty based finite sets. An object in $\mathcal{O}^{\otimes}$ is a pair $(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O}))$ consisting of a based finite set I_+, and a map $O_- : I \to \mathrm{obj}(\mathcal{O})$, $i \mapsto O_i \in \mathcal{O}$ that selects out an object of $\mathcal{O}$ for each $i \in I$. We might suppress notation and refer to an object, $(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O}))$, as the list $(O_i)_{i\in I}$ or even just (O_i). A morphism of objects
$$(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O})) \to (J_+, J \xrightarrow{P_-} \mathrm{obj}(\mathcal{O}))$$
consists of a map of based finite sets, $I_+ \xrightarrow{f} J_+$, and for each $j \in J$, a multimorphism $g_j \in \mathcal{O}((O_i)_{i\in f^{-1}(j)}; P_j)$.

Observation A.0.41. Given a multicategory $\mathcal{O}$, the functor $\pi_{\mathcal{O}}$ from Construction A.0.40 is inert coCartesian. Namely, for $f : I_+ \to J_+$ an inert morphism, and $(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O})) \in \mathcal{O}^{\otimes}$, we have
$$f_!(I_+, I \xrightarrow{O_-} \mathrm{obj}(\mathcal{O})) \simeq (J_+, J \xrightarrow{O_{f^{-1}(-)}} \mathrm{obj}(\mathcal{O})).$$
That is, $f_!((O_i)_{i\in I}) \simeq (O_{f^{-1}(j)})_{j\in J}$. Furthermore, Construction A.0.40 actually produces an ∞-operad.

An extremely important example of Construction A.0.40 is the following.

Example A.0.42. Let X be a topological space. Let $\mathsf{open}(X)$ denote the poset whose objects are open sets in X with partial order given by inclusion of open sets. This can be thought of as a multicategory by declaring that the collection of multimorphisms from a list of opens $(U_i)_{i\in I}$ to another open V is a singleton if $U_i \subset V$ for each $i \in I$ and $U_i \cap U_{i'} = \emptyset$ for each $i \neq i' \in I$, and otherwise the collection of multimorphisms is the emptyset. Construction A.0.40 produces an ∞-operad $\mathsf{open}(X)^{\otimes}$. We can think of an object of $\mathsf{open}(X)^{\otimes}$ as an I indexed list of open sets in X. We will typically denote such objects by $(I_+, (U_i))$. Note that a morphism $(I_+, (U_i)) \xrightarrow{f} (J_+, (V_j))$ is a map of based finite sets $I_+ \xrightarrow{f} J_+$ satisfying the condition that for each $j \in J$, the collection $(U_i)_{f^{-1}(j)}$ is a pairwise disjoint collection of open sets of V_j.

Observation A.0.43. Note that $\mathsf{open}(X)^{\otimes}_{|1_+} \simeq \mathsf{open}(X)$.

There is a special class of ∞-operads that play an important role in our arguments.

Definition A.0.44. A *symmetric monoidal ∞-category* is an ∞-operad $\mathcal{O}^{\otimes} \xrightarrow{\pi} \mathsf{Fin}_*$ for which π is a coCartesian fibration.

Remark A.0.45. Ordinary symmetric monoidal categories can be thought of multicategories where the collection of multimorphisms is given by the collection of maps out of the tensor product. In this way, one can again use Construction A.0.40 to produce a symmetric monoidal ∞-category from an ordinary symmetric monoidal category.

Tensor products and bifunctors

As mentioned in the introduction, one of the key ideas underlying the proof of genreal additivity is that the ∞-category of ∞-operads possesses a tensor product with the property that

$$\mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes}, \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes})) \simeq \mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes} \otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \ .$$

The defining feature of the tensor product of ∞-operads is such that there is an equivalence of ∞-categories

$$\mathsf{Fun}_{\mathsf{opd}}(\mathsf{open}(X)^{\otimes} \otimes \mathsf{open}(Y)^{\otimes}, \mathcal{V}^{\otimes}) \simeq \mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes}) \ .$$

We now spell this out a little more, and in particular, define the ∞-category

$$\mathsf{BiFun}(\mathsf{open}(X)^{\otimes}, \mathsf{open}(Y)^{\otimes}; \mathcal{V}^{\otimes})$$

of bifunctors between ∞-operads. We refer the interested reader to [21] for more details.

Define the smash product functor of based finite sets as follows:

$$\mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$$

$$I_+, J_+ \mapsto I_+ \wedge J_+ := (I \times J)_+$$

and for $f : I_+ \to K_+$, $g : J_+ \to L_+$, define

$$f \wedge g : I_+ \wedge J_+ \to K_+ \wedge L_+$$

by

$$a|J| + b - |J| \mapsto \begin{cases} *, & \text{if } f(a) = * \text{ or } g(b) = * \\ f(a)|L| + g(b) - |L|, & \text{otherwise.} \end{cases}$$

Definition A.0.46. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$, and $\mathcal{Q}^{\otimes}$ be ∞-operads. A *bifunctor of operads* is a functor

$$\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \xrightarrow{\varphi} \mathcal{Q}^{\otimes}$$

such that

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "\varphi"] \arrow[d] & \mathcal{Q}^{\otimes} \arrow[d] \\
\mathsf{Fin}_* \times \mathsf{Fin}_* \arrow[r, "\wedge"] & \mathsf{Fin}_*.
\end{tikzcd}$$

1. the following diagram commutes
2. φ takes pairs of inert coCartesian morphisms to inert coCartesian morphisms.

Definition A.0.47. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$, and $\mathcal{Q}^{\otimes}$ be ∞-operads, and let $\varphi : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ be a bifunctor. The bifunctor φ *exhibits* $\mathcal{Q}^{\otimes}$ *as a tensor product of* $\mathcal{O}^{\otimes}$ *and* $\mathcal{P}^{\otimes}$ if for any ∞-operad, $\mathcal{C}^{\otimes}$, the functor

$$\mathsf{Fun}_{\mathsf{opd}}(\mathcal{Q}^{\otimes}, \mathcal{C}^{\otimes}) \to \mathrm{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{C}^{\otimes})$$

given by precomposition with φ is an equivalence.

We attempted to directly work with the tensor product, but even for the relatively simple ∞-operads like $\mathsf{open}(X)^{\otimes}$ and $\mathsf{disk}(X)^{\otimes}$, we encountered trouble explicitly identifying the tensor product. The tensor product of ∞-operads as given above should be a generalization of the Boardman-Vogt tensor product of ordinary operads. This is another interesting aspect of the tensor product that we have yet to unravel.

Left Kan extension

We use left Kan extensions in a variety of contexts throughout the body of this work. In this section we recall the basic definitions of ordinary left Kan extension and operadic left Kan extension. Additionally, we provide basic results that we utilize.

Ordinary left Kan extension

Given a diagram of categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\rho"] & \mathcal{D} \\
\mathcal{E} &
\end{tikzcd},$$

one might wish for an extension of F. That is, a functor $\widetilde{F} : \mathcal{E} \to \mathcal{D}$ that fills the above diagram. Often, such a filler will not exist. However, a left Kan extension is a natural approximation to a filler arrow. In fact, it is the initial approximation.

Definition A.0.48. Given a diagram of categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\rho"] & \mathcal{D} \\
\mathcal{E} &
\end{tikzcd} ,$$

the *left Kan extension of F along ρ* is a functor $\rho_!F : \mathcal{E} \to \mathcal{D}$ and a natural transformation $\varepsilon : F \to \rho_!F \circ \rho$. This data satisfies the following universal property: given another functor $G : \mathcal{E} \to \mathcal{D}$ and natural transformation $\beta : F \to G \circ \rho$, there exists a unique natural transformation $\sigma : \rho_!F \to G$ such that the following diagram commutes

$$\begin{tikzcd}
F \arrow[r, "\varepsilon"] \arrow[dr, "\beta"] & \rho_!F \circ \rho \arrow[d, "\sigma_{\rho(-)}"] \\
& G \circ \rho .
\end{tikzcd}$$

Example A.0.49. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If it exists, the left Kan extension of F along the unique functor $\mathcal{C} \to *$ is given by $\operatorname{colim} F$.

The following proposition tells us that when they exisst, left Kan extensions compose.

Proposition A.0.50. *Assume we have a solid diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\alpha"] & \mathcal{D} \\
\mathcal{B} \arrow[d, "\beta"] & \\
\mathcal{C} . &
\end{tikzcd}$$

If the left Kan extensions $\alpha_!F$ and $(\beta \circ \alpha)_!F$ exist, there is an equivalence of functors

$$(\beta \circ \alpha)_!F \simeq \beta_!(\alpha_!F) .$$

The following proposition is useful for working with left Kan extensions along coCartesian fibrations, such as projections. It tells us that the left Kan extension along a coCartesian fibration evaluates as a fiberwise colimit.

Proposition A.0.51 ([20] Proposition 4.3.3.10)**.** *Given a diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathcal{E} \\
\mathcal{D} &
\end{tikzcd}$$

for which π is a coCartesian fibration, if the left Kan extension $\pi_! F$ of F along π exists, then for all $D \in \mathcal{D}$, the left Kan extension evaluates as

$$\pi_! F(D) \simeq \operatorname{colim}\left(\mathcal{C}_{|_D} \xrightarrow{F} \mathcal{E}\right)$$

a colimit indexed by the fiber over D.

Proposition A.0.52. *Consider a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "F"] \arrow[d, "\rho"] & \mathcal{V} \arrow[d, "\pi"] \\
\mathcal{E} \arrow[r, "p"] & \mathcal{B} .
\end{tikzcd}$$

Provided that for all $e \in \mathcal{E}$, the canonical morphism in $\mathcal{B}$

$$\pi\left(\operatorname{colim}(\mathcal{E}_{0/e} \to \mathcal{E}_0 \xrightarrow{F} \mathcal{V})\right) \to p(e) \tag{A.3}$$

is an equivalence, then the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\rho_! F"] \arrow[d, "p"] & \mathcal{V} \arrow[dl, "\pi"] \\
\mathcal{B}
\end{tikzcd}$$

canonically commutes. If, in addition, $\mathcal{V} \xrightarrow{\pi} \mathcal{B}$ is a coCartesian fibration, then for all $e \in \mathcal{E}$, there exists a canonical equivalence

$$\rho_! F(e) \simeq \operatorname{colim}\left(\mathcal{E}_{0/e} \to \mathcal{E}_{0/p(e)} \xrightarrow{F} \mathcal{V}_{/p(e)} \xrightarrow{(-)_!} \mathcal{V}|_{p(e)}\right) .$$

Proof. For the first statement, recall that for $e \in \mathcal{E}$, the left Kan extension is given by $\rho_! F(e) = \operatorname{colim}(\mathcal{E}_{0/e} \to \mathcal{E}_0 \xrightarrow{F} \mathcal{V})$. Observe the commutative diagram

$$\begin{tikzcd}
\mathcal{E}_{0/e} \arrow[r] \arrow[dr] & \mathcal{E}_0 \arrow[r, "F"] & \mathcal{V} \\
& \mathcal{E}_{0/p(e)} \arrow[u] \arrow[r] & \mathcal{V}_{/p(e)} \arrow[u]
\end{tikzcd} .$$

Since $\mathcal{V}_{/p(e)} \xrightarrow{\mathsf{fgt}} \mathcal{V}$ preserves and detects colimits, we see that

$$\operatorname{colim}\left(\mathcal{E}_{0e} \to \mathcal{E}_{0/p(e)} \to \mathcal{V}_{/p(e)}\right) \simeq \left(\rho_! F(e), \pi(\rho_! F(e)) \to p(e)\right) .$$

The inclusion $\mathcal{V}|_{p(e)} \hookrightarrow \mathcal{V}_{/p(e)}$ is fully faithful with image consisting of those objects whose morphism to $p(e)$ is an equivalence. Thus, the hypothesis of equation (A.3) guarantees that $\rho_! F(e) \in \mathcal{V}|_{p(e)}$, which completes the first statement.

Now, since π is a coCartesian fibration, Proposition A.0.36 tells us that

$$\mathcal{V}_{|_b} \overset{\mathsf{R}}{\hookrightarrow} \mathcal{V}_{/_b}$$

is a right adjoint. Now, the first part of this proposition showed that $\rho_!\mathcal{F}(e) \in \mathcal{V}_{|_{p(e)}} \simeq \mathsf{im}(\mathsf{R})$, and thus $\rho_!\mathcal{F}(e) = \mathsf{R}(e')$ for some $e' \in \mathcal{V}_{|_{p(e)}}$. By definition of an adjunction, the following diagram commutes

$$\begin{tikzcd}
\mathsf{R}(e') \arrow[r, "\mathsf{unit}"] \arrow[dr, "\mathsf{id}"'] & \mathsf{RLR}e' \arrow[d, "\mathsf{R}(\mathsf{counit})"] \\
& \mathsf{R}e' \ .
\end{tikzcd}$$

Note that the right vertical arrow is an equivalence since R is a right adjoint. Since R is fully faithful, this implies that the unit

$$\rho_!\mathcal{F}(e) \to \mathsf{RL}\rho_!\mathcal{F}(e)$$

is an equivalence. Since left adjoints preserve colimits, we see that

$$\mathsf{L}\rho_!\mathcal{F}(e) \simeq \operatorname{colim}\left(\mathcal{E}_{0/e} \to \mathcal{E}_{0/p(e)} \xrightarrow{F} \mathcal{V}_{/p(e)} \xrightarrow{(-)_!} \mathcal{V}|_{p(e)}\right) \ ,$$

which completes the proof. ☐

Operadic left Kan extension

Central to several of our proofs is an operadic version of left Kan extension. The general theory of operadic left Kan extension is detailed in section 3.1.2 of [21]. In this subsection we establish a colimit formula for computing operadic left Kan extensions within the context of this paper. Namely, we prove the following formula:

Proposition A.0.53. *Let $\iota : \mathcal{D}^{\otimes} \hookrightarrow \mathcal{O}^{\otimes}$ be a fully faithful functor between ∞-operads with $\mathcal{D}^{\otimes}$ unital. Given a morphism of ∞-operads, $\mathcal{F} : \mathcal{D}^{\otimes} \to \mathcal{V}^{\otimes}$, with target a $\otimes$-presentable symmetric monoidal ∞-category, the left Kan extension of $\mathcal{F}$ along ι evaluates as*

$$\iota_!\mathcal{F}((I_+,(O_i))) \simeq \operatorname{colim}\left(\mathcal{D}^{\otimes}_{/^{act}_{(I_+,(O_i))}} \xrightarrow{fgt} \mathcal{D}^{\otimes}_{/^{act}_{I_+}} \xrightarrow{\mathcal{F}_{/I_+}} \mathcal{V}^{\otimes}_{/^{act}_{I_+}} \xrightarrow{\otimes} \mathcal{V}^{\otimes}_{|_{I_+}}\right).$$

Proof. Proposition 4.3.2.17 in [20] establishes the adjunction of functors over Fin_* :

$$\iota_! : \mathsf{Fun}_{\mathsf{Fin}_*}(\mathcal{D}^{\otimes},\mathcal{V}^{\otimes}) \rightleftarrows \mathsf{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes},\mathcal{V}^{\otimes}) : \iota^*.$$

It remains to check that $\iota_!\mathcal{F}$ takes inert-coCartesian morphisms to inert-coCartesian morphisms. Let $f : I_+ \to J_+$ be an inert morphism and consider $(I_+,(O_i)_I) \xrightarrow{f_!} (J_+,(O_j)_J)$ the coCartesian morphism in $\mathcal{D}^{\otimes}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}} \arrow[r] \arrow[d, "f_!"] & \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \arrow[r] \arrow[d, "f_!"] & \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \arrow[r] \arrow[d, "f_!"] & \mathcal{V}^{\otimes}|_{I_+} \arrow[d, "f_!"] \\
\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(J_+,(O_j))}} \arrow[r] & \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{J_+}} \arrow[r] & \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{J_+}} \arrow[r] & \mathcal{V}^{\otimes}|_{J_+}.
\end{tikzcd}$$

Since f is inert, each of the vertical functors is a projection. Further, the leftmost vertical arrow is final, as justified through Quillen's Theorem A, which we have recorded as Theorem A.0.62. To invoke this theorem, we must show that for each $(K_+,(D_k)) \xrightarrow{\alpha} (J_+,(O_j))$ in $\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(J_+,(O_j))}}$ the classifying space of the undercategory

$$\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}}\right)^{(K_+,(D_k))\xrightarrow{\alpha}(J_+,(O_j))/}$$

is contractible. Since $f : I_+ \to J_+$ is inert, $f^{-1}(j)$ is a singleton for each $j \in J$. This defines a section $\sigma : J_+ \to I_+$ of the map $I_+ \xrightarrow{f} J_+$. This enables us to canonically consider $(K_+,(D_k)) \xrightarrow{\alpha} (J_+,(O_j))$ as an object of $\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}}$ via the coCartesian lift along σ

$$(K_+,(D_k)) \xrightarrow{\alpha} (J_+,(O_j)) \xrightarrow{\sigma_!} (I_+,(U_i)) ,$$

where $U_i := O_j$ if $f(i) = j$ and $U_i := \emptyset$, the initial object of $\mathcal{D}^{\otimes}$, otherwise. This implies the undercategory

$$\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}}\right)^{(K_+,(D_k))\xrightarrow{f}(J_+,(O_j))/}$$

has an initial object, and thus by Observation A.0.24, its classifying space is contractible. Therefore,

$$\begin{aligned}
\iota_!\mathcal{F}(f_!(I_+,(O_i))) &\simeq \iota_!\mathcal{F}(J_+,(O_j)) \\
&\simeq \operatorname{colim}\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{J_+,(O_j))}} \to \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{J_+}} \to \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{J_+}} \to \mathcal{V}^{\otimes}_{|_{J_+}}\right) \\
&\simeq \operatorname{colim}\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}} \to \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{|_{I_+}} \xrightarrow{f_!} \mathcal{V}^{\otimes}_{|_{J_+}}\right) \\
&\simeq f_!\left(\operatorname{colim}\left(\mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{(I_+,(O_i))}} \to \mathcal{D}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{/^{\mathrm{act}}_{I_+}} \to \mathcal{V}^{\otimes}_{|_{I_+}}\right)\right),
\end{aligned}$$

where the last equivalence follows because the projection

$$f_! : \mathcal{V}^J \simeq \mathcal{V}^{\otimes}_{|_{J_+}} \to \mathcal{V}^{\otimes}_{|_{I_+}} \simeq \mathcal{V}^I$$

preserves colimits. □

Cosheaves

Factorization algebras are functors that satisfy a local-to-global property. This is codified in the idea of a cosheaf.

Grothedieck topologies

Informally, equipping a category, $\mathcal{C}$, with a Grothendieck topology specifies a notion of 'cover' for the objects in $\mathcal{C}$. This enables us to make sense of particular coherent systems of data on $\mathcal{C}$, namely (co)sheaves.

Definition A.0.54. For $C \in \mathcal{C}$, a *sieve* is a fully faithful functor $\mathcal{U} \hookrightarrow \mathcal{C}_{/C}$ such that for each $(D \xrightarrow{f} C) \in \mathcal{U}$ and $(E \xrightarrow{g} D) \in \mathcal{C}(E, D)$, we have $(E \xrightarrow{g} D \xrightarrow{f} C) \in \mathcal{U}$.

Intuitively, we think of a sieve as specifying the allowable ways of accessing the object C.

Definition A.0.55. A *Grothendieck topology*, τ, on $\mathcal{C}$ is

- for each $C \in \mathcal{C}$, a collection of *covering sieves* for C, denoted $\tau(C)$,

such that

1. for each $C \in \mathcal{C}$, $\mathcal{C}_{/C} \overset{=}{\hookrightarrow} \mathcal{C}_{/C}$ is in $\tau(C)$;
2. for $\mathcal{U} \in \tau(C)$, and $f : D \to C$ a morphism in $\mathcal{C}$, we have $f^*\mathcal{U} \in \tau(D)$;
3. if $\mathcal{U}$ is any sieve on $C \in \mathcal{C}$ such that the sieve

$$\bigcup_D \{f : D \to C \mid f^*\mathcal{U} \in \tau(D)\} \in \tau(C),$$

then in fact, $\mathcal{U} \in \tau(C)$.

Example A.0.56. Let X be a topological space. There is a standard Grothendieck topology on $\mathsf{open}(X)$ the poset of open sets in X where for $O \in \mathsf{open}(X)$, a sieve $\mathcal{U} = \{U_\alpha \hookrightarrow O\} \in \tau_{\mathsf{std}}(\mathsf{open}(X))$ is a cover iff for each $x \in O$ there exists some U_α containing x. Thus, the covering sieves are precisely the (complete) standard open covers.

A category can be equipped various different Grothendieck topologies, similar to how a set can be endowed with various distinct topologies. For example, consider the following family of topologies on $\mathsf{open}(X)$.

Example A.0.57. Let X be a topological space and consider the category $\mathsf{open}(X)$. For each integer $r > 0$ the is a topology on $\mathsf{open}(X)$ called the J_r-topology, let us denote this by τ_{J_r}. Given an open set $O \in \mathsf{open}(X)$, if a collection of open sets $\mathcal{U} = \{U_\alpha\}$ is a cover of O in the J_r-topology on $\mathsf{open}(X)$, then for each subset $S \subset O$ with cardinality at most r, there exists some $U_\alpha \in \mathcal{U}$ that contains S. As an example of the distinction between these topologies, consider the collection $\mathcal{U} := \{(-\infty, 1), (-1, \infty)\}$ of open subsets of $\mathbb{R}$. This is a naive J_1-cover since it is an ordinary open cover, however it is *not* a naive J_2-cover. To see this, consider the cardinality 2 subset $S = \{-2, 2\}$ of $\mathbb{R}$. Note that S is not contained in either element of $\mathcal{U}$.

Definition A.0.58. Let $(\mathcal{C}, \tau)$ be a site. A *basis*, $\mathcal{B}$, is a full subcategory of $\mathcal{C}$ with the property that every $C \in \mathcal{C}$ admits a τ-covering by objects in $\mathcal{B}$.

Example A.0.59. Let X be a topological n-manifold. The full subcategory $\mathsf{disk}(X)_{\mathsf{c}} \subset \mathsf{open}(X)$ consisting of those $U \in \mathsf{open}(X)$ for which $U \cong \mathbb{R}^n$ is a basis for the standard topology on $\mathsf{open}(X)$. However, $\mathsf{disk}(X)_{\mathsf{c}}$ is in general *not* a basis for any J_r-topology on $\mathsf{open}(X)$ when $r > 1$. Rather, consider $\mathsf{disk}(X) \subset \mathsf{open}(X)$ the full subcategory consisting of those $U \in \mathsf{open}(X)$ for which $U \cong \coprod_{\text{finite}} \mathbb{R}^n$ is homeomorphic to a finite disjoint union of open disks. Then $\mathsf{disk}(X)$ is a basis for every J_r-topology on $\mathsf{open}(X)$.

Cosheaves

Recall the *right cone* of a category, $\mathcal{U}$, is given by

$$\mathcal{U}^{\triangleright} := \mathcal{U} \times \{0, 1\} \coprod_{\mathcal{U} \times \{1\}} *.$$

For $\mathcal{U} \subset \mathcal{C}_{/C}$, observe the functor

$$\mathcal{U}^{\triangleright} \to \mathcal{C}_{/C}$$

given by sending the cone point to $(C \xrightarrow{\text{id}} C)$, and sending each morphism $(C' \to C) \xrightarrow{!} *$ to the obvious square.

Definition A.0.60. Let $(\mathcal{C}, \tau)$ be a site. The category of *($\mathcal{S}$-valued) cosheaves (w.r.t. τ)* is the full subcategory

$$\mathsf{cShv}^{\tau}(\mathcal{C}) \subset \mathsf{Fun}(\mathcal{C}, \mathcal{S})$$

consisting of those functors that have the property that for all $C \in \mathcal{C}$ and all covering sieves $\mathcal{U} \subset \mathcal{C}_{/C}$, the composite

$$\mathcal{U}^{\triangleright} \to \mathcal{C}_{/C} \xrightarrow{\text{fgt}} \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{S}$$

is a colimit diagram.

Remark A.0.61. The right cone in the above definition allows us to keep track of the map that is part of the data of an object in $\mathcal{C}_{/C}$. Note that there is a terminal object in the above diagram, namely $\mathcal{F}(C)$, and by the diagram being a colimit, we mean that this terminal point is the colimit of the diagram with the terminal point removed.

Quillen's theorems

Quillen's Theorem A is a useful tool for computing (co)limits, as it provides a way to check if a functor is final or intial. This is relevant for us because we often must analyze colimits via the colimit formula for (operadic) left Kan extension. We refer the reader to [4] for more information on an ∞-categorical treatment of Quillen's Theorems A and B.

Theorem A.0.62 (Quillen's Theorem A)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. The functor F is final if and only if for each $D \in \mathcal{D}$, the classifying space*

$$B(\mathcal{C}^{D/}) \simeq *$$

is contractible. The functor F is initial if and only if for each $D \in \mathcal{D}$, the classifying space

$$B(\mathcal{C}_{/D}) \simeq *$$

is contractible.

Quillen's Theorem B is designed precisely to check if the classifying space of a fiber sequence is again a fiber sequence.

Theorem A.0.63 (Quillen's Theorem B)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If for each morphism $D \to D'$ in $\mathcal{D}$, the functor $\mathcal{C}_{/D} \to \mathcal{C}_{/D'}$ induces an equivalence between classifying spaces $B(\mathcal{C}_{/D} \xrightarrow{\simeq} B(\mathcal{C}_{/D'}$, then for each $D \in \mathcal{D}$ the diagram of classifying spaces*

$$\begin{tikzcd}
B(\mathcal{C}_{/D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\mathcal{C} \arrow[d] \\
* \arrow[r, "\langle D \rangle"] & B\mathcal{D}
\end{tikzcd}$$

is a pullback.

Lemma A.0.64 ([10] Lemma 4.3.1)**.** *Let*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "\pi"] & \mathcal{E}' \arrow[d, "\pi'"] \\
\mathcal{B} \arrow[r] & \mathcal{B}'
\end{tikzcd}$$

be a pullback of ∞-categories. If π' satisfies the hypotheses of Quillen's Theorem B, then so does π.

Miscellaneous

In this section we compile a collection of miscellaneous facts that we reference in this dissertation.

Theorem A.0.65 ([21] Theorem A.3.1)**.** *Let* X *be a paracompact Hausdorff topological space and let* $\mathcal{U} \xrightarrow{F} \mathsf{open}(X)$ *be a functor from a poset into the poset of open sets in* X*. For each* $x \in X$*, consider the full subcategory*

$$\mathcal{U}_x := \{U \in \mathcal{U} \mid x \in F(U)\} \subset \mathcal{U} \ .$$

If for all $x \in X$*, the classifying space*

$$\mathsf{B}\mathcal{U}_x \simeq *$$

is contractible, then the map

$$\mathsf{hocolim}\Big(\mathcal{U} \xrightarrow{F} \mathsf{open}(X) \to \mathsf{Top}\Big) \xrightarrow{\simeq} X$$

is a weak homotopy equivalence. Furthermore, if $F(U) \simeq *$ *for each* $U \in \mathcal{U}$*, then*

$$\mathsf{B}\mathcal{U} \xleftarrow{\simeq} \mathsf{hocolim}\Big(\mathcal{U} \xrightarrow{F} \mathsf{open}(X) \to \mathsf{Top}\Big) \xrightarrow{\simeq} X \ .$$

Theorem A.0.66 ([19])**.** *The inclusion*

$$\mathsf{Homeo}(\mathbb{R}^n) \hookrightarrow \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$$

of the space of self-homeomorphisms of $\mathbb{R}^n$ *into the space of self-embeddings is a homotopy equivalence.*