ON CONFIGURATION CATEGORIES

by

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ABSTRACT

We approach manifold topology by examining configurations of finite subsets of manifolds within the homotopy-theoretic context of ∞-categories by way of stratified spaces. Through these higher categorical means, we identify the homotopy types of such configuration spaces in the case of the circle and Euclidean space.

1

INTRODUCTION

Approach

Broadly, we ask:

Question 1.0.1. To what extent do the homotopy types of configuration spaces of a manifold recover that manifold?

In [23], Salvatore and Longoni show, by example of certain Lens spaces, that the homotopy type of configuration spaces can distinguish some manifolds that are homotopy equivalent and yet not homeomorphic. In this thesis, we use the theory of *stratified spaces* following that of Lurie in [26] and Ayala, Francis, and Tanaka in [6] to package configurations of finite subsets of manifolds. For a manifold M, the *configuration space* of k points of M is

$$\mathrm{Conf}_k(M) := \{(x_1, ..., x_k) \in M^k \mid x_i \neq x_j \text{ if } i \neq j\}.$$

There is an evident action of Σ_k on $\mathrm{Conf}_k(M)$ given by permuting the order of the indexing. The resulting quotient space

$$\mathrm{Conf}_k(M)_{\Sigma_k} := \{S \subset M \mid |S| = k\}$$

is called the *unordered configuration space* of finite subsets of k points of M.

For a fixed manifold M, the collection of $\mathrm{Conf}_k(M)_{\Sigma_k}$ for all $k \in \mathbb{N}$ naturally organizes as a topological space called the *Ran space* of M, denoted $\mathrm{Ran}(M)$. As defined by Lurie (5.5.1.1 in [26]), given a smooth, non-empty, connected manifold M, we define

$$\text{Ran}(M) := \{U \subset M \mid U \neq \emptyset \text{ is finite}\}.$$

Following Beilinson and Drinfeld [9], we endow Ran(M) with the finest topology for which the map $M^k \to \text{Ran}(M)$, given by $(m_1, ..., m_k) \mapsto \{m_1, ..., m_k\}$, is continuous for each $k \geq 1$. Furthermore, Ran(M) naturally emits a *stratification* over the natural numbers by cardinality, so that each stratum is an unordered configuration space of M.

Definition 1.0.2. A *stratified space* $S : X \to P$ is a paracompact, Hausdorff topological space X together with a poset P and a continuous surjection S such that for each $p \in P$, the *p-stratum* $S^{-1}(p)$ is connected. A *stratified map* is a continuous map between stratified spaces that respects the stratification.

Here we regard a poset P as a topological space by defining $U \subset P$ to be open if and only if it is closed upwards; that is, if $a \in U$, then every $b \geq a$ is also in U.

Notably, we equip the topological p-simplex

$$\Delta^p := \{(t_0, ..., t_p) \in [0,1]^{p+1} \mid \sum_{i=0}^{p} t_i = 1 \, \forall i\}$$

with the standard stratification $Delta^p \to [p]$ over the p-simplex $[p] := \{0 < \cdots < p\}$ given by $(t_0, ..., t_p) \mapsto \max\{i \mid t_i \neq 0\}$.

The homotopy types of the unordered configuration spaces of M form the underlying ∞-groupoid of Exit(Ran(M)), the *exit-path* ∞*-category* of Ran(M); this composite of constructions defines our notion of a *configuration category.* A similar construction is given in [1] and further developed in [11].

More generally, for any stratified space X, we define the exit-path ∞-category of X as the following simplicial set:

Definition 1.0.3 (A.6 in [26]). For a stratified space $X \to P$, the *exit-path ∞-category* of X, $\mathrm{Exit}(X)$, is the simplicial set whose value on $[p]$ is the set

$$\{\Delta^p \xrightarrow{f} K \mid f \text{ is a stratified map}\}.$$

Informally then, an object of $\mathrm{Exit}(X)$ is a point in X and a morphism is a path in X which is allowed to 'exit' from a deeper stratum to a less-deep stratum, but not vice-versa.

In summary, we approach the homotopy types of configurations of finite subsets of a manifold by examining the exit-path ∞-category of the Ran space of that manifold. Modifying Question (1.0.1), this thesis broadly asks:

Question 1.0.4. For a manifold M, to what extent does the ∞-category $\mathrm{Exit}(\mathrm{Ran}(M))$ recover M?

Motivation: Configuration spaces and embeddings

It is natural to probe spaces of embeddings with configuration spaces of points, since embeddings are, in particular, injective. The broad motivating question for this thesis is:

Question 1.0.5. How well are embedding spaces approximated by functors between $\mathrm{Exit}(\mathrm{Ran}(-))$?

Specifically, we approach embedding spaces between Euclidean spaces and the embedding space of knots by specializing Question (1.0.4) to the manifolds $\mathbb{R}^n$ and $\mathbb{S}^1$.

Embedding spaces between Euclidean spaces

We consider the categories $\mathbf{\Theta}_n$. This family of categories was introduced by Joyal in [20] for the purpose of defining (∞, n)-categories. Later, Rezk used $\mathbf{\Theta}_n$ to define (∞, n)-categories as a generalization of complete Segal spaces [32]. His work followed Berger's defintion of $\mathbf{\Theta}_n^{\mathrm{op}}$ as the n-fold wreath product of $\mathbf{\Delta}^{\mathrm{op}}$, where $\mathbf{\Theta}_1 := \mathbf{\Delta}$ [10]. Following Rezk, $\mathbf{\Theta}_2$ is the full sub-category of the category of 2-categories, Cat_2, in which an object is denoted $[m]([n_1], ..., [n_m])$, where $[m], [n_1], ..., [n_m] \in \mathbf{\Delta}$. Such an object can be described by a 'pasting diagram'. For example, the object $[3]([1], [0], [2])$ corresponds to:

In general, $\mathbf{\Theta}_n$ is the full sub-category of the category of strict n-categories, Cat_n, in which an object is a pasting diagram.

In [7], Ayala and Hepworth show that $\mathbf{\Theta}_n$ naturally encodes the homotopy type of configuration spaces of ordered points in $\mathbb{R}^n$. Specifically, they show a homotopy equivalence between $\mathrm{Conf}_k(\mathbb{R}^n)$ and the classifying space of the fiber over $\{1, ..., k\}_+$ of the functor $\mathbf{\Theta}_n^{\mathrm{op}} \to \mathrm{Fin}_*$, which is given by the Yoneda functor $\mathbf{\Theta}_n(-, c_n)$, which canonically factors through Fin_*, the category of based finite sets. The value of this functor on a pasting diagram is the set of top-dimensional cells therein, adjoined a disjoint base point. They conjecture that $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$ is a localization of a subcategory of $\mathbf{\Theta}_n$. Their methods do not extend toward a proof of this conjecture, so a new idea is required, which is supplied in this thesis. We introduce the subcategory $\mathbf{\Theta}_n^{\mathrm{Exit}}$ of $\mathbf{\Theta}_n$ and show that it localizes to $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$. In fact, this result follows from Theorem (1.0.6) of my thesis, a generalized affirmation of their conjecture, which is explained next.

The work of Ayala, Francis and Rozenblyum in [4] extends the definition of the exit-path ∞-category of a stratified space to a definition of the exit-path ∞-category of a presheaf on stratified spaces. For a connected manifold M, we consider $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$, the exit-path ∞-category of the unital Ran space of M. The superscript 'u' stands for 'unital', which, through Theorem (1.0.6), makes reference to the role of degeneracy morphisms in $\mathbf{\Theta}_n$. Explicitly, $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ is the simplicial space in which an object is a finite, possibly empty subset S of M, and a morphism from $S \subset M$ to $T \subset M$ is a map between sets $T \to S$ together with an injection $(S \coprod_{T\times\{0\}} (T \times \Delta^1)) \hookrightarrow M \times \Delta^1$ over Δ^1. So the objects of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ are all those objects in $\mathrm{Exit}(\mathrm{Ran}(M))$ together with the empty subset, and heuristically, the morphisms of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ are all those exit-paths in $\mathrm{Exit}(\mathrm{Ran}(M))$ together with morphisms that are not quite exit-paths, in that they allow points in the source to 'disappear'. Note then, that $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ contains $\mathrm{Exit}(\mathrm{Ran}(M))$ as a ∞-subcategory.

Theorem 1.0.6. *For each $n \geq 1$, there is a localization*

$$\mathbf{\Theta}_n^{act} \to Exit(Ran^u(\mathbb{R}^n))$$

from the subcategory $\mathbf{\Theta}_n^{act}$ of $\mathbf{\Theta}_n$ of active morphisms to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

This result is related to, and motivated by, Dunn's additivity [15] which asserts that the natural map between operads $\mathcal{E}_n \otimes \mathcal{E}_m \to \mathcal{E}_{n+m}$ from the Boardman-Vogt tensor product is an equivalence. In [16], it was shown that the configuration spaces of a product of parallelizable manifolds are recovered from the configuration spaces of each factor in terms of the Boardman-Vogt tensor product of modules over the little cubes operad. The next conjecture, which should follow from Theorem (1.0.6), would be a similar result, except that it does not require a framing of the manifolds.

Conjecture 1.0.7. *For connected smooth manifolds M and N, the wreath product of the exit-path ∞-category of the Ran space of M with the exit-path ∞-category of the Ran space of N localizes to the exit-path ∞-category of the Ran space of the product $M \times N$*

$$\mathit{Exit}(\mathit{Ran}^u(M)) \wr \mathit{Exit}(\mathit{Ran}^u(N)) \longrightarrow \mathit{Exit}(\mathit{Ran}^u(M \times N)).$$

Next, we present Corollary (1.0.8) of Theorem (1.0.6) which proves the aforementioned conjecture from [7]. The categories $\mathbf{\Theta}_n^{\mathrm{Exit}}$ are subcategories of $\mathbf{\Theta}_n$ which are defined in this thesis. Here, we define this subcategory for the case $n = 1$: $\mathbf{\Theta}_1^{\mathrm{Exit}}$ is the subcategory of $\mathbf{\Delta}$ in which objects are all those except $[0]$ and morphisms are all those that are injective and active.

Corollary 1.0.8. *For each $n \geq 1$, there is a localization*

$$\mathbf{\Theta}_n^{\mathit{Exit}} \to \mathit{Exit}(\mathit{Ran}(\mathbb{R}^n))$$

from the subcategory $\mathbf{\Theta}_n^{\mathit{Exit}}$ of $\mathbf{\Theta}_n^{\mathit{act}}$ to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

Technique The key idea behind Theorem (1.0.6) is that Exit$(-)$ carries refinements of stratified spaces to localizations. Though this idea does not directly apply to the case at hand, since $\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)$ is a presheaf on stratified spaces rather than a stratified space, it motivates our method of proof. Namely, we identify an ∞-category $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ that behaves like the exit-path ∞-category of a refinement of $\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)$, if such a thing were to exist, and then, we prove an equivalence from the exit-path ∞-category of that refinement to $\mathbf{\Theta}_n^{\mathrm{act}}$.

Finite-type knot invariants

In the case of the embedding space of knots, the application of configuration spaces saw progress in the development of configuration space integrals (Bott-Taubes integrals). Though the original motivation was to find new classical invariants [19], [8], [12], as it turned out, these integrals are 'finite-type knot invariants' and are in fact universal in the following sense: All finite-type invariants are constructed from a generalization of configuration space integrals [35].

In this thesis, we bring a direct, novel approach to the application of configuration spaces in the study of the embedding space of knots via the configuration categories $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$ and $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^3))$. The following theorem, which are proven in this thesis, yields a purely combinatorial perspective on the homotopy type of configuration spaces of the circle, as organized by the exit-path ∞-category.

Theorem 1.0.9. *There is equivalence of ∞-categories*

$$Exit(Ran(\mathbb{S}^1)) \simeq (\mathbf{\Delta}_{\circlearrowleft}^{surj})^{op}$$

between the exit-path ∞-category of the Ran space of the circle and the opposite of the subcategory of the parasimplex category consisting of surjective morphisms.

Theorem (1.0.9) together with Corollary (1.0.8) motivates identifying the relationship between this work and Goodwillie-Weiss embedding calculus as it relates to knots, which we briefly explain now. The manifold calculus of functors due to Goodwillie and Weiss defines a notion of 'polynomial approximation' of functors between suitable ∞-categories [38], [18]. Promising connections have been made between the embedding calculus and the study of finite-type invariants [37], [36], [13], [14]. Recently, the strongest evidence to date was established which supports the conjecture that the nth polynomial approximation of the Goodwillie-Weiss embedding tower is a universal type-n knot invariant over the integers [14]. The homotopy-theoretic context together with the role of the configuration space of points of the tower strongly suggest a close connection between it and the functor space arising from applying Exit(Ran(−)) to the space of knots.

Conjecture 1.0.10. *There is an equivalence between the space of functors over* $\mathbb{N}$ *from* $Exit(Ran(\mathbb{S}^1))$ *to* $Exit(Ran(\mathbb{R}^3))$ *and*

$$Map_{Cat_\infty/\mathbb{N}}(Exit(Ran(\mathbb{S}^1)), Exit(Ran(\mathbb{R}^3))) \simeq colim P_n(Emb(\mathbb{S}^1, \mathbb{R}^3))$$

the colimit of the nth-polynomial approximation of the Goodwillie-Weiss embedding calculus tower for knots.

The advantage of Conjecture (1.0.10) is that factorization homology of Ayala and Francis [3] can then be imported to yield a purely algebraic setting. Theorem (1.0.9) and Theorem (1.0.8) indicate that every finite-type knot invariant is captured through the Hochschild chain complex, the factorization homology which is the global observables on S^1 of a 1-dimensional TQFT,

$$\int_{S^1} A = \mathsf{HC}_\bullet(A) \ .$$

Conjecture 1.0.11. *For two knots* K *and* K' *that are distinguished by a finite-type knot invariant, there is an* $\mathcal{E}_3$*-algebra* A *in* $Mod_{\mathbb{K}}$ *for a commutative ring* $\mathbb{K}$*, for which the associated maps* $HC_\bullet(A) \to A$ *are not equivalent, where* $HC_\bullet(A)$ *is the Hochschild chain complex of* A*.*

<u>Technique</u> The key tool for showing the equivalence in Theorem (1.0.9) is the universal cover of $\mathbb{S}^1$, $\mathbb{R} \to \mathbb{S}^1$ given by exponentiation. Indeed, to give a brief

indication of the role of the universal cover in relating the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ and the parasimplex category, observe that the fiber in $\mathbb{R}$ over a finite subset of points of $\mathbb{S}^1$ is a parasimplex, the linear order of which is inherited from $\mathbb{R}$ and the $\mathbb{Z}$-action of which comes from the fact that the quotient space of $\mathbb{R}$ by the natural action of $\mathbb{Z}$ is $\mathbb{S}^1$.

Use of ∞-category/category theory

In this thesis, we use $(\infty, 1)$-*categories* to package the homotopy type of configuration spaces. In the last decade, the theory of $(\infty, 1)$-categories, which we refer to simply as ∞-categories, has been greatly developed, notable for the scope of this thesis is the work of Lurie in [26] on the theory of *quasi-categories* and the work of Rezk in [31] on the theory of *complete Segal spaces*.

Quasi-categories

We use Joyal's quasi-category model of ∞-categories [21]. Namely, a quasi-category is a called a *simplicial set*, that is, a functor from the opposite of the simplex category $\mathbf{\Delta}^{\mathrm{op}}$ (Definition (3.0.1)) to the category of sets Set, that satsifies a certain condition called the *inner-horn filling condition*. For the definition of this condition, together with other basic notions regarding quasi-catgeories which are present in the background of this work, notably ∞-*groupoids* and *geometric realization* see Rezk's friendly exposition [33].

Complete Segal spaces

We use Rezk's complete Segal spaces to model ∞-categories [31]. A complete Segal space (Definition (3.0.14)) is a *simplicial space*, that is, a functor from the opposite of the simplex category $\mathbf{\Delta}^{\mathrm{op}}$ to the ∞-category of spaces, Spaces.

Definition 1.0.12. The ∞-category of spaces, Spaces is the localization of the category of topological spaces that admit a CW structure and continuous maps thereof, localized on (weak) homotopy equivalences.

Terminology 1.0.13. We say *space* to refer to an object in Spaces; that is, a CW complex.

We refer the reader to [31] for any additional information regarding complete Segal spaces.

Notation 1.0.14. The value of a complete Segal space $\mathcal{C}$ on $[1]$ is the *arrow space* of $\mathcal{C}$, denoted $\mathrm{Ar}(\mathcal{C})$.

Model independence

In this thesis, we work model independently, which, by the work of Joyal and Tierney in [22], is a valid approach, since quasi-categories are shown to be equivalent to complete Segal spaces. Model independence is exercised in this thesis, for example, in that the hom-∞-groupoid with fixed source and target of a quasi-category is equivalent to a *space*, by which we mean a CW-complex, by way of the equivalence between quasi-categories and complete Segal spaces. Throughout this work, we are liberal with our use of model independence and typically do not give forewarning of its implementation.

Category theory

We use fundamental notions of category theory frequently and at leisure in this thesis. For the reader not familiar with categories, we suggest [34].

The nerve functor

There is a construction which takes an ordinary category $\mathcal{C}$ and produces an ∞-category $\mathcal{NC}$ called the *nerve* of $\mathcal{C}$. This construction is explicated by a fully faithful functor from the category of categories to the category of simplicial sets, through which each category is carried to a quasi-category. In light of the fully faithfulness of this functor, we refer to an ordinary category as an ∞-category without any reference to its nerve, whenever appropriate within the context. For a definition of the nerve, see 3.1 in [33].

Outline

The main result of Chapter 2 is Theorem 1.0.9, wherein we show an equivalence between the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ and the parasimplex category. The first half of the chapter is devoted to defining all those notions present in the statement of the main theorem, notable is a *stratified space*, the *Ran space*, and the *exit-path ∞-category*. Then, we prove Theorem (1.0.9) in two parts: First we define the functor in Construction (2.0.21) and then we show that this functor is an equivalence in Lemma (2.0.23).

The main result of Chapter 3 is Theorem (1.0.6), wherein we show that the category Θ_n^{act} localizes to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$. First, we define all those notions present in the statement of the main theorem. We prove Theorem (1.0.6) in two lemmas (3.0.57) and (3.0.63).

Lemma (3.0.57) states that there is an equivalence between the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ and $\mathbf{\Theta}_n^{\text{act}}$. We prove this equivalence in two parts: First we define the functor in Construction (3.0.58) and then we show that this functor is an equivalence in Lemma (3.0.59).

The second lemma used for proving Theorem (1.0.6), (3.0.63), states that the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ localizes to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$. The proof of this lemma is largely technical, rooted in category theory. We build the argument using Theorem (3.0.68) from [27] and prove two lemmas, (3.0.71) and (3.0.101), both of which rely on Quillen's Theorem B and the notion of a *Cartesian fibration*.

Lastly, we prove one corollary to Theorem (1.0.6), (3.0.103), wherein we show that a subcategory of $\mathbf{\Theta}_n$ localizes to the exit-path ∞-category of the Ran space of $\mathbb{R}^n$.

CONFIGURATION SPACES OF THE CIRCLE

In this chapter we focus on understanding an ∞-category that encodes the topology of finite subsets of the circle together with paths that witness subdivision of points on the circle, called the *exit-path ∞-category of the Ran space of the circle.* The main result of this chapter identifies that this ∞-category, which encodes all of this topological data, has a purely combinatorial description.

Preliminary definitions

We begin with the preliminary definitions of *Ran space*, *stratified space*, and the *exit-path ∞-category* as defined in ([26]) and the parasimplex category as defined in ([24]).

Definition 2.0.1. Given a manifold M, the *configuration space* of finite subsets of k points of M is the topological space

$$\mathrm{Conf}_k(M) := \{(x_1, ..., x_k) \in M^k \mid x_i \neq x_j \text{ if } i \neq j\}$$

under the subspace topology.

There is an evident action of Σ_k on $\mathrm{Conf}_k(M)$ given by permuting the order of the indexing. The resulting quotient space

$$\mathrm{Conf}_k(M)_{\Sigma_k} := \{S \subset M \mid |S| = k\}$$

is called the *unordered configuration space* of finite subsets of k points of M.

Intuitively, the configuration spaces for a fixed manifold naturally organize together as a topological space. For example, a half-open path in $\mathrm{Conf}_2(M)$ in which the two points approach one another limits to a point in $\mathrm{Conf}_1(M)$. A similar procedure intuits that for any two natural numbers k and r, $\mathrm{Conf}_k(M)$ is 'nearby' $\mathrm{Conf}_r(M)$ in some larger space; this larger space is defined next.

Definition 2.0.2. Given a smooth, non-empty, connected manifold M we denote $\mathrm{Ran}(M)$, called the *Ran space* of M, to be the set of nonempty, finite subsets of M

$$\mathrm{Ran}(M) := \{S \subset M \mid S \text{ is finite and non-empty}\}$$

with the finest topology for which the map

$$M^k \xrightarrow{I_k} \mathrm{Ran}(M)$$

given by $(m_1, ..., m_k) \mapsto \{m_1, ..., m_k\}$, is continuous for each $k \geq 1$.

There is a natural filtration of $\mathrm{Ran}(M)$ by $\mathrm{Conf}_k(M)_{\Sigma_k}$ for all $k \in \mathbb{N}$; such additional structure is made rigorous by a *stratified space*, which we define next.

Note 2.0.3. We regard a poset P as a topological space by defining $U \subset P$ to be open if and only if it is closed upwards; that is, if $a \in U$, then every $b \geq a$ is also in U.

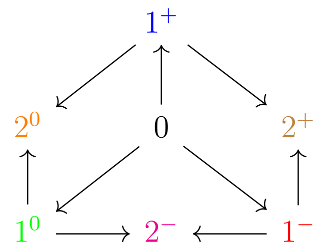
Definition 2.0.4 (2.1.3 in [6])**.** A *stratified space* $S : X \to P$ is a paracompact, Hausdorff topological space X together with a poset P and a continuous surjection S such that for each $p \in P$, the *p-stratum* $X_p := S^{-1}(p)$ is connected.

For stratified spaces $X \xrightarrow{S} P$ and $Y \xrightarrow{S'} P'$, a map $X \xrightarrow{f} Y$ is a *stratified map* if f respects the stratified structures of X and Y, i.e.,

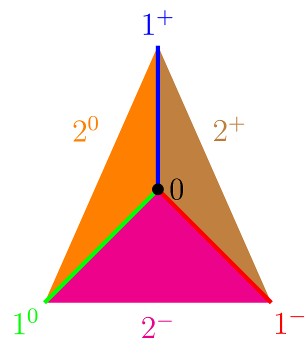
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S'} \\ P & \longrightarrow & Q \end{array}$$

is a commutative diagram.

Example 2.0.5. We stratify $\mathbb{R}^2$ over the poset P, defined to be



where the arrows indicate the ordering. The following figure depicts the stratification $\mathbb{R}^2 \to P$, where the positioning and coloring of the figure and P are coordinated as to indicate the stratification.



Example 2.0.6. The standard stratification of the topological p-simplex

$$\Delta^p := \{(t_0, ..., t_p) \in \mathbb{R}^{p+1} \mid \Sigma_{i=0}^{p} t_i = 1,\ t_i \geq 0\ \forall i\} \to [p] := \{0 < \cdots < p\}$$

is given by

$$(t_0, ..., t_p) \mapsto \max\{i \mid t_i \neq 0\}.$$

In this thesis, we focus on the Ran space of a manifold as a stratified space.

Example 2.0.7. $\mathrm{Ran}(M)$ naturally emits a stratification over the natural numbers by cardinality, $\mathrm{Ran}(M) \to \mathbb{N}$ given by $S \mapsto |S|$.

The k-stratum $\mathrm{Ran}(M)_k$ is the unordered configuration space of finite subsets of k points of M, $\mathrm{Conf}_k(M)_{\Sigma_k}$.

Notation 2.0.8. Typically, we denote a stratified space $X \xrightarrow{S} P$ by its underlying topological space X, if we expect S and P to be understood.

We are interested in understanding the homotopy type of the stratified space $\mathrm{Ran}(\mathbb{S}^1)$; the next tool provides a way to do just that. For a manifold M, the homotopy types of configuration spaces of M form the underlying ∞-groupoid of the *exit-path ∞-category* of $\mathrm{Ran}(M)$. In general, for any stratified space X, we define the exit-path ∞-category of X as follows:

Definition 2.0.9. For a stratified space $X \to P$, the *exit-path ∞-category* of X, $\mathrm{Exit}(X)$, is the simplicial set whose value on $[p]$ is the set

$$\{\Delta^p \xrightarrow{f} X \mid f \text{ is a stratified map}\}.$$

Informally, then, an object of $\mathrm{Exit}(X)$ is a point in X and a morphism is a path in X which is allowed to 'exit' from a deeper stratum to a less-deep stratum, but not vice-versa.

Remark 2.0.10. Definition (2.0.4) is not strong enough to guarantee that the simplicial set defining the exit-path ∞-category of a stratified space will satsify the inner-horn filling condition. Rather, a stronger notion of a stratified space, namely a *conically stratified space* (A.5 in [26]) is needed to guarantee that the exit-path ∞-category construction yields an ∞-category. This is shown in A.6 in [26]. The defining feature of a conically stratified space is that locally it is Euclidean space cross the cone of some stratified space of a lower dimension. The difference between Definition (2.0.4) and this stronger notion of a stratified space is technical and would distract the reader from the focus of this thesis. Extensive treatment of conically stratified spaces is given in [26] and [6] and we refer the reader to either source.

In 3.7 in [6], it is shown that the Ran space of a manifold is a conically stratified space. Hence, the exit-path ∞-category of the Ran space of a manifold is in fact an ∞-category.

In this chapter, we seek to understand the exit-path ∞-category of the Ran space of the circle, $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$. Heuristically, an object is a finite subset S of $\mathbb{S}^1$ and a morphism is an 'exit-path' in $\mathrm{Ran}(\mathbb{S}^1)$ i.e., a morphism $S \xrightarrow{f} T$ is given by a stratified map $\Delta^1 \to \mathrm{Ran}(\mathbb{S}^1)$ whose value at $(1,0) \in \Delta^1$ is S and at $(0,1)$ is T, and further, $|S| \leq |T|$. The following definition equips us to have an alternative definition of a morphism of f.

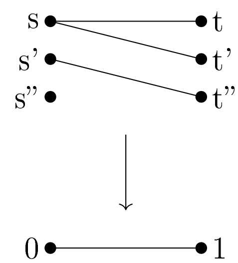
Definition 2.0.11 ([4])**.** The *reversed cylinder* of a map between finite sets $T \to S$ is

$$\mathrm{cylr}(T \to S) := S \coprod_{T \times \{0\}} T \times \Delta^1.$$

More generally, the reversed cylinder of a composable sequence of maps between finite sets $S_p \to S_{p-1} \to \cdots \to S_0$ is

$$\mathrm{cylr}(S_p \to S_{p-1} \to \cdots \to S_0) := S_0 \coprod_{S_1 \times \{0\}} S_1 \times \Delta^1 \coprod_{S_2 \times \Delta^1} \ldots \coprod_{S_p \times \Delta^{p-1}} S_p \times \Delta^p.$$

Example 2.0.12. Consider the map of sets $T = \{t, t', t''\} \to S = \{s, s', s''\}$ given by $t, t' \mapsto s$ and $t'' \mapsto s'$. The reverse cylinder $\mathrm{cylr}(T \to S)$ over Δ^1 is depicted by the following figure:

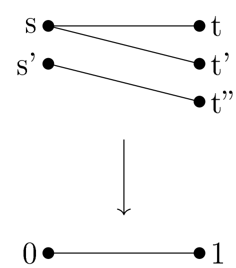


Observation 2.0.13. Given a connected manifold M, a point $\Delta^p \to \mathrm{Ran}(M)$ in the set of $[p]$-values of $\mathrm{Exit}(\mathrm{Ran}(M))$ is an embedding over Δ^p

$$\mathrm{cylr}(S_p \twoheadrightarrow \cdots \twoheadrightarrow S_0) \hookrightarrow M \times \Delta^p$$

for some sequence of surjective maps of finite sets $S_p \twoheadrightarrow \cdots \twoheadrightarrow S_0$.

Example 2.0.14. Consider $M = \mathbb{R}^2$. An embedding of the reversed cylinder $\mathrm{cylr}(T \twoheadrightarrow S)$ depicted by



into $\mathbb{R}^2 \times \Delta^1$ is an exit-path in $\mathrm{Ran}(\mathbb{R}^2)$ which starts at (the embedded image of) $S \subset \mathbb{R}^2$ and ends at (the embedded image of) $T \subset \mathbb{R}^2$. The image of the embedding is the graph of this exit-path in $\mathbb{R}^2$.

The parasimplex category and configuration spaces of $\mathbb{S}^1$

The main result of this chapter shows that the homotopy type of configurations of finite subsets of $\mathbb{S}^1$ as organized by the stratified space $\mathrm{Ran}(\mathbb{S}^1)$, codified by the exit-path ∞-category of that stratified space, $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$ is encoded by the following combinatorially defined category.

The parasimplex category

Definition 2.0.15. The *parasimplex category* $\mathbf{\Delta}_{\circlearrowleft}$ is the category in which an object is a *parasimplex*: A nonempty, linearly ordered set Λ equipped with a $\mathbb{Z}$-action,

$$+ : \Lambda \times \mathbb{Z} \to \Lambda$$

that satsifies two properties

i) for each $\lambda \in \Lambda, \lambda < \lambda + 1$

ii) for each $\lambda, \lambda' \in \Lambda, \{\mu \in \Lambda \mid \lambda < \mu < \lambda'\}$ is finite.

A morphism $\Lambda \xrightarrow{f} \Lambda'$ is a *paracyclic map*: A nondecreasing and $\mathbb{Z}$-equivariant map of parasimplicies.

Example 2.0.16. For each $n \in \mathbb{Z}_+$, the set $\frac{1}{n}\mathbb{Z} := \left\{ \frac{i}{n} \,\middle|\, i \in \mathbb{Z} \right\}$ is a parasimplex, the linear order and $\mathbb{Z}$-action of which is inherited as a subset of $\mathbb{R}$ with linear order given by positive orientation and the natural action of $\mathbb{Z}$ by addition.

Observation 2.0.17 (4.2.3. in [24])**.** Each parasimplex Λ is isomorphic to the parasimplex $\frac{1}{n}\mathbb{Z}$ for a unique $n \in \mathbb{Z}_+$.

Definition 2.0.18. The category $\mathbf{\Delta}_{\circlearrowleft}^{\text{surj}}$ is the subcategory of the parasimplex category whose objects are the same as those of $\mathbf{\Delta}_{\circlearrowleft}$ and whose morphisms are all those of $\mathbf{\Delta}_{\circlearrowleft}$ that are surjective.

Identifying the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ as $\mathbf{\Delta}_{\circlearrowleft}^{\text{surj}}$

The next theorem, the main result of this chapter, articulates a sense in which the homotopy type of configurations of unordered points in $\mathbb{S}^1$ is encoded by an ordinary, combinatorially defined category, the *opposite* of $\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{surj}}$. First, we define the notion of an opposite category.

Definition 2.0.19. Given a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is the category whose objects are the same as those of $\mathcal{C}$ and in which there is a morphism from d to c precisely when there is a morphism from c to d in $\mathcal{C}$. Composition is given by composition in $\mathcal{C}$.

Theorem 2.0.20. *There is an equivalence of ∞-categories*

$$Exit(Ran(\mathbb{S}^1)) \xrightarrow{\simeq} (\mathbf{\Delta}_{\circlearrowleft}^{surj})^{op}$$

from the exit-path ∞-category of the Ran space of the circle to the opposite of the category $\mathbf{\Delta}_{\circlearrowleft}^{surj}$.

While intuitive, this result in particular implies that the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ is, in fact, an ordinary category. We break Theorem (2.0.20) up into two parts: In Part 1, Construction (2.0.21) defines a functor from the exit-path ∞-catetgory of the Ran space of $\mathbb{S}^1$. In Part 2, Lemma (2.0.23) states that the functor defined in Construction (2.0.21) is an equivalence. The main tool we use is the universal cover $\mathbb{R} \to \mathbb{S}^1$.

Part 1: Defining the functor Υ

Construction 2.0.21. There is a functor

$$\Upsilon : \mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1)) \to (\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{surj}})^{\mathrm{op}}$$

from the exit-path ∞-category of the Ran space of the circle to the opposite of the category $\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{surj}}$.

Note 2.0.22. A functor from an ∞-category to the nerve of a (small) category is completely determined by its assignment on objects, morphisms and the requirement that composition is respected. This is due to the fact that a simplicial set given by the nerve of a small category is completely determined by its values on $[i]$ for $0 \leq i \leq 2$. (See the proof of Lemma 3.5 in [17] for more details.)

Proof. By Note (2.0.22), because the target of Υ is an ordinary category, it suffices to name Υ on objects and morphisms and check that composition is preserved.

In defining Υ, we make use of the universal cover $\mathbb{R} \xrightarrow{\exp} \mathbb{S}^1$ given by the exponentiation map $e^{2\pi i -}$. The value of Υ on an object $S \subset \mathbb{S}^1$ of $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$ is the preimage of exp over S, $\exp^{-1}(S)$ which, as a subset of $\mathbb{R}$ inherits both a linear structure and an action of $\mathbb{Z}$ from $\mathbb{R}$ with linear order on $\mathbb{R}$ given by positive orientation and the natural $\mathbb{Z}$-action of addition.

By Observation (2.0.13), a morphism $S \to T$ in $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$ is an embedding $\mathrm{cylr}(T \xrightarrow{f} S) \overset{E}{\hookrightarrow} \mathbb{S}^1 \times \Delta^1$ over Δ^1. For each $\lambda \in \exp^{-1}(T)$, consider the path given by the restriction of E to the reverse cylinder of the assignment of the singleton $\exp(\lambda) \mapsto f(\exp(\lambda))$ under f

$$E_| : \mathrm{cylr}(\exp(\lambda) \mapsto f(\exp(\lambda))) \hookrightarrow \mathbb{S}^1 \times \Delta^1.$$

Denote this path in $\mathbb{S}^1$ by $\gamma_{E,\lambda}$. Now we are prepared to name the value of Υ on $S \to T$: The image $\exp^{-1}(T) \xrightarrow{\Upsilon(f)} \exp^{-1}(S)$ is defined by

$$\lambda \mapsto \tilde{\gamma}_{E,\lambda}(1)$$

where $\tilde{\gamma}_{E,\lambda}$ is the unique path lift of the reverse of $\gamma_{E,\lambda}$ such that $\tilde{\gamma}_{E,\lambda}(0) = \lambda$:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{\gamma}_{E,\lambda}}{\nearrow} & \downarrow \exp \\ [0,1] & \xrightarrow{\bar{\gamma}_{E,\lambda}} & \mathbb{S}^1 \end{array}$$

where $\bar{\gamma}_{E,\lambda}$ denotes the reverse of the path $\gamma_{E,\lambda}$.

It is straightforward to check that $\Upsilon(f)$ is nondecreasing. $\mathbb{Z}$-equivariance of $\Upsilon(f)$ follows from the bijective correspondance between $\pi_1(\mathbb{S}^1)$ and the set of deck transformations of $\mathbb{R}$ as the universal cover of $\mathbb{S}^1$. Thus, $\Upsilon(f)$ is a paracyclic map. Furthermore, surjectivity of $\Upsilon(f)$ is straightforward to verify.

Consider a commutative triangle in Exit(Ran($\mathbb{S}^1$))

$$\begin{array}{ccc} S & & \\ \downarrow & \searrow & \\ T & \longrightarrow & R \end{array} \tag{2.0.1}$$

given by the embedding

$$\operatorname{cylr}(R \xrightarrow{g} T \xrightarrow{f} S) \overset{E}{\hookrightarrow} \mathbb{S}^1 \times \Delta^2$$

over Δ^2. Denote the subembeddings of E which define each arrow in (2.0.1) by

$$\operatorname{cylr}(T \xrightarrow{f} S) \overset{E_f}{\hookrightarrow} \mathbb{S}^1 \times \Delta^1, \ \operatorname{cylr}(R \xrightarrow{g} T) \overset{E_g}{\hookrightarrow} \mathbb{S}^1 \times \Delta^1 \text{ and } \operatorname{cylr}(R \xrightarrow{f \circ g} S) \overset{E_{f\circ g}}{\hookrightarrow} \mathbb{S}^1 \times \Delta^1.$$

Note that for each $\lambda \in R$, E restricted to the reverse cylinder of maps of singletons

$$\operatorname{cylr}(\exp(\lambda) \mapsto g(\exp(\lambda)) \mapsto f(g(\exp(\lambda))))$$

is a homotopy of paths in $\mathbb{S}^1$ between the concatenation $\gamma_{E_f, \tilde{\gamma}_{E_g,\lambda}(1)} * \gamma_{E_g,\lambda}$ and $\gamma_{E_{f\circ g},\lambda}$.

The image of (2.0.1) under Υ commutes if this homotopy of paths $\Delta^1 \to \mathbb{S}^1$ has a lift to $\mathbb{R}$ under exp for the fixed lift $\{0\} \mapsto \lambda$. Indeed, because Δ^1 is simply connected, such a lift is guaranteed by the homotopy lifting property.

□

We have just constructed a functor from the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ to the opposite of the subcategory $\mathbf{\Delta}_{\circlearrowleft}^{\text{surj}}$ of $\mathbf{\Delta}_{\circlearrowleft}$ consisting of only surjective morphisms. The next lemma states that this functor names an equivalence of ∞-categories.

Part 2: Showing Υ is an equivalence

Lemma 2.0.23. *The functor Υ defined in Lemma (2.0.21) is an equivalence*

$$\Upsilon : \mathit{Exit}(\mathit{Ran}(\mathbb{S}^1)) \xrightarrow{\simeq} (\mathbf{\Delta}_{\circlearrowleft}^{\mathit{surj}})^{op}$$

from the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ to the opposite of the category $\mathbf{\Delta}_{\circlearrowleft}^{\mathit{surj}}$.

Definition 2.0.24. A functor of ∞-categories $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is

- *essentially surjective* if for each object in $\mathcal{D}$, there exists an object c in $\mathcal{C}$ whose value under F is canonically isomorphic to d.
- *fully faithful* if for each pair of objects c and c' in $\mathcal{C}$, the map induced by F between hom-spaces $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$ is a weak equivalence.
- an *equivalence of ∞-categories* if F is essentially surjective and fully faithful.

Proof. (2.0.23) We will show that Υ is essentially surjective and fully faithful. Essential surjectivity of Υ follows from (2.0.17). Indeed, given a parasimplex Λ, it is isomorphic to $\frac{1}{n}\mathbb{Z}$ for a unique $n \in \mathbb{Z}$. The image of the object $S := \{e^{\frac{2\pi i m}{n}} \mid 0 \leq m \leq n-1\}$ in $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$ under Υ is $\frac{1}{n}\mathbb{Z}$.

We will show fully faithfulness by showing that the map induced between hom-spaces with fixed source and target

$$\mathrm{Hom}_{\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))}(S,T) \xrightarrow{\Upsilon} \mathrm{Hom}_{(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{surj}})^{\mathrm{op}}}(\exp^{-1}(S), \exp^{-1}(T)) \tag{2.0.2}$$

is a weak homotopy equivalence of spaces; that is, a surjection on path components with contractible fibers (contractible fibers because $\mathbf{\Delta}_{\circlearrowleft}$ as an ordinary cateogry has discrete hom-spaces). A weak homotopy equivalence implies a homotopy equivalence in this context because hom-spaces of ∞-categories are equivalent to CW-complexes.

Fix a morphism $\exp^{-1}(T) \xrightarrow{f} \exp^{-1}(S)$ in the target of (3.0.19). For each $\lambda \in \exp^{-1}(T)$, let $F_\lambda : [0,1] \longrightarrow \mathbb{R}$ be the straight line path in $\mathbb{R}$ from $f(\lambda)$ to λ. Postcomposing with exp yields a path g_λ in $\mathbb{S}^1$ from $\exp(f(\lambda)) \in S$ to $\exp(\lambda) \in T$. Note that $g_\lambda = g_{\lambda'}$ whenever $\lambda, \lambda' \in \exp^{-1}(t)$ for some $t \in T$. The set of distinct g_λ, which we denote by $\{g_t\}_{t \in T}$, defines a morphism g in $\mathrm{Exit}(\mathrm{Ran}(\mathbb{S}^1))$ from S to T whose image under Υ is f. Thus, (3.0.19) is a surjection on connected components.

Let $\alpha : \mathbb{S}^n \to \Upsilon^{-1}(f)$ be continuous and based at the morphism g defined previously. We will show that α is homotopic to the constant map at g. For each $r \in \mathbb{S}^n$, let $E_{\alpha(r)}$ denote the embedding defining the morphism $\alpha(r)$. For each $r \in \mathbb{S}^n$ and $t \in T$, consider the set of lifted paths $\{\tilde{\gamma}_{E_{\alpha(r)},\lambda}\}_{\lambda \in \exp^{-1}(t)}$ in $\mathbb{R}$ via exp. For each triple (r,t,λ) such that $\lambda \in \exp^{-1}(t)$, define the straight-line homotopy from $\tilde{\gamma}_{E_{\alpha(r)},\lambda}$ to the straight-line path F_λ defined previously

$$[0,1] \times [0,1] \xrightarrow{H'_{r,t,\lambda}} \mathbb{R}$$

by

$$(v, u) \mapsto (1 - v)\tilde{\gamma}_{E_{\alpha(r)},\lambda}(1 - u) + vF_\lambda(u).$$

Exponentiating yields a straight-line homotopy from $\gamma_{E_{\alpha(r)},\lambda}$ to g_t in $\mathbb{S}^1$. Note that this homotopy is independent of the choice of $\lambda \in \exp^{-1}(t)$. Thus, for each pair (r, t), define $H_{r,t} := \exp \circ H'_{r,t,\lambda}$, for any $\lambda \in \exp^{-1}(t)$. Now we are equipped to define a homotopy $[0, 1] \times \mathbb{S}^n \xrightarrow{H} \Upsilon^{-1}(f)$ from γ to the constant map at g. For each $(v, r) \in [0, 1] \times \mathbb{S}^n$, the set of paths in $\mathbb{S}^1$ $\{H_{r,t}(v, -)\}_{t \in T}$ define a morphism in $\Upsilon^{-1}(f)$. We define this morphism to be the value of (v, r) under H. Thus, a fiber of Υ is contractible and hence, Υ is an equivalence.

$\square$

In summary, we have shown that the exit-path ∞-category of the Ran space of $\mathbb{S}^1$ is equivalent to an ordinary category, the opposite of the subcategory of the parasimplex category consisting of all the same objects and only surjective morphisms. In so doing, we establish a sense in which the homotopy type of configurations of unordered points of $\mathbb{S}^1$ is encoded by the combinatorially defined parasimplex category.

CONFIGURATION SPACES OF EUCLIDEAN SPACE

The main result of this chapter generalizes the result in [7] that $\mathbf{\Theta}_n$ naturally encodes the homotopy type of configuration spaces of ordered points in $\mathbb{R}^n$. We will show that $\mathbf{\Theta}_n$ encodes the homotopy type of configuration spaces of unordered points in $\mathbb{R}^n$ by showing the subcategory $\mathbf{\Theta}_n^{\mathrm{act}}$ of $\mathbf{\Theta}_n$ consisting of all those active morphisms of $\mathbf{\Theta}_n$ localizes to the 'unital' version of the exit-path ∞-category of the Ran space of $\mathbb{R}^n$; an ∞-category in which an object is a finite subset of $\mathbb{R}^n$, including the empty subset, and in which a morphism is an exit-path, including those that allow points to vanish.

The exit-path ∞-category of the unital Ran Space

In [4], Ayala, Francis and Rozenblyum extend the definition of the exit-path ∞-category of a stratified space to a definition of the exit-path ∞-category of a stack of stratified spaces. With this definition in hand, we may articulate the sense in which we encode the homotopy type of unordered configurations of points in $\mathbb{R}^n$ with an ∞-category; namely, the 'unital' version of the exit-path ∞-category of the Ran space of $\mathbb{R}^n$, which we define next, after reviewing two important categories and two important concepts in category theory.

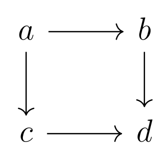
Definition 3.0.1. The *simplex category* $\mathbf{\Delta}$ is the category in which an object is a non-empty, finite, linearly ordered set and in which a morphism is a non-decreasing map of sets. Composition is composition of maps between sets.

Notation 3.0.2. For each object S of $\mathbf{\Delta}$, there is a unique non-negative integer p such that S is canonically isomorphic to the linearly ordered set $[p] := \{0 < \cdots < p\}$. We call $[p]$ the *p-simplex* and will henceforth refer to the objects of $\mathbf{\Delta}$ as p-simplicies.

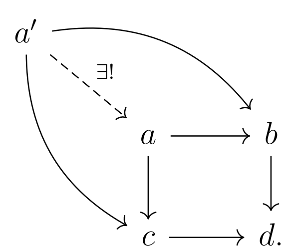
Notation 3.0.3. The category of posets Poset has an evident fully faithfully embedding into the category of categories Cat. In light of this fully faithful functor, we refer to $[p]$ as either a linearly ordered set or as the category whose objects are $\{0, 1, ..., p\}$ and in which there is a unique morphism from i to j precisely when $i \leq j$, and no morphism otherwise.

Definition 3.0.4. Fin is the category in which an object is a finite set and a morphism is a map of sets; composition is evident.

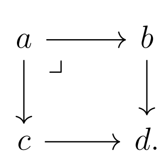
Definition 3.0.5. Given a category $\mathcal{C}$, the *pullback* of the morphisms $c \to d$ and $b \to d$ in $\mathcal{C}$ is a triple consisting of an object a in $\mathcal{C}$ together with two morphisms $a \to c$ and $a \to b$ in $\mathcal{C}$ for which the following diagram in $\mathcal{C}$



commutes and is universal in the following sense: For any triple consisting of an object $a' \in \mathcal{C}$ and morphisms $a' \to c$ and $a' \to b$ in $\mathcal{C}$ for which the following diagram commutes, there is a unique morphism $a' \to a$ in $\mathcal{C}$ satsifying:



We denote a pullback diagram by



Definition 3.0.6. Given a category $\mathcal{C}$ over Fin_* via a functor F and a category $\mathcal{D}$, the *wreath product* $\mathcal{C} \wr \mathcal{D}$ is the pullback of categories

$$\begin{array}{ccc} \mathcal{C} \wr \mathcal{D} & \longrightarrow & \mathrm{Fin}_* \wr \mathcal{D} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathrm{Fin}_* \end{array}$$

where the vertical arrow on the right is the forgetful functor from Observation (3.0.22).

Observation 3.0.7. The nerve of $\mathrm{Fin}^{\mathrm{op}}$ is a presheaf on $\mathbf{\Delta}$; that is,

$$\mathcal{N}\mathrm{Fin}^{\mathrm{op}} : \mathbf{\Delta} \to \mathsf{Spaces}\,.$$

Thus, we may define the ∞-category $\mathbf{\Delta}$ slice over $\mathrm{Fin}^{\mathrm{op}}$, $\mathbf{\Delta}_{/\mathrm{Fin}^{\mathrm{op}}}$

$$\begin{array}{ccc} \mathbf{\Delta}_{/\mathrm{Fin}^{\mathrm{op}}} & \longrightarrow & \mathrm{PShv}(\mathbf{\Delta})_{/\mathrm{Fin}^{\mathrm{op}}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{\Delta} & \xrightarrow{\text{Yoneda}} & \mathrm{PShv}(\mathbf{\Delta}) \end{array}$$

as the pullback of ∞-categories.

Definition 3.0.8. For a smooth, connected manifold M, the *exit-path ∞-category of the unital Ran space of M*, $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$, is the simplicial space over $\mathrm{Fin}^{\mathrm{op}}$ representing the presheaf on $\mathbf{\Delta}_{/\mathrm{Fin}^{\mathrm{op}}}$ whose value on an object

$$[p] \xrightarrow{<\sigma>} \mathrm{Fin}^{\mathrm{op}}$$

which selects out a sequence of maps among finite sets $\sigma : S_p \to \cdots \to S_0$, is the space of embeddings

$$\mathrm{cylr}(\sigma) \hookrightarrow M \times \Delta^p$$

over Δ^p equipped with the compact-open topology; the structure maps are evident.

Remark 3.0.9. The functor that names $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ over $\mathrm{Fin}^{\mathrm{op}}$,

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M)) \xrightarrow{\phi} \mathrm{Fin}^{\mathrm{op}}$$

is the evident forgetful functor; its value on an object $S \hookrightarrow M$ is S and its value on a morphism $\mathrm{cylr}(J \xrightarrow{\sigma} S) \hookrightarrow M \times \Delta^1$ is σ.

Observation 3.0.10. Explicitly, an object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ in the fiber over the finite (possibly empty) set S is an embedding,

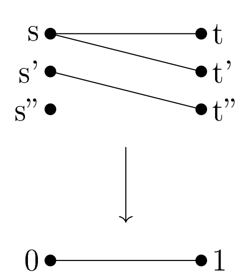
$$S \hookrightarrow M.$$

A morphism from $S \overset{e}{\hookrightarrow} M$ to $T \overset{d}{\hookrightarrow} M$ over the map of finite sets $T \xrightarrow{\sigma} S$ is an embedding,

$$\mathrm{cylr}(T \xrightarrow{\sigma} S) \overset{E}{\hookrightarrow} M \times \Delta^1$$

over Δ^1 such that $E_{|S} = e$ and $E_{|T\times 1} = d$.

Example 3.0.11. Consider $M = \mathbb{R}^2$. An embedding of the reversed cylinder $\mathrm{cylr}(T \to S)$ depicted by



into $\mathbb{R}^2 \times \Delta^1$ is a morphism in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^2))$ which starts at (the embedded image of) $S \subset \mathbb{R}^2$ and ends at (the embedded image of) $T \subset \mathbb{R}^2$.

The image of the embedding in Example (3.0.11) is the graph of a set of paths in $\mathbb{R}^2$ that almost name an 'exit-path' in $\mathrm{Ran}(\mathbb{R}^2)$, except that they do not because the point s'' vanishes. The failure of this embedding to name a morphism in $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^2))$ is that the map of finite sets $T \to S$ is not surjective.

Informally then, we see that the objects of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ are all those in $\mathrm{Exit}(\mathrm{Ran}(M))$ together with the empty subset, and its morphisms are all those exit-paths in $\mathrm{Exit}(\mathrm{Ran}(M))$ together with morphisms that are not quite exit-paths, in that they allow points in the source to disappear. This feature explains the he superscript 'u' - it stands for 'unital', which, through Theorem (3.0.46), makes reference to the role of degeneracy morphisms in $\mathbf{\Theta}_n$.

Evidently then, $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ 'contains' $\mathrm{Exit}(\mathrm{Ran}(M))$ as a sub-simplicial space. Indeed, our observations in (2.0.13) indicate that $\mathrm{Exit}(\mathrm{Ran}(M))$ defines a simplicial space over $(\mathrm{Fin}^{\mathrm{surj}}_{\neq\emptyset})^{\mathrm{op}}$, the value of which on $[p]$ over $S_p \twoheadrightarrow \cdots \twoheadrightarrow S_0$ is the space of embeddings of $\mathrm{cylr}(S_p \twoheadrightarrow \cdots \twoheadrightarrow S_0)$ into $M \times \Delta^p$ over Δ^p, equipped with the compact-open topology. Observation (3.0.12) explicates precisely how $\mathrm{Exit}(\mathrm{Ran}(M))$ is a sub-simplicial space of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$.

Observation 3.0.12. The exit-path ∞-category of the Ran space of $\mathbb{R}^n$, $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$, is a sub-simplicial space of the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$, $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$, described as the following pullback of simplicial spaces:

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)) & \hookrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)) \\ \downarrow & \lrcorner & \downarrow \\ (\mathrm{Fin}^{\mathrm{surj}}_{\neq\emptyset})^{\mathrm{op}} & \hookrightarrow & \mathrm{Fin}^{\mathrm{op}} \end{array} \tag{3.0.1}$$

where the downward arrows are given by the forgetful functor ϕ_n (3.0.9), which just remembers the underlying set data.

Remark 3.0.13. As previously mentioned, an alternative, but equivalent definition of the exit-path ∞-category of the unital Ran space of a manifold M is as the value of a certain presheaf on stratified spaces under the functor $\mathrm{Exit}(-)$; this functor is defined in [4]. In this thesis, we choose to not introduce this presheaf as it is not explicitly used and further, would distract the reader from the core ideas.

The exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ is an ∞-category

Of central, technical importance in this thesis is that, for a smooth, connected manifold M, $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ is, in fact, an ∞-category. Such a fact, however, does not follow from [26], as does the case of the exit-path ∞-category of the stratified space $\mathrm{Ran}(M)$, because $\mathrm{Ran}^{\mathrm{u}}(M)$ is not stratified space, as discussed in Remark (3.0.13). This subsection is devoted to proving the technical result that $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ is a complete Segal space.

Definition 3.0.14. A simplicial space $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{F} \mathsf{Spaces}$ is a *complete Segal space* if it satsifies the following two conditions:

i) (Segal Condition) For each $p > 1$, the diagram of spaces

$$\begin{array}{ccc} F[p] & \longrightarrow & F\{p-1<p\} \\ \downarrow & & \downarrow \\ F\{0<\cdots<p-1\} & \longrightarrow & F\{p-1\} \end{array}$$

is a pullback.

ii) (Completeness Condition) The diagram of spaces

$$
\begin{array}{ccccccc}
 & & & F(*) & & & \\
 & \swarrow & & \downarrow & & \searrow & \\
F(*) & & & F[3] & & & F(*) \\
 & \searrow & \swarrow & & \searrow & \swarrow & \\
 & & F\{0<2\} & & F\{1<3\} & &
\end{array}
$$

is a limit.

Proposition 3.0.15. *The simplicial space* $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ *satisfies the Segal and completeness conditions.*

Corollary 3.0.16. *The simplicial space* $\mathrm{Exit}(\mathrm{Ran}(M))$ *satisfies the Segal and completeness conditions.*

Proof. In (3.0.12), we observed the pullback diagram among simplicial spaces

$$
\begin{array}{ccc}
\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)) & \hookrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)) \\
\downarrow & \lrcorner & \downarrow \\
(\mathrm{Fin}^{\mathrm{surj}}_{\neq\emptyset})^{\mathrm{op}} & \hookrightarrow & \mathrm{Fin}^{\mathrm{op}}
\end{array}
$$

The result follows because the full ∞-subcategory of simplicial spaces consisting of the complete Segal spaces is closed under the formation of pullbacks.

□

We allow ourselves, in this subsection, to freely use notation and results from [4]. The idea for this proof is to witness the simplicial space $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$ as one derived through formal constructions among complete Segal spaces from a complete Segal space Bun, defined in §6 of [4].

Namely, the simplicial space $\mathsf{Bun}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Spaces}$ is that for which the value on $[p]$ is the moduli space of *constructible bundles* over Δ^p with its *standard stratification*. The simplicial structure maps are implemented by base change of constructible bundles. Section §6 of [4] is devoted to the proof that this simplicial space satisfies the Segal and completeness conditions, which is to say Bun is an ∞-category.

So the space of objects in Bun is the moduli space of constructible bundles over $\Delta^0 = *$, which is simply the moduli space of stratified spaces. So an object of Bun is simply a stratified space. In particular, a finite set is an example of an object in Bun, and a smooth manifold is an example of an object in Bun as well. Lemma 6.3.11 of [4] constructs a fully faithful functor

$$\mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathsf{Bun} \ ,$$

whose image consists of finite sets. In particular, there is a composite monomorphism between ∞-categories:

$$\mathrm{Fin}^{\mathrm{op}} \xrightarrow{(-)_+} \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathsf{Bun} \ . \tag{3.0.2}$$

Note that a $[p]$-point $X \to \Delta^p$ of Bun factors through this monomorphism (3.0.2) if and only if $X \to \Delta^p$ is a finite proper constructible bundle.

Lemma 3.31 of [5] constructs, for each dimension k, the *k-skeleton* functor

$$\mathsf{sk}_k\colon \mathsf{Bun} \longrightarrow \mathsf{Bun} \ .$$

Explicitly, the value on a stratified space X is the proper constructible stratified subspace $\mathsf{sk}_k(X) \subset X$ that is the union of the strata whose dimension is at most k.

The value of sk_k on a $[p]$-point $X \to \Delta^p$ of Bun is the constructible bundle

$$\mathsf{sk}_k^{\mathsf{fib}}(X) \longrightarrow \Delta^p$$

which is the *fiberwise k-skeleton* of X – it is the union of those strata of X whose projection to Δ^p have fiber-dimension at most k.

Note, then, that sk_0 factors through Fin_*:

$$\mathsf{sk}_0\colon\ \mathsf{Bun} \longrightarrow \mathrm{Fin}_*^{\mathrm{op}} \hookrightarrow \mathsf{Bun}\ \ .$$

Consider the ∞-category $\mathcal{S}\mathsf{trat}^{\mathsf{ref}}$ underlying the topological category in which an object is a stratified space and the space of morphisms is that of refinements. Section §6.6 of [4] constructs the *open cylinder* functor between ∞-categories

$$\mathsf{Cylo}\colon \mathcal{S}\mathsf{trat}^{\mathsf{ref}} \longrightarrow \mathsf{Bun}$$

which is an equivalence on spaces of objects. Theorem 6.6.15 of [4] verifies that this functor is a monomorphism. So each refinement between stratified spaces defines a morphism in Bun. As a matter of notation, a morphism in Bun that is in the image of this functor is called a *refinement*; the ∞-category of *refinement arrows* in Bun is the full ∞-subcategory

$$\mathrm{Ar}^{\mathsf{ref}}(\mathsf{Bun})\ \subset\ \mathrm{Ar}(\mathsf{Bun})$$

consisting of the refinements arrows. Evaluation at source-target defines a functor

$$(\mathsf{ev}_s, \mathsf{ev}_t)\colon \mathrm{Ar}^{\mathsf{ref}}(\mathsf{Bun}) \longrightarrow \mathsf{Bun}\times\mathsf{Bun}\ \ .$$

Denote the pullback ∞-category:

$$\begin{array}{ccccc}
\mathcal{R}\mathsf{ef}^0(M) & \longrightarrow & & & \mathrm{Ar}^{\mathsf{ref}}(\mathsf{Bun}) \\
\downarrow & & & & \downarrow{\scriptstyle (\mathsf{ev}_s,\mathsf{ev}_t)} \\
 & & & & \mathsf{Bun}\times\mathsf{Bun} \\
 & & & & \downarrow{\scriptstyle \mathsf{sk}_{n-1}\times\mathrm{Id}} \\
\mathrm{Fin}^{\mathrm{op}} & \xrightarrow{=} & \mathrm{Fin}^{\mathrm{op}}\times * & \xrightarrow{(3.0.2)\times\langle M\rangle} & \mathsf{Bun}\times\mathsf{Bun}\,.
\end{array}$$

Unpacking this definition (and using the *open cylinder* construction of [4] referenced above) $\mathcal{R}\mathsf{ef}^0(M)$ is a simplicial space whose value on $[p]\in\mathbf{\Delta}$ is the moduli space of

- constructible bundles

  $$X \longrightarrow \Delta^p$$

  for which the $(n-1)$-skeleton of each fiber of which is a finite set,

- together with a refinement

  $$X \xrightarrow{\text{refinement}} M\times\Delta^p$$

  over Δ^p.

We will denote such a $[p]$-point of $\mathcal{R}\mathsf{ef}^0(M)$ simply as $(X \xrightarrow{\mathsf{ref}} M\times\Delta^p)$.

Remark 3.0.17. Informally, an object in $\mathcal{R}\mathsf{ef}^0(M)$ is a refinement of M in which the $(n-1)$-skeleton of the domain is a finite set, and a morphism in $Ref^0(M)$ is a path of such refinements of M witnessing anti-collision of strata and disappearences of strata.

Lemma 3.0.18. *There is a canonical equivalence between simplicial spaces over Fin:*

$$\mathcal{R}\mathsf{ef}^0(M) \;\simeq\; Exit(Ran^u(M))$$

Proof. A rightward morphism is implemented by, for each $[p] \in \mathbf{\Delta}$, the assignment,

$$\big(X \xrightarrow{\mathsf{ref}} M \times \Delta^p\big) \mapsto \big(\mathsf{sk}^{\mathsf{fib}}_{n-1}(X) \hookrightarrow X \to M \times \Delta^p\big) ~,$$

whose value is the embedding over Δ^p from the fiberwise $(n-1)$-skeleton, which maps to Δ^p as a finite proper constructible bundle. A leftward morphism is implemented by, for each $[p] \in \mathbf{\Delta}$, the assignment,

$$\big(\mathsf{Cylr}(\sigma) \hookrightarrow M \times \Delta^p\big) \mapsto \Big(\big(\mathsf{Cylr}(\sigma) \subset M \times \Delta^p\big) \xrightarrow{\mathsf{ref}} M \times \Delta^p\Big) ~,$$

whose value is the coarsest refinement of $M \times \Delta^p$ for which the embedding from $\mathsf{Cylr}(\sigma)$ is a proper and constructible. (Such a refinement exists because the image of this embedding is, by definition, a properly embedded stratified subspace.)

It is straightforward to verify that these two assignments are mutually inverse to one another.

□

Proof of Proposition 3.0.107. Being an ∞-category, the simplicial space $\mathcal{R}\mathsf{ef}^0(M)$ satisfies the Segal and completeness conditions. Through the equivalence of Lemma 3.0.18, then so does the simplicial space $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(M))$. □

The category $\mathbf{\Theta}_n$

We use the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ to codify the homotopy type of configurations of finite (possibly empty) subsets of $\mathbb{R}^n$. The main objective of this chapter is to show that the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ is a localization of a combinatorially defined category; namely, a subcategory of the category $\mathbf{\Theta}_n$. This section is devoted to defining this subcategory.

We define Joyal's category $\mathbf{\Theta}_n$ using Berger's definition from [10] as the n-fold *wreath product* of the *simplex category* $\mathbf{\Delta}$ with itself. Notation largely follows that of [7].

Wreath product

Definition 3.0.19. Fin_* is the category whose objects are finite, pointed sets and its morphisms are pointed maps; composition is evident.

Notation 3.0.20. Given a finite set S, let S_* denote the finite, pointed set $S \amalg \{*\}$.

Definition 3.0.21. The *wreath product* $\mathrm{Fin}_* \wr \mathcal{D}$ for an arbitrary category $\mathcal{D}$ is the category defined as follows: An object is a symbol $S(d_s)$ where S is a finite set and $(d_s)_{s \in S}$ is a tuple of objects in $\mathcal{D}$ indexed by S. A morphism $S(d_s) \to T(e_t)$ consists of a pair of data:

i) A morphism $S_* \xrightarrow{\delta} T_*$ in Fin_*

ii) For each pair $(s \in S, t \in T)$ such that $\delta(s) = t$, a morphism $d_s \xrightarrow{\delta_{st}} e_t$ in $\mathcal{D}$.

Composition is given by composition in Fin_* and $\mathcal{D}$.

Observation 3.0.22. There is a forgetful functor $\mathrm{Fin}_* \wr \mathcal{D} \to \mathrm{Fin}_*$ given by $S(d_s) \mapsto S_*$; its value on morphisms is evident.

Observation 3.0.23. A category $\mathcal{C}$ is canonically equivalent to the opposite of its opposite category, $\mathcal{C} \cong (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Thus, upon taking opposites of a functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}$, there canonically results a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}^{\mathrm{op}}$, which we will also denote by F. With this fact then, we could just as well defined the wreath product of a category $\mathcal{C}$ over $\mathrm{Fin}_*^{\mathrm{op}}$.

In the next section, we take advantage of the previous observation and define Joyal's category $\mathbf{\Theta}_n$ over $\mathrm{Fin}_*^{\mathrm{op}}$ inductively as the n-fold wreath product of the simplex category $\mathbf{\Delta}$ with itself.

Definition of $\mathbf{\Theta}_n$

Definition 3.0.24. The *assembly functor*

$$\mathrm{Fin}_* \wr \mathrm{Fin}_* \xrightarrow{\nu} \mathrm{Fin}_*$$

is given by the wedge sum. Explicitly, the value of ν on an object $S((T_s)_*)$ is the wedge sum $\bigvee_{s \in S} (T_s)_*$. Its value on a morphism $S((T_s)_*) \to S'((T'_{s'})_*)$ given by

i) A morphism $S \xrightarrow{\delta} S'$

ii) For each pair (s, s') such that $\delta(s) = s'$, a morphism $(T_s)_* \xrightarrow{\delta_{ss'}} (T'_s)_*$, is

$$\bigvee_{s \in S} (T_s)_* \to \bigvee_{s' \in S'} (T'_{s'})_*$$

defined by $t \in T_s \mapsto \delta_{ss'}(t)$ for every pair (s, s') such that $\delta(s) = s'$.

Definition 3.0.25. The *simplicial circle* is the functor

$$\mathbf{\Delta} \xrightarrow{\gamma} \mathrm{Fin}_*^{\mathrm{op}}$$

the value of which on an object $[p]$ is the quotient morphism set $\mathbf{\Delta}([p], [1])/\{\{0\}, \{1\}\}$, where $\{i\}$ denotes the constant map at i; $\{0\} \sim \{1\}$ is the evident basepoint of the image. The value of γ on a morphism $[p] \xrightarrow{f} [q]$ is precomposition with f, $(- \circ f)$.

Observation 3.0.26. The map induced by γ between each hom-set is injective. This observation comes down to the fact that on morphisms γ is given by precomposition and composition is unique in categories.

Observation 3.0.27. There is an evident isomorphism $\gamma([p]) \xrightarrow{\cong} \{1, ..., p\}_*$ in Fin_*. Let ν_j denote the morphism that assigns each $0 \leq i \leq j-1$ to 0 and each $j \leq i \leq p$ to 1 (i.e., a unique composite of degeneracy maps). The value of ν_j under the isomorphism is j. Its assignment on morphisms then is evident.

Terminology 3.0.28. In light of the previous observation, we will freely refer to a non-basepoint value in the pointed set $\gamma([p])$ by j for some $1 \leq j \leq p$.

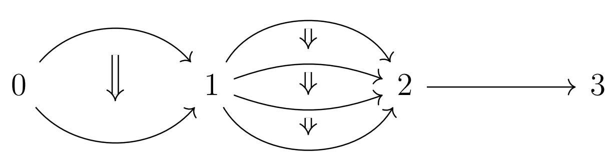
Definition 3.0.29. For each integer $n \geq 1$, the categories $\mathbf{\Theta}_n$ are defined inductively by setting

$$\mathbf{\Theta}_1 := \mathbf{\Delta} \quad \text{and} \quad \mathbf{\Theta}_n := \mathbf{\Theta}_1 \wr \mathbf{\Theta}_{n-1}$$

where the *assembly functors* $\mathbf{\Theta}_n \xrightarrow{\gamma_n} \mathrm{Fin}_*^{\mathrm{op}}$ are also defined inductively by setting

$$\gamma_1 := \gamma \quad \text{and} \quad \gamma_n := \nu \circ (\gamma_1 \wr \gamma_{n-1}).$$

Example 3.0.30. Following Rezk in [32], the object $[3]([1],[3],[0])$ in $\mathbf{\Theta}_2$ corresponds to



Observation 3.0.31. Because the wreath product is associtative, equivalently $\mathbf{\Theta}_n := \mathbf{\Theta}_{n-1} \wr \mathbf{\Theta}_1$.

Remark 3.0.32. $\mathbf{\Theta}_n$ is the full sub-category of the category of strict n-categories, Cat_n, in which an object is a pasting diagram ([32]).

A description of the objects of $\mathbf{\Theta}_n$ is given in terms of *planar level trees* in [7]. We define this notion and present an example next.

Definition 3.0.33.

- A *level tree* is a directed tree with a specified vertex called the *root* which induces a direction on each edge such that there is a unique directed path from each vertex to the root.

- For each vertex, we may equip the set of edges directed toward the vertex with a linear order. A tree is called a *planar level tree* if such an order is specified with respect to each vertex.

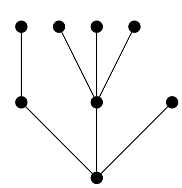
- A vertex is at *level i* if the directed path from the vertex to the root counts i edges.

- A tree has *height n* if the maximum level of all the verticies is n.

- A vertex is a *leaf* if it has no edges directed towards it.

- A planar level tree of height n is *healthy* if all of its leaves are at level n.

Example 3.0.34. Planar level trees of height n naturally depict the objects of $\mathbf{\Theta}_n$. For example, the object $[3]([1],[3],[0])$ in $\mathbf{\Theta}_2$ corresponds to the following planar level tree:



Terminology 3.0.35. We will say *tree* to mean planar level tree.

Informally then, in terms of trees, the functor $\Theta_n \xrightarrow{\gamma_n} \mathrm{Fin}_*^{\mathrm{op}}$ assigns a tree to its set of leaves formally unioned with a basepoint.

The category $\mathbf{\Theta}_n^{\mathrm{act}}$

In [7], Ayala and Hepworth show that $\mathbf{\Theta}_n$ naturally encodes the homotopy type of configuration spaces of ordered points in $\mathbb{R}^n$. Specifically, they show a homotopy equivalence between $\mathrm{Conf}_k(\mathbb{R}^n)$ and the classifying space of the full subcategory of $\mathbf{\Theta}_n$ consisting of all of those objects whose value under γ_n is $\{1, ..., k\}_*$. In this chapter, we present a generalization of this result to unordered configurations of points of $\mathbb{R}^n$, wherein we show that the following subcategory of $\mathbf{\Theta}_n$, $\mathbf{\Theta}_n^{\mathrm{act}}$ encodes the homotopy type of unordered configuration of $\mathbb{R}^n$ as organized by the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

Definition 3.0.36. Given a category $\mathcal{C} \xrightarrow{F} \mathrm{Fin}_*$ over based, finite sets, a morphism σ in $\mathcal{C}$ is *active* if $(F(\sigma))^{-1}(\{*\}) = \{*\}$. The subcategory $\mathcal{C}^{\mathrm{act}}$ of $\mathcal{C}$ is defined to be the pullback

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{act}} & \hookrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow F \\ \mathrm{Fin} & \hookrightarrow & \mathrm{Fin}_*. \end{array}$$

Observation 3.0.37. For each category $\mathcal{C} \to \mathrm{Fin}_*$, the restriction $\mathcal{C}^{\mathrm{act}} \to \mathrm{Fin}_*$ canonically factors through Fin.

Notation 3.0.38. In light of the previous observation, for a functor $\mathcal{C} \xrightarrow{F} \mathrm{Fin}_*$, we denote $\mathcal{C}^{\mathrm{act}} \xrightarrow{F} \mathrm{Fin}$ by F as well.

Definition 3.0.39. For each integer $n \geq 1$, the categories $\mathbf{\Theta}_n^{\mathrm{act}}$ are the subcategories of $\mathbf{\Theta}_n$ defined inductively by setting

$$\mathbf{\Theta}_1^{\mathrm{act}} := \mathbf{\Delta}^{\mathrm{act}}$$

and defining $\mathbf{\Theta}_n^{\mathrm{act}} := \mathbf{\Theta}_1^{\mathrm{act}} \wr \mathbf{\Theta}_{n-1}^{\mathrm{act}}$, i.e., the pullback

$$\begin{array}{ccc} \mathbf{\Theta}_n^{\mathrm{act}} & \longrightarrow & \mathrm{Fin}^{\mathrm{op}} \wr \mathbf{\Theta}_{n-1}^{\mathrm{act}} \\ \downarrow & \lrcorner & \downarrow \mathrm{frgt} \\ \mathbf{\Theta}_1^{\mathrm{act}} & \xrightarrow{\gamma_1} & \mathrm{Fin}^{\mathrm{op}}. \end{array} \tag{3.0.3}$$

Informally then, in terms of trees, the functor $\Theta_n^{\mathrm{act}} \xrightarrow{\gamma_n} \mathrm{Fin}^{\mathrm{op}}$ assigns a tree precisely to its set of leaves.

Observation 3.0.40. For each n, there is a natural forgetful functor

$$\mathbf{\Theta}_n^{\mathrm{act}} \xrightarrow{\mathrm{tr}} \mathbf{\Theta}_{n-1}^{\mathrm{act}}.$$

The value on an object $T_{n-1}([m_k]) \in \mathbf{\Theta}_{n-1}^{\mathrm{act}} \wr \mathbf{\Theta}_1^{\mathrm{act}} =: \mathbf{\Theta}_n^{\mathrm{act}}$ is T_{n-1}.

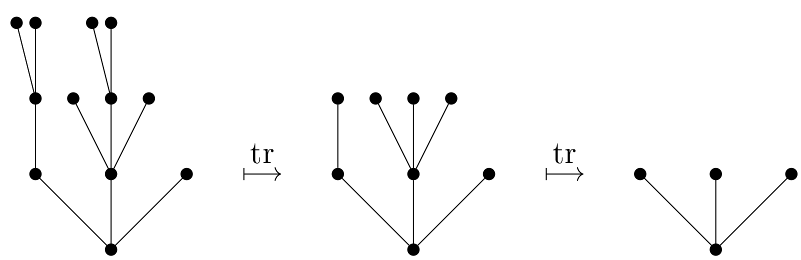
Let $T_{n-1}([m_k]) \xrightarrow{\sigma} W_{n-1}([p_l])$ be a morphism in $\mathbf{\Theta}_n^{\mathrm{act}}$ defined by

i) a morphism $T_{n-1} \xrightarrow{\sigma'} W_{n-1}$ in $\mathbf{\Theta}_{n-1}^{\mathrm{act}}$ and,

ii) a morphism $[m_k] \xrightarrow{\sigma_k} [p_l]$ in $\mathbf{\Theta}_1^{\mathrm{act}}$ for each pair (k, l) such that $\gamma_{n-1}(l) = k$.

The value of σ under tr is σ'.

Example 3.0.41. Let T be the object of $\mathbf{\Theta}_3$ depicted as the far left tree in the figure below. We depict two iterations of the truncation functor tr on T:



Notation 3.0.42. For each $1 \leq i \leq n-1$, denote the $(n-i)$-fold composite of the truncation functor tr by $\mathrm{tr}_i : \Theta_n^{\mathrm{act}} \to \Theta_i^{\mathrm{act}}$.

Observation 3.0.43. For each $1 \leq i \leq n-1$, there is a natural transformation from $\Theta_n^{\mathrm{act}} \xrightarrow{\gamma_n} \mathrm{Fin}^{\mathrm{op}}$ to the composite $\gamma_i \circ \mathrm{tr}_i$,

$$\begin{array}{ccc} \Theta_n^{\mathrm{act}} & & \\ {\scriptstyle \mathrm{tr}_i}\downarrow & \searrow^{\gamma_n} & \\ \Theta_i^{\mathrm{act}} & \xrightarrow{\gamma_i} & \mathrm{Fin}^{\mathrm{op}}. \end{array}$$

For each tree T, the natural transformation ϵ is given by the natural map $\gamma_n(T) \xrightarrow{\epsilon_T} \gamma_i(\mathrm{tr}_i(T))$ from the leaves of T to the leaves of the truncation of T to height i, the assignment of which is the evident one given by the structure of the tree T. It is straightforward to verify that ϵ does indeed define a natural transformation.

Definition 3.0.44. Given a category $\mathcal{C}$, we define $\mathrm{Fun}(\{1 < \cdots < n\}, \mathcal{C})$ to be the category in which an object is a functor $\{1 < \cdots < n\} \to \mathcal{C}$ which selects out a sequence of composable morphisms in $\mathcal{C}$:

$$c_1 \to c_2 \to \cdots \to c_n$$

and a morphism from $c_1 \to c_2 \to \cdots \to c_n$ to $d_1 \to d_2 \to \cdots \to d_n$ is a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccc} c_1 & \longrightarrow & d_1 \\ \downarrow & & \downarrow \\ c_2 & \longrightarrow & d_2 \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ c_n & \longrightarrow & d_n. \end{array}$$

Composition is evident.

Observation 3.0.45. We use the previous observation (3.0.43) to define the natural functor

$$\mathbf{\Theta}_n^{\mathrm{act}} \xrightarrow{\tau_n} \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$$

the value of which on an object T is the functor which selects out the sequence of composable maps of sets

$$\gamma_n(T) \xrightarrow{\epsilon_T} \gamma_{n-1}(\mathrm{tr}(T)) \xrightarrow{\epsilon_{\mathrm{tr}(T)}} \gamma_{n-2}(\mathrm{tr}_{n-2}(T)) \to \cdots \to \gamma_1(\mathrm{tr}_1(T))$$

and on a morphism $T \xrightarrow{f} S$ is the diagram of finite sets

$$\begin{array}{ccc}
\gamma_n(S) & \xrightarrow{\gamma_n(f)} & \gamma_n(T) \\
\Big\downarrow{\scriptstyle \epsilon_S} & & \Big\downarrow{\scriptstyle \epsilon_T} \\
\gamma_{n-1}(\mathrm{tr}_{n-1}(S)) & \xrightarrow{\gamma_{n-1}(\mathrm{tr}_{n-1}(f))} & \gamma_{n-1}(\mathrm{tr}_{n-1}(T)) \\
\Big\downarrow & & \Big\downarrow \\
\vdots & & \vdots \\
\Big\downarrow & & \Big\downarrow \\
\gamma_1(\mathrm{tr}_1(S)) & \xrightarrow{\gamma_1(\mathrm{tr}_1(f))} & \gamma_1(\mathrm{tr}_1(T))
\end{array}$$

which is guaranteed to commute in Fin because the downward arrows in the diagram are given by the natural transformation ϵ from (3.0.43).

<u>Identifying the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$</u>

We are now equipped to state the main result of the chapter, Theorem (3.0.46), which articulates a sense in which the homotopy type of configurations of unordered points in $\mathbb{R}^n$ are encoded by (a localization of) the category $\boldsymbol{\Theta}_n^{\mathrm{act}}$.

Theorem 3.0.46. *For $n \geq 1$, there is a localization of ∞-categories*

$$\boldsymbol{\Theta}_n^{act} \to Exit(Ran^u(\mathbb{R}^n))$$

over Fin^{op} from the category $\boldsymbol{\Theta}_n^{act}$ to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

Our procedure for proving Theorem (3.0.46) is motivated by Theorem 3.3.12 in [4] which states that a *refinement* of stratified spaces is carried by Exit$(-)$ to a localization of ∞-categories. Heuristically, a refinement of a stratified space Y is a stratified space X whose stratification is a finer version of the stratification of Y.

Definition 3.0.47. A map of stratified spaces $X \xrightarrow{f} Y$ is a *refinement* if f is a homeomorphism between the underlying topological spaces, and if the restriction of f on each stratum X_p of X is an embedding into Y.

Generally then, in showing a localization of a category $\mathcal{C}$ to the exit-path ∞-category of a stratified space X, one might hope to identify a refinement of the stratified space X whose exit-path ∞-category is equivalent to $\mathcal{C}$. Then, in light of the fact that $\mathrm{Exit}(-)$ carries refinements to localizations (3.3.12 in [4]), the desired localization of $\mathcal{C}$ to the exit-path ∞-category of X would be established.

In the situation at hand however, the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ is not the exit-path ∞-category of a stratified space. Rather, recall that it is the exit-path ∞-category of a stack on stratified spaces (3.0.13), within the context of which there is no clear notion of a refinement; that is, it is not clear how to define a refinement of stacks on stratified spaces. Thus, we cannot directly apply that $\mathrm{Exit}(-)$ carries refinements to localizations to prove the localization of Theorem (3.0.46).

Nonetheless, motivated by Theorem 3.3.12 of [4], we define an ∞-category that behaves like the exit-path ∞-category of a refinement of the unital Ran space of $\mathbb{R}^n$, if such a thing were to exist. Suggestively, we name this ∞-category the *exit-path ∞-category of the fine unital Ran space of* $\mathbb{R}^n$. Then, in Lemma (3.0.57), we will show that this ∞-category is equivalent to the category $\mathbf{\Theta}_n^{\mathrm{act}}$; this is the first of two main lemmas which together imply Theorem (3.0.46). The second lemma, (3.0.63), shows that this ∞-category localizes to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

The exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$

We define the exit-path ∞-category $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$, the defining feature of which is that its morphisms are exit-paths that are controled by coordinate coinsidence in addition to cardinality.

Observation 3.0.48. Similar to Observation (3.0.7), the nerve of

$$\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$$

is a presheaf on $\mathbf{\Delta}$, and thus, we may define the ∞-category $\mathbf{\Delta}$ slice over $\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$,

$$\mathbf{\Delta}_{/\mathrm{Fun}(\{1<\cdots<n\},\mathrm{Fin}^{\mathrm{op}})}.$$

Definition 3.0.49. The *exit-path ∞-category of the fine unital Ran space of* $\mathbb{R}^n$ $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ is the simplicial space over $\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$ representing the presheaf on $\mathbf{\Delta}_{/\mathrm{Fun}(\{1<\cdots<n\},\mathrm{Fin}^{\mathrm{op}})}$ whose value on an object

$$[p] \to \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$$

which selects a diagram of finite sets

$$\begin{array}{cccccc}
\sigma_n : & S_n^p & \longrightarrow & \cdots & \longrightarrow & S_n^0 \\
& \downarrow & & & & \downarrow \\
& \vdots & & \vdots & & \vdots \\
& \downarrow & & & & \downarrow \\
\sigma_1 : & S_1^p & \longrightarrow & \cdots & \longrightarrow & S_1^0
\end{array} \tag{3.0.4}$$

is the space of compatible embeddings

$$\begin{array}{ccc}
\mathrm{cylr}(\sigma_n) & \xhookrightarrow{E_n} & \mathbb{R}^n \times \Delta^p \\
\downarrow & & \downarrow \mathrm{pr}_{<n} \times \mathrm{id}_{\Delta^p} \\
\mathrm{cylr}(\sigma_{n-1}) & \xhookrightarrow{E_{n-1}} & \mathbb{R}^{n-1} \times \Delta^p \\
\downarrow & & \downarrow \mathrm{pr}_{<n-1} \times \mathrm{id}_{\Delta^p} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{cylr}(\sigma_1) & \xhookrightarrow{E_1} & \mathbb{R} \times \Delta^p
\end{array} \tag{3.0.5}$$

where each embedding is over Δ^p and the downward arrows on the lefthand side are induced by the downward arrows of (3.0.4). This embedding space is given the compact-open topology; the structure maps are evident.

Observation 3.0.50. Explicitly, an object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ over the sequence of finite sets,

$$S_n \xrightarrow{\tau_{n-1}} S_{n-1} \xrightarrow{\tau_{n-2}} \cdots \to S_1 \tag{3.0.6}$$

is a sequence of embeddings,

$$\begin{array}{ccc}
S_n & \xhookrightarrow{e_n} & \mathbb{R}^n \\
\tau_{n-1} \downarrow & & \downarrow \mathrm{pr}_{<n} \\
S_{n-1} & \xhookrightarrow{e_{n-1}} & \mathbb{R}^{n-1} \\
\tau_{n-2} \downarrow & & \downarrow \mathrm{pr}_{<n-1} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
S_1 & \xhookrightarrow{e_1} & \mathbb{R}.
\end{array} \tag{3.0.7}$$

When the context is clear, we denote (3.0.7) by $\underline{S} \overset{\underline{e}}{\hookrightarrow} \underline{\mathbb{R}}^n$ or just $\underline{e}$. A morphism from

$\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ to $\underline{T} \overset{d}{\hookrightarrow} \underline{\mathbb{R}}^n$ over the diagram of finite sets,

$$\begin{array}{ccc} T_n & \xrightarrow{\sigma_n} & S_n \\ {\scriptstyle \omega_{n-1}}\downarrow & & \downarrow{\scriptstyle \tau_{n-1}} \\ T_{n-1} & \xrightarrow{\sigma_{n-1}} & S_{n-1} \\ {\scriptstyle \omega_{n-2}}\downarrow & & \downarrow{\scriptstyle \tau_{n-2}} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ T_1 & \xrightarrow{\sigma_1} & S_1 \end{array} \tag{3.0.8}$$

is a sequence of embeddings,

$$\begin{array}{ccc} \mathrm{cylr}(\sigma_n) & \overset{E_n}{\hookrightarrow} & \mathbb{R}^n \times \Delta^1 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<n} \times \mathrm{id}_{\Delta^1}} \\ \mathrm{cylr}(\sigma_{n-1}) & \overset{E_{n-1}}{\hookrightarrow} & \mathbb{R}^{n-1} \times \Delta^1 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<n-1} \times \mathrm{id}_{\Delta^1}} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathrm{cylr}(\sigma_1) & \overset{E_1}{\hookrightarrow} & \mathbb{R} \times \Delta^1 \end{array} \tag{3.0.9}$$

over Δ^1 such that $E_{i|S_i} = e_i$ and $E_{i|T_i \times \{1\}} = d_i$, for each $1 \leq i \leq n$. When the context is clear, we denote (3.0.9) by $\mathrm{cylr}(\underline{\sigma}) \overset{\underline{E}}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^1$ or $\underline{E}$.

Heuristically, a morphism in the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ is a sequence of finite sets of paths in $\mathbb{R}^i$ for each $1 \leq i \leq n$, compatible under projection, that are allowed to witness anticollision of points, including vanishing of points, but are not allowed to witness collision of points. In particular, this means that for each $2 \leq j \leq n$ and each $0 \leq i \leq j-1$, projecting off the last i-coordinates of the finite set of paths in $\mathbb{R}^j$, given by some fixed morphism in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$, yields a finite set of paths in $\mathbb{R}^{j-i}$ that, too, witness anticollision of points, but not collision; this projection is part of, but not necessarily all of the data of the morphism at the $\mathbb{R}^{j-i}$ level.

Notation 3.0.51. In general, we denote a point in the $[p]$-space of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ over (3.0.8) by $\mathrm{cylr}(\underline{\sigma}) \overset{\underline{E}}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^p$.

Observation 3.0.52. The exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ is an ∞-category for similar technical reasons that the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ is an ∞-category, as shown in Proposition (3.0.107).

Observation 3.0.53. There is a natural forgetful functor

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \to \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$$

over $\mathrm{Fin}^{\mathrm{op}}$ induced by the functor from $\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$ to $\mathrm{Fin}^{\mathrm{op}}$ that evaluates on $\{n\}$. The value of a $[p]$-value $\mathrm{cylr}(\underline{\sigma}) \overset{\underline{E}}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^p$ over (3.0.4) under this forgetful functor is the embedding of $\underline{E}$ at the $\mathbb{R}^n$ level

$$\mathrm{cylr}(\sigma_n) \overset{E_n}{\hookrightarrow} \mathbb{R}^n \times \Delta^p$$

over $\sigma_n : S_n^p \to \cdots \to S_n^0$.

Observation 3.0.54. There is a natural forgetful functor

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \overset{\rho}{\to} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$$

that forgets all but the first coordinate data. The image of a $[p]$-value

$$\mathrm{cylr}(\underline{\sigma}) \overset{E}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^p$$

over (3.0.4) under ρ is

$$\mathrm{cylr}(\sigma_1) \overset{E_1}{\hookrightarrow} \mathbb{R} \times \Delta^p$$

defined over $\sigma_1 : S_1^p \to \cdots \to S_1^0$.

Observation 3.0.55. For each $1 \leq i \leq n$, there is a natural forgetful functor to finite sets

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \xrightarrow{\phi_i} \mathrm{Fin}^{\mathrm{op}}$$

that forgets all but the set data at the $\mathbb{R}^i$ level. Its value on an object $\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ is S_i and on a morphism

$$\mathrm{cylr}(\underline{T} \overset{\sigma}{\to} \underline{S}) \overset{E}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^1$$

from $\underline{S} \hookrightarrow \underline{\mathbb{R}}^n$ to $\underline{T} \hookrightarrow \underline{\mathbb{R}}^n$ is the map of finite sets

$$T_i \xrightarrow{\sigma_i} S_i.$$

The collection of functors $\{\phi_i\}_{i=1}^n$ naturally compile to name a functor

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \xrightarrow{\Phi_n} \mathrm{Fun}(\{1 < \cdots n\}, \mathrm{Fin}^{\mathrm{op}})$$

which just remembers the underlying set data, i.e., its value on an object $\underline{S} \hookrightarrow \underline{\mathbb{R}}^n$ is the functor which selects out the composable sequence of maps of finite sets $\underline{S}$ and its value on a morphism $\mathrm{cylr}(\underline{T} \overset{\sigma}{\to} \underline{S}) \overset{E}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^1$ is the commutative diagram of finite sets $\underline{T} \overset{\sigma}{\to} \underline{S}$. In otherwords, Φ_n is simply the downward arrow in the definition of

$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ (3.0.49).

Observation 3.0.56. There is a natural functor

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \xrightarrow{\pi} \mathrm{Fin}^{\mathrm{op}} \wr \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$$

the value of which on an object $\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ is

$$S_1((\underline{S})_s \xhookrightarrow{e_{|(\underline{S})_s}} \underline{\mathbb{R}}^{n-1})$$

where for each $s \in S_1$, $(\underline{S})_s \xhookrightarrow{e_{|(\underline{S})_s}} \underline{\mathbb{R}}^{n-1}$ denotes the object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$ determined by the compatible sequence of embeddings of pullbacks over s:

$$\begin{array}{ccccc}
(S_n)_s & \hookrightarrow & S_n & \xhookrightarrow{e_n} & \mathbb{R}^n \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \tau_{n-1}} & & \downarrow{\scriptstyle \mathrm{pr}_{<n}} \\
(S_{n-1})_s & \hookrightarrow & S_{n-1} & \xhookrightarrow{e_{n-1}} & \mathbb{R}^{n-1} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \tau_{n-2}} & & \downarrow{\scriptstyle \mathrm{pr}_{<n-1}} \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
(S_2)_s & \hookrightarrow & S_2 & \xhookrightarrow{e_2} & \mathbb{R}^2 \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \mathrm{pr}_{<2}} \\
\{s\} & \hookrightarrow & S_1 & \xhookrightarrow{e_1} & \mathbb{R}.
\end{array} \tag{3.0.10}$$

Note that for each $1 \le i \le n$, each embedding $(S_i)_s \hookrightarrow S_i \hookrightarrow \mathbb{R}^i$ agrees on its first coordinate and thus, canonically factors through $\mathbb{R}^{i-1}$, which in particular means that (3.0.10) yields an object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$.

The value of π on a morphism $\mathrm{cylr}(\underline{\sigma}) \overset{\underline{E}}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^1$ is:

i. the morphism $T_1 \xrightarrow{\sigma_1} S_1$ in Fin

ii. for each pair $r \in T_1, s \in S_1$ such that $\sigma_1(r) = s$, the morphism

$$\begin{array}{ccccc}
\mathrm{cylr}(\sigma_n)_{|\sigma_1 : r \mapsto s} & \hookrightarrow & \mathrm{cyl}(\sigma_n) & \hookrightarrow & \mathbb{R}^n \\
\downarrow & \lrcorner & \downarrow & & \downarrow \mathrm{pr}_{<n} \\
\mathrm{cylr}(\sigma_{n-1})_{|\sigma_1 : r \mapsto s} & \hookrightarrow & \mathrm{cylr}(\sigma_{n-1}) & \hookrightarrow & \mathbb{R}^{n-1} \\
\downarrow & \lrcorner & \downarrow & & \downarrow \mathrm{pr}_{<n-1} \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{cylr}(\sigma_2)_{|\sigma_1 : r \mapsto s} & \hookrightarrow & \mathrm{cyl}(\sigma_2) & \hookrightarrow & \mathbb{R}^2 \\
\downarrow & \lrcorner & \downarrow & & \downarrow \mathrm{pr}_{<2} \\
\mathrm{cylr}(\sigma_{1_{|r}}) \simeq \Delta^1 & \hookrightarrow & \mathrm{cyl}(\sigma_1) & \hookrightarrow & \mathbb{R}
\end{array} \tag{3.0.11}$$

in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$, where for each $1 \leq i \leq n$, each embedding $\mathrm{cylr}(\sigma_i)_{|\sigma_1 : r \mapsto s} \hookrightarrow \mathbb{R}^i \times \Delta^1$ canonically factors through $\mathbb{R}^{i-1} \times \Delta^1$ and thus, diagram (3.0.11) yields a morphism in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$.

<u>Identifying the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ as $\boldsymbol{\Theta}_n^{\mathrm{act}}$</u>

The next lemma is the first of two main lemmas which together prove the main result of this chapter, Theorem (3.0.46), which states that the category $\boldsymbol{\Theta}_n^{\mathrm{act}}$ localizes to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$. This lemma articulates a sense in which $\boldsymbol{\Theta}_n^{\mathrm{act}}$ encodes the homotopy type of configurations of finite (possibly empty) subsets of $\mathbb{R}^n$ which are organized according to cardinality and coordinate coinsidence, heuristically, as a stratified space.

Lemma 3.0.57. *There is an equivalence of ∞-categories*

$$Exit(Ran^u(\underline{\mathbb{R}}^n)) \simeq \boldsymbol{\Theta}_n^{act}$$

over $Fun(\{1 < \cdots < n\}, Fin^{op})$, between the exit-path ∞-category of the fine unital

Ran space of $\mathbb{R}^n$ *and the category* $\mathbf{\Theta}_n^{act}$.

We prove Lemma (3.0.57) in two parts: In Part 1, we construct a functor $\mathcal{G}_n$ in Construction (3.0.58) from the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ to the category $\mathbf{\Theta}_n^{\mathrm{act}}$. In Part 2, Lemma (3.0.59) shows that the functor $\mathcal{G}_n$ from Construction (3.0.58) is an equivalence.

<u>Part 1: The functor $\mathcal{G}_n$</u>

Construction 3.0.58. For $n \geq 1$, there is a functor

$$\mathcal{G}_n : \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \to \mathbf{\Theta}_n^{\mathrm{Act}}$$

over $\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$, from the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ to the category $\mathbf{\Theta}_n^{\mathrm{act}}$.

Proof. By induction on n.

BASECASE. For $n = 1$, we seek to define a functor $\mathcal{G}_1$ over $\mathrm{Fin}^{\mathrm{op}}$:

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}})) & \overset{\mathcal{G}_1}{\dashrightarrow} & \mathbf{\Delta}^{\mathrm{act}} \\ & \searrow^{\phi_1} & \downarrow{\gamma_1} \\ & & \mathrm{Fin}^{\mathrm{op}}. \end{array}$$

$\mathbf{\Delta}^{\mathrm{act}}$ is an ordinary category and thus, we employ (2.0.22) and simply define $\mathcal{G}_1$ on objects and morphisms and check that composition is respected.

Let $S \overset{e}{\hookrightarrow} \mathbb{R}$ be an object in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$. The value of $\mathcal{G}_1$ on e is the linearly ordered set of connected components of the complement of $e(S)$ in $\mathbb{R}$

$$\mathcal{G}_1 : e \mapsto \pi_o(\mathbb{R} - e(S))$$

the linear order of which is inherited from the linear order on $\mathbb{R}$.

Let $\mathrm{cylr}(T \xrightarrow{\sigma} S) \overset{E}{\hookrightarrow} \mathbb{R} \times \Delta^1$ be a morphism from $S \overset{e}{\hookrightarrow} \mathbb{R}$ to $T \overset{d}{\hookrightarrow} \mathbb{R}$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$. Let C_E denote the compliment of the image of the embedding of E,

$$C_E := (\mathbb{R} \times \Delta^1) - E(\mathrm{cylr}(\sigma)).$$

Before we name the value of $\mathcal{G}_1$ on E, we make three observations:

1. Consider the inclusion $\iota_1 : (\mathbb{R} - d(T)) \hookrightarrow C_E$ given by $x \mapsto (x, \{1\})$. Taking connected components induces an inclusion of sets

   $$\pi_o(\iota_1) : \pi_o(\mathbb{R} - d(T)) \hookrightarrow \pi_o(C_E).$$

   It is easy to see that $\pi_o(\iota_1)$ is, in particular, a bijection. We denote its inverse $\pi_o(\iota_1)^{-1}$.

2. Taking connected components of the inclusion $\iota_0 : (\mathbb{R} - e(S)) \hookrightarrow C_E$ given by $x \mapsto (x, \{0\})$ induces a map between sets

   $$\pi_o(\iota_0) : \pi_o(\mathbb{R} - e(S)) \hookrightarrow \pi_o(C_E).$$

   Note that $\pi_o(\iota_0)$ is not necessarily injective nor surjective because σ is not necessairly injective nor surjective.

3. $\pi_o(\iota_1)$ determines a linear order on $\pi_o(C_E)$ and thus, $\pi_o(C_E)$ is an object in $\mathbf{\Delta}$.

Then, the value of $\mathcal{G}_1$ on $\mathrm{cylr}(T \xrightarrow{\sigma} S) \overset{E}{\hookrightarrow} \mathbb{R} \times \Delta^1$ is the composite

$$\begin{array}{ccc} \pi_o(\mathbb{R} - e(S)) & \xrightarrow{\pi_o(\iota_0)} & \pi_o(C_E) \\ {\scriptstyle \mathcal{G}_1(E)} \downarrow & \swarrow {\scriptstyle \pi_o(\iota_1)^{-1}} & \\ \pi_o(\mathbb{R} - d(T)) & & \end{array} \tag{3.0.12}$$

in $\mathbf{\Delta}^{\text{act}}$. It must be checked that $\mathcal{G}_1(E)$ is linear and active. We do this by verifying that each morphism in the composite is linear and active. $\pi_o(\iota_1)^{-1}$ is a linear map because it defines the linear order of $\pi_o(C_E)$. Bijectivity of $\pi_o(\iota_1)^{-1}$ implies that it is active. Similarly, it is easy to see that $\pi_o(\iota_0)$ is order-preserving and sends unbounded components to unbounded components thereby being active.

Next, we will show that $\mathcal{G}_1$ respects composition by showing that the diagram of ∞-categories (3.0.13) commutes on the level of objects and morphisms:

$$\begin{array}{ccc} \text{Exit}(\text{Ran}^{\text{u}}(\underline{\mathbb{R}})) & \overset{\mathcal{G}_1}{\dashrightarrow} & \mathbf{\Delta}^{\text{act}} \\ & \searrow^{\phi_1} & \downarrow{\scriptstyle \gamma_1} \\ & & \text{Fin}^{\text{op}}. \end{array} \tag{3.0.13}$$

Indeed, if (3.0.13) commutes, then faithfulness of γ_1 together with functorality of ϕ_1 guarantee that $\mathcal{G}_1$ respects composition. More precisely, let

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \searrow^{h} & \downarrow{\scriptstyle g} \\ & & c \end{array} \tag{3.0.14}$$

denote a commutative triangle in $\text{Exit}(\text{Ran}^{\text{u}}(\underline{\mathbb{R}}))$. We will show that if (3.0.13) commutes, then $\mathcal{G}_1$ carries the composite h in (3.0.14) to $\mathcal{G}_1(g) \circ \mathcal{G}_1(f)$. First, note that functorality of ϕ_1 implies that $\phi_1(h) = \phi_1(g) \circ \phi_1(f)$. Commutativity of (3.0.13) guarantees that the morphisms

$$\gamma_1(\mathcal{G}_1(h)),\ \gamma_1(\mathcal{G}_1(f)) \text{ and } \gamma_1(\mathcal{G}_1(g))$$

are equivalent (upto composition with canonical isomorphisms) to

$$\phi_1(h),\ \phi_1(f) \text{ and } \phi_1(g)$$

respectively, in Fin. Thus,

$$\gamma_1(\mathcal{G}_1(h)) = \gamma_1(\mathcal{G}_1(g)) \circ \gamma_1(\mathcal{G}_1(f)) = \gamma_1(\mathcal{G}_1(g) \circ \mathcal{G}_1(f)).$$

Then, faithfulness of γ_1 guarantees that $\mathcal{G}_1(h) = \mathcal{G}_1(g) \circ \mathcal{G}_1(f)$, as desired.

Now, we verify commutativity of (3.0.13) on objects and morphisms. Let $S \overset{e}{\hookrightarrow} \mathbb{R}$ be an object in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$. There is a canonical bijection of sets

$$\gamma_1(\mathcal{G}_1(e)) \xrightarrow{\cong} \phi_1(e) := S \tag{3.0.15}$$

in Fin given by

$$(\pi_o(\mathbb{R} - e(S)) \xrightarrow{\alpha} [1]) \mapsto \inf\{x \in \coprod_{U \in \alpha^{-1}(\{1\})} U\}$$

verifying commutativity of (3.0.13) on objects.

Let $\mathrm{cylr}(T \overset{\sigma}{\to} S) \overset{E}{\hookrightarrow} \mathbb{R} \times \Delta^1$ be a morphism from $S \overset{e}{\hookrightarrow} \mathbb{R}$ to $T \overset{d}{\hookrightarrow} \mathbb{R}$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$. We consider the canonical bijections of the source and target of $\gamma_1(\mathcal{G}_1(E))$, and the coresponding composite, α, in Fin from T to S :

$$\begin{array}{ccc} \gamma_1(\mathcal{G}_1(d)) & \xrightarrow{\gamma_1(\mathcal{G}_1(E))} & \gamma_1(\mathcal{G}_1(e)) \\ \cong\uparrow & & \cong\downarrow \\ T & \xrightarrow{\alpha} & S. \end{array} \tag{3.0.16}$$

By definition, the value of α on $r \in T$ is

$$\alpha(r) := \inf\{x \in \coprod_{U \in S^r} U\},$$

where $S^r := \{U \in \pi_0(\mathbb{R} - e(S)) | \inf\{y \in \mathcal{G}_1(E)(U)\} \geq r\}$. The composite α agrees with $\phi_1(E) := \sigma$, as desired. Indeed, if $U \in S^r$, then

$$\inf\{x \in U\} = \sigma(r) \text{ or } \inf\{x \in U\} = \sigma(r'),$$

for some $r' > r$. But $\sigma(r') \geq \sigma(r)$ whenever $r' > r$, which implies

$$\inf\{x \in \coprod_{U \in S^r} U\} = \sigma(r).$$

In summary, we have just shown that (3.0.13) commutes on objects and morphisms, which, as previously argued, implies that $\mathcal{G}_1$ respects composition. Therefore, $\mathcal{G}_1$ is a functor, and moreover is defined naturally over $\mathrm{Fin}^{\mathrm{op}}$.

General Case. In the inductive step, we assume the existence of a functor over $\mathrm{Fun}(\{1 < \cdots < n-1\}, \mathrm{Fin}^{\mathrm{op}})$

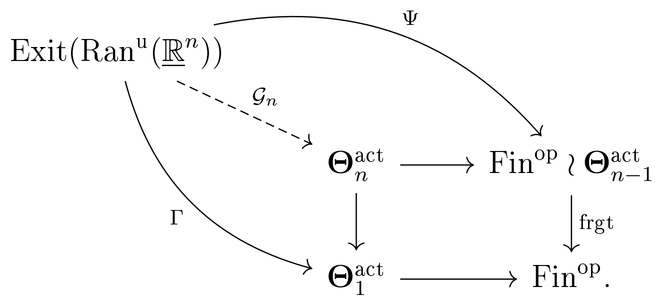
$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1})) & \xrightarrow{\mathcal{G}_{n-1}} & \mathbf{\Theta}_{n-1}^{\mathrm{act}} \\ & \searrow^{\Phi_{n-1}} & \downarrow{\scriptstyle \tau_{n-1}} \\ & & \mathrm{Fun}(\{1 < \cdots < n-1\}, \mathrm{Fin}^{\mathrm{op}}). \end{array}$$

In particular, this implies that $\mathcal{G}_{n-1}$ is over $\mathrm{Fin}^{\mathrm{op}}$ for each $1 \leq i \leq n-1$; i.e., the following diagram commutes

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1})) & \xrightarrow{\mathcal{G}_{n-1}} & \mathbf{\Theta}_{n-1}^{\mathrm{act}} \\ & & \downarrow{\scriptstyle \mathrm{tr}_i} \\ & \searrow^{\phi_i} & \mathbf{\Theta}_{i}^{\mathrm{act}} \\ & & \downarrow{\scriptstyle \gamma_i} \\ & & \mathrm{Fin}^{\mathrm{op}} \end{array} \tag{3.0.17}$$

for each $1 \leq i \leq n-1$, where, recall that tr_i denotes the $(n-1-i)$-fold self-composite of the truncation map (3.0.40).

We define the functor $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \xrightarrow{\mathcal{G}_n} \mathbf{\Theta}_n^{\mathrm{act}}$ by defining Ψ and Γ such that



Γ is defined to be the composite of the forgetful functor ρ followed by $\mathcal{G}_1$, $\mathcal{G}_1 \circ \rho$, where ρ was the natural forgetful functor defined in (3.0.54).

Ψ is defined by the composite of π (defined in (3.0.54)) followed by the functor

$$\mathrm{Fin}^{\mathrm{op}} \wr \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1})) \to \mathrm{Fin}^{\mathrm{op}} \wr \mathbf{\Theta}_{n-1}^{\mathrm{act}}$$

determined by the identity on $\mathrm{Fin}^{\mathrm{op}}$ and the functor $\mathcal{G}_{n-1}$ given by the inductive step. Thus, $\mathcal{G}_n$ is a functor.

In unwinding the above definition of $\mathcal{G}_n$, an inductive description of $\mathcal{G}_n$ is made available. We explicate this inductive description on objects and morphisms: Let $\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ be an object in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$. Its value under $\mathcal{G}_n$ is inductively defined as

$$\mathcal{G}_n(\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n) := \mathcal{G}_1(S_1 \overset{e_1}{\hookrightarrow} \mathbb{R})(\mathcal{G}_{n-1}((\underline{S})_s \xhookrightarrow{e|_{(\underline{S})_s}} \underline{\mathbb{R}}^{n-1}))$$

where $(\underline{S})_s \xhookrightarrow{e|_{(\underline{S})_s}} \underline{\mathbb{R}}^{n-1}$ denotes the object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$ determined by (3.0.10).

Let $\mathrm{cylr}(\underline{T} \overset{\sigma}{\to} \underline{S}) \overset{E}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^1$ be a morphism from $\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ to $\underline{T} \overset{d}{\hookrightarrow} \underline{\mathbb{R}}^n$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$. Its value under $\mathcal{G}_n$ is inductively defined by:

i) the morphism $\mathcal{G}_1(S_1 \overset{e_1}{\hookrightarrow} \mathbb{R}) \xrightarrow{\mathcal{G}_1(\mathrm{cylr}(\sigma_1) \overset{E_1}{\hookrightarrow} \mathbb{R}\times\Delta^1)} \mathcal{G}_1(T_1 \overset{d_1}{\hookrightarrow} \mathbb{R})$ in Δ^{act}

ii) for each pair $(t \in T_1, s \in S_1)$ such that $\sigma_1(t) = s$, the morphism given by the image of (3.0.11) under $\mathcal{G}_{n-1}$ in $\mathbf{\Theta}_{n-1}^{\mathrm{act}}$.

Next, we will show that for each $1 \leq i \leq n$,

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) & \xrightarrow{\mathcal{G}_n} & \mathbf{\Theta}_n^{\mathrm{act}} \\ & \searrow^{\phi_i} & \downarrow{\scriptstyle \mathrm{tr}_i} \\ & & \mathbf{\Theta}_i^{\mathrm{act}} \\ & & \downarrow{\scriptstyle \gamma_i} \\ & & \mathrm{Fin}^{\mathrm{op}}. \end{array} \tag{3.0.18}$$

For the cases $1 \leq i \leq n-1$, this diagram follows by the inductive step wherein we assume commutativity of diagram (3.0.17). For the remaining case, $i = n$, we use the inductive definitions of $\mathcal{G}_n$ and γ_n in terms of $\mathcal{G}_1$ and $\mathcal{G}_{n-1}$, and γ_1 and γ_{n-1}, respectively. Then, indeed, in employing the commutativity of (3.0.13) and (3.0.17) for $i = n-1$, we see that for the case $i = n$, diagram (3.0.18) must commute. Through Observation (3.0.55) wherein the functor Φ_n was defined in terms of ϕ_i for $1 \leq i \leq n$, commutativity of this diagram for each $1 \leq i \leq n$ compiles to prove that $\mathcal{G}_n$ is over $\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$,

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) & \xrightarrow{\mathcal{G}_n} & \mathbf{\Theta}_n^{\mathrm{act}} \\ & \searrow^{\Phi_n} & \downarrow{\scriptstyle \tau_n} \\ & & \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}}). \end{array}$$

□

We have just defined a functor from the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ to the category $\mathbf{\Theta}_n^{\mathrm{act}}$. In the next lemma, we show that this functor is an equivalence by showing that $\mathcal{G}_n$ is essentially surjective and fully faithful.

Part 2: $\mathcal{G}_n$ is an equivalence

Lemma 3.0.59. *For each $n \geq 1$, the functor*

$$\mathcal{G}_n : Exit(Ran^u(\underline{\mathbb{R}}^n)) \xrightarrow{\simeq} \mathbf{\Theta}_n^{act}$$

defined in (3.0.58) is an equivalence of ∞-categories.

Proof. By induction on n.

BASECASE. We will show that $\mathcal{G}_1$ is essentially surjective and fully faithful; the former follows easily: Let $[p] \in \mathbf{\Delta}^{\mathrm{act}}$. Define the set $T_p := \{1, 2, ..., p\}$ together with the object $T_p \overset{d}{\hookrightarrow} \mathbb{R}$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$, given by $i \mapsto i$. Then, $[p]$ is isomorphic to $\mathcal{G}_1(T_p) := \pi_o(\mathbb{R} - d(T_p))$ in $\mathbf{\Delta}$, with the isomorphism given by $i \mapsto [i + \frac{1}{2}]$.

Fix a pair of objects $S \overset{e}{\hookrightarrow} \mathbb{R}$ and $T \overset{d}{\hookrightarrow} \mathbb{R}$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))$. Showing fully faithfulness of $\mathcal{G}_1$ amounts to showing that the map induced by $\mathcal{G}_1$ between corresponding hom-spaces

$$\mathrm{Hom}_{\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))^{\mathrm{op}}}(d, e) \xrightarrow{\mathcal{G}_1} \mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{act}})^{\mathrm{op}}}(\pi_o(\mathbb{R} - d(T)), \pi_o(\mathbb{R} - e(S))) \tag{3.0.19}$$

is a surjection on connected components with contractible fibers.

Fix a morphism $\pi_0(\mathbb{R} - d(T)) \overset{\varphi}{\to} \pi_0(\mathbb{R} - e(S))$ in $\Delta^{\mathrm{act,op}}$. Any morphism

$$\mathrm{cylr}(T \xrightarrow{\gamma_1(\varphi)} S) \overset{E}{\hookrightarrow} \mathbb{R} \times \Delta^1 \tag{3.0.20}$$

in $\mathrm{Hom}_{\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}))}(d, e)$ is in the fiber of $\mathcal{G}_1$ over φ. Indeed, we observed in (3.0.26) that γ_1 is injective on hom-sets. Thus, commutativity of (3.0.13) guarantees that E is in the fiber of $\mathcal{G}_1$ over φ. Hence, (3.0.19) is a surjection on connected components.

The fiber of (3.0.19) over φ is the topological space of embeddings $\text{cylr}(\gamma_1 \circ \varphi) \xhookrightarrow{E} \mathbb{R} \times \Delta^1$ over Δ^1 such that $E_{|S} = e$ and $E_{|T\times\{1\}} = d$,

$$\mathcal{G}_1^{-1}(\varphi) \cong \text{Emb}^{e,d}_{/\Delta^1}(\text{cylr}(\gamma_1(\varphi)), \mathbb{R} \times \Delta^1)$$

under the compact-open topology. We will show that this space is contractible. Fix an embedding $\tilde{E}$ in the fiber of $\mathcal{G}_1$ over φ. Let $\mathbb{S}^k \xrightarrow{\psi} \mathcal{G}_1^{-1}(\varphi)$ be continuous and based at $\tilde{E}$. We construct a null-homotopy of ψ. For each $z \in \mathbb{S}^n$, denote the image of z under ψ by ψ_z. The straight-line homotopy, H_z, from ψ_z to $\tilde{E}$ defined by

$$H_z(x,t) = (1-t)\psi_z(x) + t\tilde{E}(x)$$

names a path from ψ_z to $\tilde{E}$ in $\mathcal{G}_1^{-1}(\varphi)$. For each $z \in \mathbb{S}^k$, we let each path H_z run simultaneously to name a null-homotopy of ψ to the constant path at $\{\tilde{E}\}$. Explicitly, the null-homotopy $\mathbb{S}^k \times [0,1] \to \mathcal{G}_1^{-1}(\varphi)$ is given by $(z,t) \mapsto H_z(-,t)$.

General Case. We will show that $\mathcal{G}_n$ is essentially surjective and fully faithful. Let $[k](T_s)$ be an object in $\mathbf{\Theta}_n^{\text{act}}$. Because $\mathcal{G}_1$ is essentially surjective, we may choose an object of $\text{Exit}(\text{Ran}^{\text{u}}(\underline{\mathbb{R}}))$ that is in the fiber of $\mathcal{G}_1$ over $[k]$:

$$\{1, ..., k\} \xhookrightarrow{e} \mathbb{R}. \tag{3.0.21}$$

Likewise, by essential surjectivity of $\mathcal{G}_{n-1}$, for each $s \in \{1, ..., k\}$, we may choose an object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n-1}))$ that is in the fiber of $\mathcal{G}_{n-1}$ over T_s:

$$\begin{array}{ccc}
(S_{n-1})_s & \xhookrightarrow{(e_{n-1})_s} & \mathbb{R}^{n-1} \\
{\scriptstyle (\tau_{n-2})_s}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<n-1}} \\
(S_{n-2})_s & \xhookrightarrow{(e_{n-2})_s} & \mathbb{R}^{n-2} \\
{\scriptstyle (\tau_{n-3})_s}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<n-2}} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
(S_1)_s & \xhookrightarrow{(e_1)_s} & \mathbb{R}.
\end{array} \tag{3.0.22}$$

The choices (3.0.21) and (3.0.22) for each s, uniquely determine an object of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^{n}))$ that is in the fiber of $\mathcal{G}_n$ over $[k](T_s)$:

$$\begin{array}{ccc}
\coprod_{1\leq s\leq k} (S_{n-1})_s & \xhookrightarrow{\coprod\{e(s)\}\times(e_{n-1})_s} & \mathbb{R}\times\mathbb{R}^{n-1} \\
{\scriptstyle \coprod(\tau_{n-2})_s}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<n}} \\
\coprod_{1\leq s\leq k} (S_{n-2})_s & \xhookrightarrow{\coprod\{e(s)\}\times(e_{n-2})_s} & \mathbb{R}\times\mathbb{R}^{n-2} \\
{\scriptstyle \coprod(\tau_{n-3})_s}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<n-1}} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\coprod_{1\leq s\leq k} (S_1)_s & \xhookrightarrow{\coprod\{e(s)\}\times(e_1)_s} & \mathbb{R}\times\mathbb{R} \\
{\scriptstyle \coprod\{s\}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{<2}} \\
\{1,...,k\} & \xhookrightarrow{e} & \mathbb{R}
\end{array} \tag{3.0.23}$$

where each map defined in terms of a coproduct is indexed over $1 \leq s \leq k$, and $\{e(s)\}$ and $\{s\}$ denote the constant maps at $e(s)$ and s, respectively.

Fix a pair of objects $\underline{T} \overset{d}{\hookrightarrow} \underline{\mathbb{R}}^n$ and $\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$. We will show fully faithfulness of $\mathcal{G}_n$ by showing that the map induced by $\mathcal{G}_n$ between hom-spaces

$$\mathrm{Hom}_{\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))}(\underline{e}, \underline{d}) \xrightarrow{\mathcal{G}_n} \mathrm{Hom}_{\boldsymbol{\Theta}_n^{\mathrm{act}}}(\mathcal{G}_n(\underline{e}), \mathcal{G}_n(\underline{d})) \tag{3.0.24}$$

is a surjection on connected components with contractible fibers.

Fix a morphism

$$\mathcal{G}_n(\underline{e}) \xrightarrow{\varphi} \mathcal{G}_n(\underline{d})$$

in $\boldsymbol{\Theta}_n^{\mathrm{act}}$. Using the inductive description of $\mathcal{G}_n$, φ is given by:

i) a morphism $\mathcal{G}_1(S_1 \xhookrightarrow{e_1} \mathbb{R}) \xrightarrow{\varphi_1} \mathcal{G}_1(T_1 \xhookrightarrow{d_1} \mathbb{R})$ in $\boldsymbol{\Delta}^{\mathrm{act}}$

ii) for each pair $(r \in T_1, s \in S_1)$ such that $\gamma_1(\varphi_1(r)) = s$, a morphism

$$\mathcal{G}_{n-1}((\underline{S})_s \xhookrightarrow{\underline{e}|_{(\underline{S})_s}} \underline{\mathbb{R}}^{n-1}) \xrightarrow{\varphi_r} \mathcal{G}_{n-1}((\underline{T})_r \xhookrightarrow{\underline{d}|_{(\underline{T})_r}} \underline{\mathbb{R}}^{n-1})$$

in $\boldsymbol{\Theta}_{n-1}^{\mathrm{act}}$.

Using the basecase and inductive step, we define a morphism that is in the fiber of $\mathcal{G}_n$ over φ: By fullness of $\mathcal{G}_1$, we may choose a morphism in the fiber of $\mathcal{G}_1$ over φ_1,

$$\mathrm{cylr}(\gamma_1(\varphi_1)) \xhookrightarrow{E_1} \mathbb{R} \times \Delta^1 \tag{3.0.25}$$

which is defined over the map of finite sets $T_1 \xrightarrow{\gamma_1 \circ \varphi_1} S_1$.

By fullness of $\mathcal{G}_{n-1}$ as assumed in the inductive step, for each pair $(r \in T_1, s \in S_1)$ such that $\gamma_1(\varphi_1(r)) = s$, we may choose a morphism in the fiber of $\mathcal{G}_{n-1}$ over φ_r,

$$\begin{array}{ccc} \mathrm{cylr}(\gamma_{n-1} \circ \varphi_r) & \xhookrightarrow{(E_n)_r} & \mathbb{R}^{n-1} \times \Delta^1 \\ \downarrow & & \downarrow\!\!\!\downarrow {\scriptstyle \mathrm{pr}_{<n-1} \times \mathrm{id}_{\Delta^1}} \\ \mathrm{cylr}(\gamma_{n-2} \circ \mathrm{tr}_{n-2} \circ \varphi_r) & \xhookrightarrow{(E_{n-1})_r} & \mathbb{R}^{n-2} \times \Delta^1 \\ \downarrow & & \downarrow\!\!\!\downarrow {\scriptstyle \mathrm{pr}_{<n-2} \times \mathrm{id}_{\Delta^1}} \\ \vdots & & \vdots \\ \downarrow & & \downarrow\!\!\!\downarrow \\ \mathrm{cylr}(\gamma_1 \circ \mathrm{tr}_1 \circ \varphi_r) & \xhookrightarrow{(E_2)_r} & \mathbb{R} \times \Delta^1 \end{array} \tag{3.0.26}$$

Note that diagram (3.0.17) guarantees that (3.0.26) must be defined over the diagram of finite sets,

$$\begin{array}{ccc} (T_n)_r & \xrightarrow{\gamma_{n-1} \circ \varphi_r} & (S_n)_s \\ {\scriptstyle \omega_{n-1}|_\bullet} \downarrow & & \downarrow {\scriptstyle \tau_{n-1}|_\bullet} \\ (T_{n-1})_r & \xrightarrow{\gamma_{n-2} \circ \mathrm{tr}_{n-1} \circ \varphi_r} & (S_{n-1})_s \\ {\scriptstyle \omega_{n-2}|_\bullet} \downarrow & & \downarrow {\scriptstyle \tau_{n-2}|_\bullet} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ (T_2)_r & \xrightarrow{\gamma_1 \circ \mathrm{tr}_1 \circ \varphi_r} & (S_2)_s. \end{array} \tag{3.0.27}$$

Using (3.0.25) and (3.0.26), we define a morphism in the fiber of $\mathcal{G}_n$ over φ:

$$
\begin{array}{ccc}
\mathrm{cylr}(\coprod_{r\in T_1}\gamma_{n-1}\circ\varphi_r) & \xhookrightarrow{\coprod\{E_1(r)\}\times(E_n)_r} & (\mathbb{R}\times\mathbb{R}^{n-1})\times\Delta^1 \\
\downarrow & & \downarrow \mathrm{pr}_{<n}\times\mathrm{id}_{\Delta^1} \\
\mathrm{cylr}(\coprod_{r\in T_1}\gamma_{n-2}\circ\mathrm{tr}_{n-2}\circ\varphi_r) & \xhookrightarrow{\coprod\{E_1(r)\}\times(E_{n-1})_r} & (\mathbb{R}\times\mathbb{R}^{n-2})\times\Delta^1 \\
\downarrow & & \downarrow \mathrm{pr}_{<n-1}\times\mathrm{id}_{\Delta^1} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{cylr}(\coprod_{r\in T_1}\gamma_{1}\circ\mathrm{tr}_{1}\circ\varphi_r) & \xhookrightarrow{\coprod\{E_1(r)\}\times(E_2)_r} & (\mathbb{R}\times\mathbb{R})\times\Delta^1 \\
\downarrow & & \downarrow \mathrm{pr}_{<2}\times\mathrm{id}_{\Delta^1} \\
\mathrm{cylr}(\gamma_1(\varphi_1)) & \xhookrightarrow{E_1} & \mathbb{R}\times\Delta^1
\end{array}
\tag{3.0.28}
$$

where $\{r\}$ and $\{E_1(r)\}$ denote the constant map at r and $E_1(r)$, respectively, and (3.0.28) is defined over the diagram of finite sets,

$$\begin{array}{ccc}
T_n = \coprod_{r \in T_1} (T_n)_r & \xrightarrow{\coprod \gamma_{n-1} \circ \varphi_r} & \coprod_{s \in S_1} (S_n)_s = S_n \\
\omega_{n-1} \big\downarrow & & \big\downarrow \tau_{n-1} \\
T_{n-1} = \coprod_{r \in T_1} (T_{n-1})_r & \xrightarrow{\coprod \gamma_{n-2} \circ \mathrm{tr}_{n-2} \circ \varphi_r} & \coprod_{s \in S_1} (S_{n-1})_s = S_{n-1} \\
\omega_{n-2} \big\downarrow & & \big\downarrow \tau_{n-2} \\
\vdots & & \vdots \\
\big\downarrow & & \big\downarrow \\
T_2 = \coprod_{r \in T_1} (T_2)_r & \xrightarrow{\coprod \gamma_1 \circ \mathrm{tr}_1 \circ \varphi_r} & \coprod_{s \in S_1} (S_2)_s = S_2 \\
\omega_1 = \coprod \{r\} \big\downarrow & & \big\downarrow \coprod \{s\} = \tau^1 \\
T_1 = \coprod_{r \in T_1} r & \xrightarrow{\gamma_1 \circ \varphi_1} & \coprod_{s \in S_1} s = S_1.
\end{array} \tag{3.0.29}$$

Lastly, we will show that each fiber of (3.0.24) is contractible. The fiber of $\mathcal{G}_n$ in (3.0.24) over φ is, under the compact-open topology, the topological space of compatible embeddings

$$\begin{array}{ccc}
\mathrm{cyl}(\gamma_n \circ \varphi) & \xhookrightarrow{E_n} & \mathbb{R}^n \times \Delta^1 \\
\big\downarrow & & \big\downarrow \mathrm{pr}_{<n} \times \mathrm{id} \\
\mathrm{cyl}(\gamma'_{n-1} \circ \mathrm{tr}_{n-1} \circ \varphi) & \xhookrightarrow{E_{n-1}} & \mathbb{R}^{n-1} \times \Delta^1 \\
\big\downarrow & & \big\downarrow \mathrm{pr}_{<n-1} \times \mathrm{id} \\
\vdots & & \vdots \\
\big\downarrow & & \big\downarrow \\
\mathrm{cyl}(\gamma_1 \circ \mathrm{tr}_1 \circ \varphi) & \xhookrightarrow{E_1} & \mathbb{R} \times \Delta^1
\end{array} \tag{3.0.30}$$

over Δ^1 over Δ^1 such that $E_{|S_n} = e_n$ and $E_{|T_n \times \{1\}} = d_n$. Note that (3.0.30) guarantees that each morphism in $\mathcal{G}_n^{-1}(\varphi)$ is defined over the diagram of finite sets

$$\begin{array}{ccc} T_n & \xrightarrow{\gamma'_n \circ \varphi} & S_n \\ {\scriptstyle \omega_{n-1}}\downarrow & & \downarrow{\scriptstyle \tau_{n-1}} \\ T_{n-1} & \xrightarrow{\gamma'_{n-1} \circ \mathrm{tr}_{n-1} \circ \varphi} & S_{n-1} \\ {\scriptstyle \omega_{n-2}}\downarrow & & \downarrow{\scriptstyle \tau_{n-2}} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ T_1 & \xrightarrow{\gamma_1 \circ \mathrm{tr}_1 \circ \varphi} & S_1. \end{array} \tag{3.0.31}$$

Fix an embedding E in the fiber of $\mathcal{G}_n$ over φ. Let $\mathbb{S}^k \xrightarrow{\psi} \mathcal{G}_n^{-1}(\varphi)$ be continuous and based at E. We construct a null-homotopy of ψ: For each $z \in \mathbb{S}^n$, denote the image of z under ψ by ψ_z. The straight-line homotopy, H_z, from ψ_z to E defined by

$$H_z(x,t) = (1-t)\psi_z(x) + tE(x)$$

names a path from ψ_z to E in $\mathcal{G}_n^{-1}(\varphi)$. For each $z \in \mathbb{S}^k$, we let each path H_z run simultaneously to name a null-homotopy of ψ to the constant path at $\{E\}$. Explicitly, the null-homotopy $\mathbb{S}^k \times [0,1] \to \mathcal{G}_n^{-1}(\varphi)$ is given by $(z,t) \mapsto H_z(-,t)$.

□

In summary, we have identified the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ as the category $\mathbf{\Theta}_n^{\mathrm{act}}$. This was the first of two main lemmas which together imply the main result of the chapter, Theorem (3.0.46), that $\mathbf{\Theta}_n^{\mathrm{act}}$ localizes to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$. Next, we prove the second main lemma, (3.0.63), which identifies the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ as a localization of the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$.

Localizing the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$

Localization Heuristically, an ∞-category $\mathcal{C}$ localizes to an ∞-category $\mathcal{D}$ if there is some collection of morphisms in $\mathcal{C}$ such that the ∞-category obtained from $\mathcal{C}$ by formally inverting each morphism in this collection is equivalent to $\mathcal{D}$. We build to the formal definition of a localization next.

Definition 3.0.60. Given an ∞-category $\mathcal{C}$, the *maximal sub ∞-groupoid* $\mathcal{C}^{\sim}$ of $\mathcal{C}$, is the ∞-subcategory of $\mathcal{C}$ such that for an ∞-groupoid $\mathcal{E}$, any functor F from $\mathcal{E}$ to $\mathcal{C}$ uniquely factors through $\mathcal{C}^{\sim}$,

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{C} \\ & \overset{\exists !}{\searrow} & \uparrow \\ & & \mathcal{C}^{\sim}. \end{array}$$

Informally, $\mathcal{C}^{\sim}$ is the ∞-subcategory of $\mathcal{C}$ consisting of all the same objects and only those morphisms that are isomorphisms.

Example 3.0.61. Let $[p]$ denote the category with objects $\{0, 1, ..., p\}$ and a unique morphism from i to j if $i \leq j$, and no morphism from i to j otherwise. The maximal sub ∞-groupoid $[p]^{\sim}$ of $[p]$ is the category with the same objects as $[p]$ and only identity morphisms.

Definition 3.0.62. Let $\mathcal{C}$ be an ∞-category and let W be a ∞-subcategory of $\mathcal{C}$ which contains the maximal sub ∞-groupoid $\mathcal{C}^{\sim}$ of $\mathcal{C}$. The *localization* of $\mathcal{C}$ on W is an ∞-category $\mathcal{C}[W^{-1}]$ and a functor $\mathcal{C} \xrightarrow{L} \mathcal{C}[W^{-1}]$ satsifying the following universal property: For any ∞-category $\mathcal{D}$, any functor F from $\mathcal{C}$ to $\mathcal{D}$ uniquely factors through L if and only if F maps each morphism in W to an isomorphism in $\mathcal{D}$; otherwise, there is no filler

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow_{\exists! \text{ or } \emptyset} & \\ \mathcal{C}[W^{-1}]. & & \end{array}$$

Statement of Lemma (3.0.63) We state the second of the two main lemmas used in our procedure for proving Theorem (3.0.46). Heuristically, this lemma articulates a sense in which the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ behaves like the exit-path ∞-category of a refinement of the unital Ran space.

Lemma 3.0.63. *The forgetful functor is a localization of ∞-categories*

$$Exit(Ran^u(\underline{\mathbb{R}}^n)) \to Exit(Ran^u(\mathbb{R}^n))$$

over Fin^{op} from the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

Our procedure for showing Lemma (3.0.63) is technical. We build the argument using a result from [27], Theorem (3.0.68) below, and premised on that result, show two lemmas, (3.0.71) and (3.0.101) which we will use to prove Lemma (3.0.63).

A technical theorem by Aaron Mazel-Gee First, we recall some definitions needed for stating Theorem (3.0.68). We define a construction which associates to each ∞-category $\mathcal{C}$ a CW-complex B$\mathcal{C}$ called the *classifying space* of $\mathcal{C}$. Heuristically, this is a space obtained by taking a vertex for each object of $\mathcal{C}$, attaching a 1-simplex for each morphism of $\mathcal{C}$, attaching a 2-simplex for each commutative triangle in $\mathcal{C}$, etc. It makes use of the *geometric realization* which is denoted $|-|$; we refer the reader to 15.18 in [33].

Definition 3.0.64. The *groupoid completion* $\mathcal{C}^{\simeq}$ of an ∞-catgory $\mathcal{C}$ is the localization of $\mathcal{C}$ on $\mathcal{C}$,

$$\mathcal{C}^{\simeq} := \mathcal{C}[\mathcal{C}^{-1}].$$

The idea of the groupoid completion is to simply formally invert every morphism.

Definition 3.0.65. Given an ∞-category $\mathcal{C}$, the *classifying space of* $\mathcal{C}$ $\mathrm{B}\mathcal{C}$ is the geometric realization of the groupoid completion of $\mathcal{C}$

$$\mathrm{B}\mathcal{C} := |\mathcal{C}^{\simeq}|.$$

We are nearly equipped to state Theorem (3.0.68), the key tool in our argument for proving Lemma (3.0.63). It establishes a way to identify the localization of an ∞-category in terms of the classifying space of the following ∞-category.

Definition 3.0.66. Given an ∞-category $\mathcal{C}$ and an ∞-subcategory $W \hookrightarrow \mathcal{C}$, $\mathrm{Fun}^W([p], \mathcal{C}))$ is defined to be the pullback of ∞-categories

$$\begin{array}{ccc} \mathrm{Fun}^W([p], \mathcal{C}) & \longrightarrow & \mathrm{Fun}([p], \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}([p]^{\simeq}, W) & \hookrightarrow & \mathrm{Fun}([p]^{\simeq}, \mathcal{C}). \end{array}$$

Observation 3.0.67. In the case $p = 0$, $\mathrm{Fun}^W([0], \mathcal{C})$ is equivalent to W. Indeed, an object is a functor $[0] \to \mathcal{C}$ selects out an object of W, which is precisely an object of $\mathcal{C}$; a morphism is a natural transformation between any two such functors, which is precisely determined by a morphism in W.

A similar examination of the $p = 1$ case identifies that $\mathrm{Fun}^W([1], \mathcal{C})$ is the ∞-category whose objects are morphisms of $\mathcal{C}$ and whose morphisms are all those natural transformations given by morphisms in W, i.e., a morphism from $c \to d$ in $\mathcal{C}$ to $c' \to d'$

in $\mathcal{C}$ is a commutative square in W

$$\begin{array}{ccc} c & \longrightarrow & c' \\ \downarrow & & \downarrow \\ d & \longrightarrow & d' \end{array}$$

such that both horizontal arrows are morphisms in W.

Finally, we state the key theorem used in proving Lemma (3.0.63).

Theorem 3.0.68 (3.8 in [27]). *For an ∞-category $\mathcal{C}$ and a ∞-subcategory containing the maximal sub ∞-groupoid of $\mathcal{C}$, $\mathcal{C}^{\sim} \subset W \subset \mathcal{C}$, if the classifying space of $Fun^{W}([\bullet], \mathcal{C})$ is a complete Segal space, then it is equivalent as a simplicial space to the localization of $\mathcal{C}$ on W,*

$$BFun^{W}([\bullet], \mathcal{C}) \simeq \mathcal{C}[W^{-1}].$$

We wish to emply Theorem (3.0.68) to prove that there is a localization of the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$ (Lemma (3.0.63)). Before we outline our procedure, we define the sub ∞- category of the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ upon which we localize to obtain the exit-path ∞-category of the unital Ran space.

The localizing ∞-subcategory W_n

Definition 3.0.69. W_n is the ∞-subcategory of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ defined to be the pullback

$$\begin{array}{ccc} W_n & \hookrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \\ \downarrow & \lrcorner & \downarrow \phi \\ \mathrm{Fun}^{\text{n-bij}}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}}) & \hookrightarrow & \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}}) \end{array}$$

where $\mathrm{Fun}^{\text{n-bij}}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$ is the subcategory of $\mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$ in which the objects are the same and a morphism must satsify that its value under evaluation at n is a bijection, and ϕ is the forgetful functor from (3.0.55) that simply remembers the underlying data of sets at each level $1 \leq i \leq n$.

Heuristically, W_n has the same objects as $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ and all those morphisms whose values under ϕ_n (3.0.55) are bijections. Intuitively then, $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ localizing on W_n to $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ is no surprise. Indeed, in formally declaring all those morphisms in W_n to be isomorphisms, we forget the restriction by coordinate coincidence which defines morphisms in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ and only remember cardinality, which is the defining restriction of morphisms in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$.

Remark 3.0.70. When convenient, we consider W_n as a subcategory of $\mathbf{\Theta}_n^{\mathrm{act}}$ in lieu of the equivalence $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \simeq \mathbf{\Theta}_n^{\mathrm{act}}$ proven in Lemma (3.0.59).

We use the following two lemmas, (3.0.71) and (3.0.101), together with Theorem (3.0.68) to prove that the localization of the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ on W_n is equivalent to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$. Recall that in so doing, we prove the second of the two lemmas, (3.0.57) and (3.0.63), which together imply the main result of the chapter, Theorem (3.0.46).

The first lemma states that the value of $\mathrm{BFun}^{W_n}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))$ on $[0]$ is the space of objects of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ and its value on $[1]$ is the space of morphisms of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$.

Lemma 3.0.71. *For $p = 0, 1$, there is an equivalence of spaces*

$$BFun^{W_n}([p], Exit(Ran^u(\underline{\mathbb{R}}^n))) \simeq Hom_{Cat_\infty}([p], Exit(Ran^u(\mathbb{R}^n))) \tag{3.0.32}$$

between the classifying space of the ∞-category $Fun^{W_n}([p], Exit(Ran^u(\underline{\mathbb{R}}^n)))$ and the hom-space in ∞-categories from $[p]$ to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

Observation 3.0.72. For an ∞-category $\mathcal{C}$, the value of the simplicial space $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathcal{C})$ on $[0]$ is the underlying maximal sub ∞-groupoid $\mathcal{C}^\sim$ of $\mathcal{C}$, and on $[1]$ is the space of morphisms $\mathrm{mor}(\mathcal{C})$ of $\mathcal{C}$, both of which are a space, that is, they are equivalent to a CW complex.

The second lemma verifies that the hypothesis of Theorem (3.0.68) is satsified for the situation at hand.

Lemma 3.0.73. *The classifying space of $Fun^{W_n}([\bullet], Exit(Ran^u(\underline{\mathbb{R}}^n)))$ is a complete Segal space.*

Proving Lemma (3.0.71) We break Lemma (3.0.71) into two parts; the $p = 0$ case and the $p = 1$ case.

$[\mathbf{p} = \mathbf{0}]$ In light of Observation (3.0.67), where we identified that for a fully faithful ∞-subcategory W of an ∞-category $\mathcal{C}$, the ∞-category $\mathrm{Fun}^W([0], \mathcal{C})$ is the ∞-subcategory W, and Observation (3.0.72), where we identified that for an ∞-category $\mathcal{C}$, the value of the simplicial space $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathcal{C})$ on $[0]$ is the maximal sub ∞-groupoid $\mathcal{C}^\sim$ of $\mathcal{C}$, we rephrase the $p = 0$ case of Lemma (3.0.71) as the following lemma.

Lemma 3.0.74. *There is an equivalence of spaces*

$$BW_n \simeq \mathit{Exit}(\mathit{Ran}^u(\mathbb{R}^n))^{\sim}$$

from the classifying space of the ∞-subcategory W_n of $\mathit{Exit}(\mathit{Ran}^u(\underline{\mathbb{R}}^n))$ to the maximal sub ∞-groupoid of the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

The proof of Lemma (3.0.74) relies on the two results that we will show next, Lemma (3.0.77) and Corollary (3.0.81). We begin with Lemma (3.0.77), in which we show that there is an *adjunction* between W and the subcategory of W consisting of healthy trees; we define this notion and this subcategory next, and then state the lemma.

Definition 3.0.75. A pair of functors

$$\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{D}$$

between ∞-categories is an *adjunction* if there exists a unit transformation $\mathrm{Id}_{\mathcal{D}} \xrightarrow{\epsilon} R \circ L$ such that for all $d \in \mathcal{D}$ and $c \in \mathcal{C}$, the induced morphism

$$\mathrm{Hom}_{\mathcal{C}}(L(d), c) \xrightarrow{R} \mathrm{Hom}_{\mathcal{D}}(R(L(d)), R(c)) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\epsilon, R(c))} \mathrm{Hom}_{\mathcal{D}}(d, R(c))$$

is an equivalence of ∞-groupoids. The functor R is *right adjoint* to L and L is *left adjoint* to R.

Definition 3.0.76. W_n^{healthy} is the subcategory of W_n defined to be the pullback

$$\begin{array}{ccc} W_n^{\text{healthy}} & \hookrightarrow & W_n \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}^{\text{n-bij}}(\{1 < \cdots < n\}, (\mathrm{Fin}^{\text{surj}})^{\text{op}}) & \hookrightarrow & \mathrm{Fun}^{\text{n-bij}}(\{1 < \cdots < n\}, \mathrm{Fin}^{\text{op}}) \end{array}$$

of categories.

Informally, the category W_n^{healthy} is the full subcategory of W_n consisting of all those objects of $\mathbf{\Theta}_n$ that are healthy trees.

Lemma 3.0.77. *The inclusion functor $W_n^{healthy} \hookrightarrow W_n$ is a right adjoint.*

In the proof of Lemma (3.0.77), we make use of the following construction..

Construction 3.0.78 (The Pruning Functor, P_n)**.** For each $n \geq 1$, we define a canonical functor

$$P_n : W_n \to W_n^{\text{healthy}}.$$

For $n = 1$, $P_1 := \mathrm{Id}_{W_1}$ since $W_1 = W_1^{\text{healthy}}$.

For $n \geq 2$, we define P_n inductively. First, for each object $T = [p](T_i) \in W_n$ for $n \geq 2$, define the sub-linearly ordered set

$$N_T := \left\{ 0 = i_0 < i_1 < \cdots < i_k \,\middle|\, \begin{matrix} i_j \in \{1, ..., p\} \forall 1 \leq j \leq k \\ T_i = \emptyset \iff \exists 1 \leq j \leq k \text{ s.t. } i = i_j \end{matrix} \right\} \subset [p].$$

For the case $n = 2$, we define

$$P_2 : T = [p]([q_i]) \mapsto N_T([q_{i_j}])$$

on objects; the value of a morphism under P_2 is given by restriction of that morphism to N_T. P_2 respects composition because restriction respects composition.

For general n, we define

$$P_n : T = [p](T_i) \mapsto N_T(P_{n-1}(T_{i_j}))$$

on objects; the value of a morphism under P_n is again determined by restriction of that morphism to N_T together with P_{n-1}. Composition is preserved by P_n because restriction and P_{n-1} both respect composition.

Proof. (Lemma (3.0.77)) We use Lemma 2.17 from [2] which states that for a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ of ∞-categories, F is a right adjoint if and only if for each object $d \in \mathcal{D}$, the ∞-undercategory $\mathcal{C}^{d/}$ has an initial object, and verify that for each $T \in W_n$, the undercategory $W^{\text{healthy}T/}$ has an initial object. To define such an initial object, we use the canonical morphism defined as follows:

For each object $T \in W_n$, we define a morphism $T \xrightarrow{\alpha_T} P_n(T)$ in W_n such that any morphism $T \xrightarrow{f} S$ in W to a healthy tree S uniquely factors through α_T. For $n = 1$, $T = P_1(T)$ for each $T \in W_1$ and thus we define $\alpha_T := \text{Id}_T$.

To define α_T for $T \in W_n$ for $n \geq 2$, we proceed by induction. Fix an object $T \in W_2$. Define $\alpha_T : T = [p]([q_i]) \to N_T([q_{i_j}])$ by

i) $[p] \to N_T$ is given by the assignment $i \mapsto \begin{cases} i_j, \text{ if } \exists 0 \leq j \leq k-1 \text{ s.t. } i_j \leq i < i_{j+1} \\ i_k, \text{ if } i \geq i_k. \end{cases}$

ii) For each pair (i, i_j) such that $i = i_j$, $[q_i] \xrightarrow{\text{Id}} [q_{i_j}]$.

For the general case, fix an object $T \in W_n$. Define $\alpha_T : T = [p](T_i) \to N_T(P_{n-1}(T_{i_j})$ by

i) $[p] \to N_T$ is the same as i) for $n = 2$.

ii) For each pair (i, i_j) such that $i = i_j$, $T_i \xrightarrow{\alpha_{T_i}} P_{n-1}(T_{i_j})$, where α_{T_i} is guaranteed by the inductive step.

Next, we observe that by design each morphism $T \xrightarrow{f} S$ in W to a healthy tree S factors through α_T via $P_n(f)$:

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ {\scriptstyle \alpha_T}\downarrow & \nearrow_{P_n(f)} & \\ P_n(T). & & \end{array} \tag{3.0.33}$$

Further, $P_n(f)$ uniquely fills (3.0.33).

We have just verified that for each fixed object $T \in W$, the initial object of ${W^{\text{healthy}}}^{T/}$ is $(P_n(T), T \xrightarrow{\alpha_T} P_n(T))$.

□

We have just verified that there is an adjunction between W_n^{healthy} and W_n, the right adjoint of which is the inclusion functor. This was the first of two results that we will use to prove Lemma (3.0.74), the second of which is Corollary (3.0.81).

Corollary (3.0.81) identifies that the configuration space of r unordered points in $\mathbb{R}^n$ is homotopy equivalent to the classifying space of the following ∞-category.

Definition 3.0.79. For $r \geq 0$, $\text{Exit}(\text{Conf}_r(\underline{\mathbb{R}}^n))_{\Sigma_r}$ is the ∞-subcategory of the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ defined to be the pullback

$$\begin{array}{ccc} \text{Exit}(\text{Conf}_r(\underline{\mathbb{R}}^n))_{\Sigma_r} & \hookrightarrow & \text{Exit}(\text{Ran}^{\text{u}}(\underline{\mathbb{R}}^n)) \\ \downarrow & \lrcorner & \downarrow \\ \text{Fun}^{r,n}(\{1 < \cdots < n\}, \text{Fin}^{\text{surj,op}})) & \hookrightarrow & \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}). \end{array}$$

where $\text{Fun}^{r,n}(\{1 < \cdots < n\}, \text{Fin}^{\text{surj,op}})$ is the subcategory of $\text{Fun}(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}})^{\text{op}})$ in which the value of an object upon evaluation at n has cardinality r.

Observation 3.0.80. It follows by definition that the colimit of ∞-categories $\coprod_{r \geq 0} \text{Exit}(\text{Conf}_r(\underline{\mathbb{R}}^n))_{\Sigma_r}$ is equivalent to the category W_n^{healthy}.

Corollary 3.0.81. *There is a homotopy equivalence*

$$B(\mathit{Exit}(\mathit{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r})) \simeq \mathit{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$$

from the classifying space of the ∞-category $\mathit{Exit}(\mathit{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r})$ to the configuration space of r unordered points in $\mathbb{R}^n$.

We explicate the sense in which Corollary (3.0.81) is a corollary to a result in [7] in which the homotopy type of the configuration space of r ordered points in $\mathbb{R}^n$ is identified in terms of the classifying space of certain subcategory of $\mathbf{\Theta}_n$. First, we introduce the following ∞-categories to translate the language of [7] to that of exit-path ∞-categories.

Definition 3.0.82. Fix a natural number r and a set S with cardinaltiy r.

- The category $\mathbf{\Theta}_n^{\text{healthy,act}}$ is the full subcategory of $\mathbf{\Theta}_n^{\text{act}}$ consisting of all those objects that are healthy trees.

- The category $\mathbf{\Theta}_n^{\text{healthy},r}$ is the subcategory of $\mathbf{\Theta}_n^{\text{healthy,act}}$ defined to be the pullback

$$\begin{array}{ccc} \mathbf{\Theta}_n^{\text{healthy},r} & \hookrightarrow & \mathbf{\Theta}_n^{\text{healthy,act}} \\ {\scriptstyle \gamma_n}\downarrow & \lrcorner & \downarrow{\scriptstyle \gamma_n} \\ (\text{Fin}_r)^{\text{op}} & \hookrightarrow & \text{Fin}^{\text{op}} \end{array}$$

  of categories, where Fin_r is the full subcategory of Fin consisting of all those finite sets with cardinality r.

- The category $\mathbf{\Theta}_n^{\text{healthy}}(S)$ is defined to be the pullback

$$\begin{array}{ccc} \mathbf{\Theta}_n^{\text{healthy}}(S) & \longrightarrow & \mathbf{\Theta}_n^{\text{healthy},r} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \gamma_n} \\ * & \xrightarrow{\langle S \rangle} & (\text{Fin}_k)^{\text{op}} \end{array}$$

of categories.

- The ∞-category $\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n))$ is defined to be the pullback

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n)) & \longrightarrow & \mathrm{Exit}(\mathrm{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{\Theta}_n^{\mathrm{healthy}}(S) & \longrightarrow & \mathbf{\Theta}_n^{\mathrm{healthy},r} \end{array}$$

of ∞-categories.

The objects of $\mathbf{\Theta}_n^{\mathrm{healthy},r}$ are healthy trees whose set of leaves has cardinality r, whereas the objects of $\mathbf{\Theta}_n^{\mathrm{healthy}}(S)$ are healthy trees whose set of leaves is labeled by the set S of cardinality r.

Heuristically, an object of $\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n))$ is an embedding of the set S into $\mathbb{R}^n$ and a morphism is a path in $\mathrm{Conf}_S(\mathbb{R}^n)$ the image of which after projecting off the last i coordinates for each $0 \leq i \leq n-1$ is an 'exit-path' in $\mathrm{Conf}_{\bullet}(\mathbb{R}^{n-i})$ in that it allows anticolision of points, but does not allow collision of points.

Observation 3.0.83. The category $\mathbf{\Theta}_n^{\mathrm{healthy}}(S)$ is equivalent to the category in [7] refered to as the *poset of n-orderings of* S, denoted $nOrd(S)$, in which an object is a healthy tree of height n whose set of leaves is labeled by the set S, and a morphism is an active morphism in $\mathbf{\Theta}_n$ which satsifies the *branching condition*; see Definition 8 in [7].

To prove Corollary (3.0.81), we need the next result, Corollary (3.0.84) which is a corollary to Theorem A of [7]. Theorem A, in light of Observation (3.0.83), gives a homotopy equivalence between the classifying space of $\mathbf{\Theta}_n^{\mathrm{healthy}}(S)$ and the configuration space of r points in $\mathbb{R}^n$ labeled by a set S with cardinality r, $\mathrm{Conf}_S(\mathbb{R}^n)$. Our result involves the following ∞-category, which translates the language of [7] to our language in terms of exit-path ∞-categories.

Corollary 3.0.84. *There is a homotopy equivalence*

$$B(Exit(Conf_S(\underline{\mathbb{R}}^n))) \simeq Conf_S(\mathbb{R}^n)$$

between the classifying space of the ∞-category $Exit(Conf_S(\underline{\mathbb{R}}^n))$ and the configuration space of points in $\mathbb{R}^n$ marked by the set S of cardinality r.

.

Proof. First, observe that the following diagram is a pullback of ∞-categories:

$$\begin{array}{ccc} \mathrm{Exit}(\mathrm{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r}) & \longrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \\ \downarrow & \lrcorner & \downarrow \mathcal{G}_n \\ \mathbf{\Theta}_n^{\mathrm{healthy},r} & \longrightarrow & \mathbf{\Theta}_n^{\mathrm{act}} \end{array} \tag{3.0.34}$$

where $\mathcal{G}_n$ was defined in Construction (3.0.58).

Combining (3.0.34) with the definition of $\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n))$, we obtain the following diagram of ∞-categories:

$$\begin{array}{ccccc} \mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n)) & \longrightarrow & \mathrm{Exit}(\mathrm{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r}) & \hookrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \mathcal{G}_n \\ \mathbf{\Theta}_n^{\mathrm{healthy}}(S) & \longrightarrow & \mathbf{\Theta}_n^{\mathrm{healthy},r} & \hookrightarrow & \mathbf{\Theta}_n^{\mathrm{act}}. \end{array} \tag{3.0.35}$$

Lemma (3.0.59) showed that the functor $\mathcal{G}_n$ is an equivalence, which implies the other two downward vertical arrows in Diagram (3.0.35) are both equivalences of ∞-categories as well. In particular, we note the equivalence $\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n)) \simeq \mathbf{\Theta}_n^{\mathrm{healthy}}(S)$. Then, using the homotopy equivalence between the classifying space of $\mathbf{\Theta}_n^{\mathrm{healthy}}(S)$ and $\mathrm{Conf}_S(\mathbb{R}^n)$ established in Theorem A of [7], we have the desired result, namely,

$$\mathrm{B}(\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n))) \simeq \mathrm{B}(\mathbf{\Theta}_n^{\mathrm{healthy}}(S)) \simeq \mathrm{Conf}_S(\mathbb{R}^n).$$

□

Using the previous corollary, we prove Corollary (3.0.81)

Proof. (Corollary 3.0.81) Fix a natural number r and a set S with cardinality r. Observe that the equivalence in Corollary (3.0.84) is Σ_S-equivariant, and thus,

$$(\mathrm{B}(\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n))))_{\Sigma_S} \simeq \mathrm{Conf}_S(\mathbb{R}^n)_{\Sigma_S}.$$

In chapter 4 of [25], it is shown that the classifying space of a colimit is equivalent to the colimit of the classifying space. Thus, since the quotient is a colimit, the quotient of the classifying space of $\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n))$ is equivalent to the classifying space of $\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n)_{\Sigma_S})$, which establishes the desired homotopy equivalence

$$\mathrm{B}(\mathrm{Exit}(\mathrm{Conf}_S(\underline{\mathbb{R}}^n)_{\Sigma_r})) \simeq \mathrm{Conf}_S(\mathbb{R}^n)_{\Sigma_r}.$$

□

With Lemma (3.0.77) and Corollary (3.0.81) in hand, we are now equipped to prove Lemma (3.0.74).

Proof. (Lemma 3.0.74) Corollary 2.1.28 in [28] states that an adjunction between ∞-categories yields an equivalence between their classifying spaces. We apply this result to the adjunction from Lemma (3.0.77) to obtain an equivalence of the classifying spaces,

$$\mathrm{B}W_n \simeq \mathrm{B}W_n^{\mathrm{healthy}}.$$

We observed in (3.0.80) that

$$\mathrm{B}W^{\mathrm{healthy}} \simeq \coprod_{r\geq 0} \mathrm{BExit}(\mathrm{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r}).$$

By Corollary (3.0.81),

$$\coprod_{r\geq 0} \mathrm{BExit}(\mathrm{Conf}_r(\underline{\mathbb{R}}^n)_{\Sigma_r}) \simeq \coprod_{r\geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}.$$

Lastly, $\coprod_{r\geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$ is, by definition, equivalent to $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))^{\sim}$ the maximal sub ∞-groupoid of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$. □

[**p = 1**] There are three main lemmas, Lemmas (3.0.95), (3.0.96), and (3.0.97), which together prove the equivalence of Lemma (3.0.71) for $p = 1$. Our approach involves some technical developments of category theory, notably *Cartesian fibrations* and Quillen's Theorem B. These however should not distract the reader from the main idea of proof: Both the source and target of Equation (3.0.32) in Lemma (3.0.71) assemble as fibrations and we use Lemma (3.0.74) to show that the natural map between them induces an equivalence of the base spaces and fibers, hence inducing a weak equivalence of the total spaces, which are CW-complexes:

$$\begin{array}{ccc} \mathrm{BFun}^W([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\mathrm{frgt}} & \mathrm{mor}(\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))) \\ {\scriptstyle \mathrm{Bev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathrm{BFun}^W([0], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \longrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))^{\simeq} \end{array} \qquad (3.0.36)$$

First, we detail the machinery needed to define and identify the fibrations Bev_0 and ev_0. Lemma (3.0.95) identifies the fibers of ev_0 and Lemmas (3.0.96) and (3.0.97) identify the fibers of Bev_0.

Definition 3.0.85 (∞-Undercategory)**.** Given a functor of ∞-categories $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and an object $d \in \mathcal{D}$, the ∞-*undercategory* $\mathcal{C}^{d/}$ of $\mathcal{C}$ under d is the pullback

$$\begin{array}{ccc} \mathcal{C}^{d/} & \longrightarrow & \mathcal{D}^{d/} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

Theorem 3.0.86 (Quillen's Theorem B)**.** *Given a functor between ∞-categories $\mathcal{C} \xrightarrow{F} \mathcal{D}$, if each morphism $d \xrightarrow{f} d'$ in $\mathcal{D}$ induces a weak equivalence $B(\mathcal{C}^{d'/}) \xrightarrow{\simeq} B(\mathcal{C}^{d/})$ between the classifying spaces of the induced ∞-undercategories, then $B(\mathcal{C}^{d/})$ is the homotopy fiber of BF over d and thus, $B(\mathcal{C}^{d/}) \hookrightarrow B\mathcal{C} \xrightarrow{BF} B\mathcal{D}$ is a fiber sequence.*

Remark 3.0.87. Quillen originally proved Theorem B in [29] for categories. Theorem 5.16 in [2] generalizes the result for ∞-categories, which is the statement of Quillen's Theorem B given above.

Definition 3.0.88 (2.1 in ([2]))**.** Let $\mathcal{E} \xrightarrow{\pi} \mathrm{B}$ be a functor between ∞-categories. A morphism $c_1 \xrightarrow{\langle e \xrightarrow{\varphi} e' \rangle} \mathcal{E}$ is π-*Cartesian* if the diagram of ∞-overcategories

$$\begin{array}{ccc} \mathcal{E}_{/e} & \xrightarrow{\phi \circ -} & \mathcal{E}_{/e'} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathrm{B}_{/\pi(e)} & \xrightarrow{\pi(e) \circ -} & \mathrm{B}_{/\pi(e')} \end{array}$$

is a pullback.

π is a *Cartesian fibration* if for every solid square

$$\begin{array}{ccc} * & \longrightarrow & \mathcal{E} \\ {\scriptstyle \langle t \rangle}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle \pi} \\ c_1 & \longrightarrow & \mathrm{B} \end{array}$$

there is a π-Cartesian filler.

Observation 3.0.89. The functor $\mathrm{Fun}^{W_n}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) \xrightarrow{\mathrm{ev}_0} W_n$ is a Cartesian fibraton. This follows upon application of example 2.5 in [2], wherein they show that for an ∞-category $\mathcal{C}$, the functor given by evaluation at 0, $\mathrm{Fun}([1], \mathcal{C}) \xrightarrow{\mathrm{ev}_0} \mathrm{Fun}([0], \mathcal{C})$ is a Cartesian fibration, to the following diagram:

$$\begin{array}{ccc} \mathrm{Fun}^{W_n}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\mathrm{frgt}} & \mathrm{mor}(\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathrm{Fun}^{W_n}([0], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \longrightarrow & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))^{\simeq}. \end{array}$$

Observation 3.0.90. Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ be a Cartesian fibration. For each object $b \in$ B there is a canonical inclusion $\pi^{-1}(b) \hookrightarrow \mathcal{E}^{b/}$ from the fiber of π over b to the undercategory of $\mathcal{E}$ under b. Its value on an object $e \in \pi^{-1}(b)$ such that $b \cong \pi(e)$ is the equivalence $b \xrightarrow{\cong} \pi(e)$. Its value on a morphism $e \xrightarrow{f} e'$ is $\pi(f)$.

Definition 3.0.91 (Cartesian Monodromy Functor)**.** Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ be a Cartesian fibration. For each morphism $b \xrightarrow{f} b'$ in $\mathcal{B}$, the induced *Cartesian monodromy functor* $f^* : \pi^{-1}(b') \to \pi^{-1}(b)$ from the fiber over b' to the fiber over b is defined to be the threefold composite

$$\begin{array}{ccc} \pi^{-1}(b') & \overset{f^*}{\dashrightarrow} & \pi^{-1}(b) \\ \downarrow & & \uparrow{\scriptstyle \mu} \\ \mathcal{E}^{b'/} & \xrightarrow{-\circ f} & \mathcal{E}^{b/} \end{array}$$

where μ is right adjoint to the inclusion functor $\mathcal{E}_{|_b} \hookrightarrow \mathcal{E}^{b/}$.

Observation 3.0.92. Given a pullback of ∞-categories

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{G} & \mathcal{E}' \\ {\scriptstyle \pi}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi'} \\ \mathrm{B} & \xrightarrow{F} & \mathrm{B}' \end{array} \tag{3.0.37}$$

in which π' is a Cartesian fibration, for each morphism $b \xrightarrow{\alpha} b'$ in B, G carries the induced monodromy functor α^* to the monodromy functor induced by $F(\alpha)$,

$$\begin{array}{ccc} \pi^{-1}(b') & \xrightarrow{\alpha^*} & \pi^{-1}(b) \\ \simeq \downarrow & & \downarrow \simeq \\ \pi'^{-1}(F(b')) & \xrightarrow{F(\alpha)^*} & \pi'^{-1}(F(b)). \end{array}$$

Note that the downward vertical arrows are equivalences between fibers precisely because (3.0.37) is a pullback.

Note 3.0.93. Lemma 2.20 in [2] guarantees that for a Cartesian fibration $\mathcal{E} \xrightarrow{\pi} \mathrm{B}$ the inclusion $\pi^{-1}(b) \hookrightarrow \mathcal{E}^{b/}$ is a left adjoint.

Notation 3.0.94. For the remainder of this thesis, we impliment the following notational changes:

- $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$: We denote an object $S \overset{e}{\hookrightarrow} \mathbb{R}^n$ by S, by which we mean the image of S in $\mathbb{R}^n$ under the embedding e.

- $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$: We denote an object $\underline{S} \overset{e}{\hookrightarrow} \underline{\mathbb{R}}^n$ by $\underline{S} = S_n \to \cdots \to S_1$, by which we mean the images of S_i under e_i for each $1 \leq i \leq n$ together with the coordinate projection data given by the sequence of maps of finite sets $S_n \to \cdots \to S_1$.

  We denote a morphism $\mathrm{cylr}(\underline{S}' \xrightarrow{\sigma} \underline{S}) \overset{E}{\hookrightarrow} \underline{\mathbb{R}}^n \times \Delta^1$ from $\underline{S}$ to $\underline{S}'$ by simply an arrow $\underline{S} \to \underline{S}'$.

Lemma 3.0.95. *The fiber of the map of spaces*

$$mor(Exit(Ran^u(\mathbb{R}^n))) \xrightarrow{ev_0} Exit(Ran^u(\mathbb{R}^n))^{\sim}$$

from the space of morphisms of $Exit(Ran^u(\mathbb{R}^n))$ *to the maximal sub* ∞*-groupoid of* $Exit(Ran^u(\mathbb{R}^n))$ *over an object* $S \subset \mathbb{R}^n$ *of* $Exit(Ran^u(\mathbb{R}^n))$ *is*

$$\coprod_{s \in S} Exit(Ran^u(T_s\mathbb{R}^n))^{\sim}$$

the product space indexed by S *of the maximal sub* ∞*-groupoid of the exit-path* ∞*-category of the unital Ran space of the tangent space of* $\mathbb{R}^n$ *at* $s \in S$.

Proof. First, recall that the maximal sub ∞-groupoid $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))^{\sim}$ of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ is equivalent to the disjoint union of configuration spaces $\coprod_{r \geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$.

Next, we define three maps.

1. Fix a continuous map

$$\epsilon : \coprod_{r \geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \to \mathbb{R}_{>0}$$

such that for each pair of distinct points $s \neq s'$ in S, the open n-dimentional cubes $\mathrm{Box}_{\epsilon(S)}(-) \cong (-\epsilon(S), \epsilon(S))^{\times n}$ of volume $(2\epsilon(S))^n$, centered at each point do not intersect,

$$\mathrm{Box}_{\epsilon(S)}(s) \cap \mathrm{Box}_{\epsilon(S)}(s') = \emptyset.$$

Note that such a continuous map exists because $\mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$ is Hausdorff.

2. For each $S \in \coprod_{r \geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$ and each $s \in S$, define a homeomorphism

$$D_{s \in S} : \mathbb{R}^n \to \mathrm{Box}_{\epsilon(S)}(s)$$

by the composite

$$\mathbb{R}^n \xrightarrow{\cong} (-\epsilon(S), \epsilon(S))^{\times n} \xrightarrow{\cong} \mathrm{Box}_{\epsilon(S)}(s) \tag{3.0.38}$$

the first homeomorphism of which is the product $\eta^{\times n}$ where

$$\mathbb{R} \xrightarrow{\eta} (-\epsilon(S), \epsilon(S))$$

is the homeomorphism given by $\frac{2\epsilon(S)}{\pi}\arctan(-)$; the second homeomorphism of (3.0.38) is translation by s, that is, $(-) + s$.

3. For $S \in \coprod_{r\geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$, let

$$\mathrm{Emb}^S_{/\Delta^1}(\mathrm{cylr}(f), R^n \times \Delta^1)$$

denote the subspace (with the subspace topology) of $\mathrm{Emb}_{/\Delta^1}(\mathrm{cylr}(f), R^n \times \Delta^1)$ with the compact-open topology, consisting of those embeddings E for which the image of $E_{|S}$ is the given subset $S \subset \mathbb{R}^n$.

Then, for each $S \in \coprod_{r\geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$, fix a continuous map

$$\delta_S : \mathrm{Emb}^S_{/\Delta^1}(\mathrm{cylr}(f), R^n \times \Delta^1) \to (0, 1]$$

such that for each point $\mathrm{cylr}(T \xrightarrow{f} S) \xhookrightarrow{E} \mathbb{R}^n \times \Delta^1$ in the domain,

$$E(f^{-1}(s) \times \delta) \subset \mathrm{Box}_{\epsilon(S)}(s)$$

for each $0 < \delta \leq \delta_S(E)$ and for each $s \in S$.

Note that such a continuous map exists because $\mathrm{Emb}^S_{/\Delta^1}(\mathrm{cylr}(f), R^n \times \Delta^1)$ is Hausdorff.

Next, fix an object S in $\coprod_{r\geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$. We will show that the map

$$\prod_{s\in S}\left(\coprod_{k\geq 0}\mathrm{Conf}_k(T_s\mathbb{R}^n)_{\Sigma_k}\right) \to \mathrm{ev}_0^{-1}(S) \tag{3.0.39}$$

given by

$$(R_s \subset T_s\mathbb{R}^n)_{s\in S} \mapsto \left(\mathrm{cylr}\left(\coprod_{s\in S} D_s(R_s) \xrightarrow{\text{index}} S\right) \xhookrightarrow{E^{\text{straight}}} \mathbb{R}^n \times \mathbf{\Delta}^1\right) \tag{3.0.40}$$

is a homotopy equivalence, where index is the map given by assigning $r \in D_s(R_s)$ to its index, $s \in S$, and E^{straight} is the embedding given by straight-line paths; namely, for each pair $(s \in S, r \in D_s(R_s))$, the embedding E^{straight} restricted to the segment $\mathrm{cylr}(r \mapsto s) \simeq \Delta^1$ into $\mathbb{R}^n$ is given by

$$t \in \Delta^1 \mapsto r(1-t) + st.$$

Consider a map in the other direction

$$\mathrm{ev}_0^{-1}(S) \to \prod_{s\in S}\left(\coprod_{k\geq 0}\mathrm{Conf}_k(T_s\mathbb{R}^n)_{\Sigma_k}\right)$$

defined by

$$(\mathrm{cylr}(T \xrightarrow{f} S) \xhookrightarrow{E} \mathbb{R}^n \times \Delta^1) \mapsto (E_{|T\times\{1\}}(f^{-1}(s)) \subset \mathbb{R}^n = T_s\mathbb{R}^n)_{s\in S}. \tag{3.0.41}$$

The homotopy from the identity of the left-hand side of (3.0.39) to the composite of (3.0.40) followed by (3.0.41) is given by applying the collection of homeomorphisms $\{D_s\}_{s\in S}$.

The homotopy from the identity on the fiber over S to the other composite is given by concatenating the following four homotopies together in the specified order. For each $\mathrm{cylr}(R \xrightarrow{f} S) \overset{E}{\hookrightarrow} \mathbb{R}^n \times \Delta^1$ in $\mathrm{ev}_0^{-1}(S)$,

1. Simultaneously run the paths of E backwards until $t = \delta_S(E)$

2. For each $s \in S$, simultaneously run straight-line paths from each $r \in E(f^{-1}(s) \times \delta_S(E)) \subset \mathbb{R}^n$ to $D_{s \in S}(r)$

3. For each $s \in S$, simultaneously run the paths given by the composite $D_s \circ E|_{\mathrm{cylr}(f^{-1}(s) \to s)}$ from $t = \delta_S(E)$ to $t = 1$

4. For each $r \in R$, simultaneously straighten each path by the Alexandar trick.

□

Lemma 3.0.96. *The classifying space of the fiber of*

$$Fun([1], Exit(Ran^u(\underline{\mathbb{R}}^n))) \xrightarrow{ev_0} Fun([0], Exit(Ran^u(\underline{\mathbb{R}}^n)))$$

over an object $\underline{S} := S_n \to S_{n-1} \to \cdots \to S_1$ *is*

$$\prod_{s \in S_n} Exit(Ran^u(T_s \mathbb{R}^n))^{\sim}$$

the product space indexed by S_n *of the maximal sub* ∞*-groupoid of the exit-path* ∞*-category of the unital Ran space of the tangent space of* $\mathbb{R}^n$ *at* $s \in S_n$*.*

Proof. Fix an object $\underline{S}$ in the base of

$txBox\mathrm{ev}_0$. We will show that there exists a refinement of the stratified space

$$\prod_{s \in S_n} \left(\coprod_{k \geq 0} \mathrm{Conf}_r(\underline{T_s \mathbb{R}^n})_{\Sigma_r} \right)$$

such that there exists an adjunction between the exit-path ∞-category of that refinment and the fiber of ev_0 over $\underline{S}$. First, we define the desired refinement. Similar to ϵ and $D_{s\in S}$ as defined in the proof of Lemma (3.0.95), we define two maps:

1. A continuous map

$$\epsilon : \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))^{\sim} \to \mathbb{R}_{>0}$$

such that for each object $\underline{S} := S_n \to \cdots \to S_1$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))^{\sim}$ and for each pair of distinct points $s \neq s'$ in S_n,

$$\mathrm{Box}_{\epsilon(\underline{S})}(s) \cap \mathrm{Box}_{\epsilon(\underline{S})}(s') = \emptyset.$$

2. For each object $\underline{S} := S_n \to \cdots \to S_1$ in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))^{\sim}$ and for each $s \in S_n$, a homeomorphism

$$D_{s\in S_n} : \mathbb{R}^n \to \mathrm{Box}_{\epsilon(\underline{S})}(s)$$

from n-Euclidean space to the box of size $(2\epsilon(\underline{S}))^n$ centered at s defined exactly as in the proof of (3.0.95).

Consider the sub-stratified space of $\coprod_{k\geq 0} \mathrm{Conf}_k(\underline{\mathbb{R}}^n)_{\Sigma_k}$,

$$\coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\underline{\mathbb{R}}^n)_{\Sigma_k} := \left\{ R \subset \coprod_{s\in S_n} \mathrm{Box}_{\epsilon(\underline{S})}(s) \;\middle|\; R \text{ is finite} \right\}.$$

For each $s \in S_n$, define a continuous map between topological spaces

$$\Phi_s : \coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\mathbb{R}^n)_{\Sigma_k} \to \coprod_{r\geq 0} \mathrm{Conf}_r(T_s\mathbb{R}^n)_{\Sigma_r}$$

by the assignment

$$R \mapsto D_s^{-1}(R \cap \mathrm{Box}_{\epsilon(\underline{S})}(s)).$$

Take the product of the Φ_s over all $s \in S_n$ to define a homeomorphism of topological spaces

$$\Phi : \coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\mathbb{R}^n)_{\Sigma_k} \to \prod_{s\in S_n}\left(\coprod_{r\geq 0}\mathrm{Conf}_r(T_s\mathbb{R}^n)_{\Sigma_r}\right)$$

the inverse of which is given by the assignment

$$(R_s \subset T_s\mathbb{R}^n)_{s\in S_n} \mapsto \coprod_{s\in S_n} D_s^{-1}(R_s).$$

Observe that each stratum of $\coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\underline{\mathbb{R}}^n)_{\Sigma_k}$ is carried by Φ into a stratum of the stratified space $\prod_{s\in S_n}\left(\coprod_{r\geq 0}\mathrm{Conf}_r(\underline{T_s\mathbb{R}^n})_{\Sigma_r}\right)$, whose stratification is given as the product of the stratified spaces $\coprod_{r\geq 0}\mathrm{Conf}_r(\underline{T_s\mathbb{R}^n})_{\Sigma_r}$. As such, the homeomorphism Φ is a refinement of $\coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\underline{\mathbb{R}}^n)_{\Sigma_k}$ to $\prod_{s\in S_n}\left(\coprod_{r\geq 0}\mathrm{Conf}_r(\underline{T_s\mathbb{R}^n})_{\Sigma_r}\right)$.

Consider the functor

$$\mathrm{Exit}\left(\coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\underline{\mathbb{R}}^n)_{\Sigma_k}\right) \overset{\iota}{\hookrightarrow} \mathrm{ev}_0^{-1}(\underline{S})$$

whose value on object $R \subset \coprod_{s\in S_n} \mathrm{Box}_{\epsilon(\underline{S})}(s)$ is

$$\mathrm{cylr}\left(\coprod_{s\in S_n}(R\cap \mathrm{Box}_{\epsilon(\underline{S})}(s)) \to S_n\right) \xrightarrow{E^{\mathrm{straight}}} \mathbb{R}^n \times \Delta^1$$

where E^{straight} is the embedding given by straight-line paths. Further, using Lemma 2.17 from [2], showing that ι is a right adjoint follows directly from Lemma (3.0.77). By 2.1.28 of [28] and Corollary 1.2.7 of [6], therefore,

$$\mathrm{Bev}_0^{-1}(\underline{S}) \simeq \mathrm{BExit}\left(\coprod_{k\geq 0} \mathrm{Conf}_k^{\epsilon(\underline{S})}(\underline{\mathbb{R}}^n)_{\Sigma_k}\right) \simeq \prod_{s\in S_n}\left(\coprod_{r\geq 0}\mathrm{Conf}_r(T_s\mathbb{R}^n)_{\Sigma_r}\right).$$

□

Lemma 3.0.97. *The fiber of the map of spaces*

$$BFun([1], Exit(Ran^u(\underline{\mathbb{R}}^n))) \xrightarrow{Bev_0} BFun([0], Exit(Ran^u(\underline{\mathbb{R}}^n)))$$

over an object $\underline{S} = S_n \to \cdots \to S_1$ *is equivalent to the classifying space of the fiber of* ev_0 *over* $\underline{S}$*,*

$$(Bev_0)^{-1}(\underline{S}) \simeq B(ev_0^{-1}(\underline{S})).$$

Proof. Fix a morphism $\underline{S} \xrightarrow{\alpha} \underline{S}'$ in $\mathrm{Fun}([0], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) \simeq W_n$. Recall the induced monodromy functor α^* defined as the composite

$$\begin{array}{ccc} \mathrm{ev}_0^{-1}(\underline{S}') & \xrightarrow{\alpha^*} & \mathrm{ev}_0^{-1}(\underline{S}) \\ \downarrow & & \uparrow \mu \\ \mathrm{Fun}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))^{\underline{S}'/} & \xrightarrow{-\circ\alpha} & \mathrm{Fun}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))^{\underline{S}/} \end{array} \tag{3.0.42}$$

where μ is right adjoint to inclusion by Lemma (2.20) of ([2]). The diagram obtained after taking classifying spaces of (3.0.109) yields an equivalence of the vertical arrows by 2.1.28 of ([28]).

$$\begin{array}{ccc} \mathrm{Bev}_0^{-1}(\underline{S}') & \xrightarrow{\mathrm{B}\alpha^*} & \mathrm{Bev}_0^{-1}(\underline{S}) \\ \simeq\downarrow & & \uparrow\simeq \\ \mathrm{BFun}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))^{\underline{S}'/} & \xrightarrow{\mathrm{B}(-\circ\alpha)} & \mathrm{BFun}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))^{\underline{S}/}. \end{array} \tag{3.0.43}$$

Thus, we see that $\mathrm{B}(-\circ\alpha)$ is an equivalence if and only if $\mathrm{B}\alpha^*$ is an equivalence. Quillen's Theorem B ([30]) states that if $\mathrm{B}(-\circ\alpha)$ is an equivalence for each morphism α, then the fiber of Bev_0 over $\underline{S}$ is the classifying space of $\mathrm{Fun}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))^{\underline{S}/}$, which, in particular, by (3.0.43), is equivalent to the classifying space of the fiber of ev_0 over $\underline{S}$. Thus, we seek to show that $\mathrm{B}\alpha^*$ is an equivalence; for in so doing, we will prove the desired result.

First, consider the diagram

$$\begin{array}{ccc} \mathrm{Fun}^{W_n}([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\mathrm{frgt}} & \mathrm{mor}(\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))) \\ \downarrow{\scriptstyle \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathrm{Fun}^{W_n}([0], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\simeq} & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))^{\simeq} \end{array} \tag{3.0.44}$$

in which each vertical arrow is a Cartesian fibration. By Observation (3.0.109),

$$\begin{array}{ccc} \mathrm{ev}_0^{-1}(\underline{S}') & \xrightarrow{\alpha^*} & \mathrm{ev}_0^{-1}(\underline{S}) \\ \downarrow{\scriptstyle \mathrm{frgt}} & & \downarrow{\scriptstyle \mathrm{frgt}} \\ \mathrm{ev}_0^{-1}(S'_n) & \xrightarrow{\mathrm{frgt}(\alpha)^*} & \mathrm{ev}_0^{-1}(S_n). \end{array} \tag{3.0.45}$$

Observe that $\mathrm{frgt}(\alpha)^*$ is equivalence precisely because the data of α at n is a bijection

$$S_n \xrightarrow{\alpha_n} S'_n.$$

We apply the universal property of localization to the canonical localization $\mathrm{ev}_0^{-1}(\underline{S}) \to \mathrm{Bev}_0^{-1}(\underline{S})$ to obtain

$$\begin{array}{ccc} \mathrm{ev}_0^{-1}(\underline{S}) & \xrightarrow{\mathrm{frgt}} & \mathrm{ev}_0^{-1}(S_n) \simeq \prod_{s \in S_n} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(T_s\mathbb{R}^n)_{\Sigma_r})^{\sim} \\ \downarrow & \overset{\exists !}{\nearrow} & \\ \mathrm{Bev}_0^{-1}(\underline{S}) \simeq \prod_{s \in S_n} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(T_s\mathbb{R}^n)_{\Sigma_r})^{\sim} & & \end{array} \tag{3.0.46}$$

and observe that such a filler must be an equivalence. We paste diagram (3.0.45) and diagram (3.0.46) for $\underline{S}$ and $\underline{S}'$ together to see that $\mathrm{B}\alpha^*$ is an equivalence:

$$\begin{array}{ccccc}
\mathrm{ev}_0^{-1}(\underline{S}') & & \xrightarrow{\alpha^*} & & \mathrm{ev}_0^{-1}(\underline{S}) \\
 & \searrow^{\mathrm{loc}} & & \swarrow^{\mathrm{loc}} & \\
\downarrow^{\mathrm{frgt}} & \mathrm{Bev}_0^{-1}(\underline{S}') & \xrightarrow{\mathrm{B}\alpha^*} & \mathrm{Bev}_0^{-1}(\underline{S}) & \downarrow^{\mathrm{frgt}} \\
 & \swarrow_{\simeq} & & \searrow^{\simeq} & \\
\mathrm{ev}_0^{-1}(S'_n) & & \xrightarrow{\simeq} & & \mathrm{ev}_0^{-1}(S_n).
\end{array}$$

□

We are now equipped to prove Lemma (3.0.71) for the case $p = 1$.

Lemma 3.0.98. *There is an equivalence of spaces*

$$BFun([1], Exit(Ran^u(\underline{\mathbb{R}}^n))) \simeq mor(Exit(Ran^u(\mathbb{R}^n)))$$

induced by the forgetful functor $Exit(Ran^u(\underline{\mathbb{R}}^n)) \to Exit(Ran^u(\mathbb{R}^n))$ *(3.0.53).*

Proof. We will show that the long exact sequences in homotopy induced by the natural diagram of fibrations

$$\begin{array}{ccc}
\mathrm{BFun}^W([1], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\mathrm{frgt}} & \mathrm{mor}(\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))) \\
\downarrow^{\mathrm{Bev}_0} & & \downarrow^{\mathrm{ev}_0} \\
\mathrm{BFun}^W([0], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\simeq} & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))^{\sim}
\end{array} \tag{3.0.47}$$

induces a weak equivalence between the total spaces. Indeed, Lemma (3.0.74) yields the equivalence of the bottom horizontal arrow and Lemmas (3.0.96), (3.0.97) and (3.0.101) yield an equivalence between fibers. □

Proving Lemma (3.0.101) Next, we prove Lemma (3.0.101) which states that $\mathrm{BFun}^{W_n}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))$ is a complete Segal space. This section is technical, rooted in category theory and thus, we remind the reader that our goal is to apply Theorem (3.0.68) to prove Lemma (3.0.63); this section checks that the hypothesis of (3.0.68) is satsified.

Lemma 3.0.99. *Given a pullback of ∞-categories*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ \downarrow{\scriptstyle \pi} & \lrcorner & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{G} & B' \end{array}$$

in which π' is a Cartesian fibration, if π' satsifies Quillen's Theorem B, then so does π.

Proof. Let $b \xrightarrow{f} b'$ be a morphism in B. The definition of the induced monodromy functor together with Observation (3.0.109) yields

$$\begin{array}{ccc} \mathcal{E}^{b'}{}_{/} & \xrightarrow{-\circ f} & \mathcal{E}^{b}{}_{/} \\ \uparrow & & \uparrow \\ \pi^{-1}(b') & \xrightarrow{f^*} & \pi^{-1}(b) \\ \simeq\downarrow & & \downarrow\simeq \\ \pi'^{-1}(G(b')) & \xrightarrow{G(f)^*} & \pi'^{-1}(G(b)) \\ \downarrow & & \downarrow \\ \mathcal{E}'^{G(b')}{}_{/} & \xrightarrow{-\circ G(f)} & \mathcal{E}'^{G(b)}{}_{/}. \end{array} \tag{3.0.48}$$

By taking the classifying space of (3), we obtain the desired result. Indeed, π' satsifying Quillen's Theorem B implies $\mathrm{B}(- \circ G(f))$ is a equivalence and thus, each horizontal arrow resulting between classifying spaces is an equivalence, which in particular means $\mathrm{B}(-\circ f)$ is an equivalence:

$$\begin{array}{ccc}
\mathrm{B}\mathcal{E}^{b'}{}_{/} & \xrightarrow{\mathrm{B}(-\circ f)} & \mathrm{B}\mathcal{E}^{b}{}_{/} \\
\simeq\uparrow & & \simeq\uparrow \\
\mathrm{B}\pi^{-1}(b') & \xrightarrow{\mathrm{B}f^{*}} & \mathrm{B}\pi^{-1}(b) \\
\simeq\downarrow & & \downarrow\simeq \\
\mathrm{B}\pi'^{-1}(G(b')) & \xrightarrow{\mathrm{B}G(f)^{*}} & \mathrm{B}\pi'^{-1}(G(b)) \\
\downarrow\simeq & & \downarrow\simeq \\
\mathrm{B}\mathcal{E}'^{G(b')}{}_{/} & \xrightarrow{\mathrm{B}(-\circ G(f))} & \mathrm{B}\mathcal{E}'^{G(b)}{}_{/}.
\end{array}$$

∎

Observation 3.0.100. The following diagram of ∞-categories is a pullback:

$$\begin{array}{ccc}
\mathrm{Fun}^{W_n}([p], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\tau} & \mathrm{Fun}^{W_n}(\{1-p<p\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) \\
\sigma\downarrow & \lrcorner & \downarrow s \\
\mathrm{Fun}^{W_n}(\{0<\cdots<p-1\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{t} & \mathrm{Fun}^{W_n}(\{p-1\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))).
\end{array} \tag{3.0.49}$$

Indeed, for an ∞-category $\mathcal{C}$, $\mathrm{Fun}([\bullet], \mathcal{C})$ satsifies the Segal condition, i.e., for each $p \geq 2$, the diagram obtained by replacing $\mathrm{Fun}^{W_n}([p], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))$ with $\mathrm{Fun}([p], \mathcal{C})$ in (3.0.53) is pullback. Using this, it is straightforward to show (3.0.53) is pullback.

In the next lemma, we verify that the hypothesis of Theorem (3.0.68) is satsified.

Lemma 3.0.101. *The classifying space* $BFun^{W}([\bullet], Exit(Ran^{u}(\underline{\mathbb{R}}^n)))$ *is a complete Segal space.*

Proof. First, we will show that $\mathrm{BFun}^{W}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ satsifies the Segal condition. Consider the diagram of spaces obtained by taking the classifying spaces of diagram (3.0.53)

$$\begin{array}{ccc} \mathrm{BFun}^{W_n}([p], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\mathrm{B}\tau} & \mathrm{BFun}^{W_n}(\{p-1<p\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) \\ \downarrow{\scriptstyle \mathrm{B}\sigma} & & \downarrow{\scriptstyle \mathrm{B}s} \\ \mathrm{BFun}^{W_n}(\{0<\cdots<p-1\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) & \xrightarrow{\mathrm{B}t} & \mathrm{BFun}^{W}(\{p-1\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))). \end{array} \tag{3.0.50}$$

To show that this diagram is a pullback, we will show that the map induced between fibers of (3.0.50) is an equivalence. By Observation (3.0.89), the functor s in (3.0.53) is a Cartesian fibration. Further, in the proof of Lemma (3.0.97), we showed that s satsifies Quillen's Theorem B. Thus, diagram (3.0.53) satsifies the hypothesis' of Lemma (3.0.99) and we identify the fibers of $\mathrm{B}\sigma$ and $\mathrm{B}s$ over the objects

$$\underline{S}_0 \to \cdots \to \underline{S}_{p-1} \text{ in } \mathrm{BFun}^{W_n}(\{0<\cdots<p-1\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))$$

and

$$\underline{S}_{p-1} \text{ in } \mathrm{BFun}^{W}(\{p-1\}, \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))$$

respectively, as the classifying spaces of the fibers of σ and s over $\underline{S}_0 \to \cdots \to \underline{S}_{p-1}$ and $\underline{S}_{p-1}$, respectively:

$$(\mathrm{B}\sigma)^{-1}(\underline{S}_0 \to \cdots \to \underline{S}_{p-1}) \simeq \mathrm{B}(\sigma^{-1}(\underline{S}_0 \to \cdots \to \underline{S}_{p-1}))$$

and

$$(\mathrm{B}s)^{-1}(\underline{S}_{p-1}) \simeq \mathrm{B}(s^{-1}(\underline{S}_{p-1})).$$

Therefore, because diagram (3.0.53) being a pullback implies an equivalence between fibers induced by τ

$$\tau_{|} : \sigma^{-1}(\underline{S}_0 \to \cdots \to \underline{S}_{p-1}) \xrightarrow{\simeq} s^{-1}(\underline{S}_{p-1})$$

there results an equivalence between fibers of (3.0.50) given by $\mathrm{B}\tau_{|}$

$$(\mathrm{B}\sigma)^{-1}(\underline{S}_0 \to \cdots \to \underline{S}_{p-1}) \simeq \mathrm{B}\sigma^{-1}(\underline{S}_0 \to \cdots \to \underline{S}_{p-1}) \xrightarrow{\simeq} \mathrm{B}s^{-1}(\underline{S}_{p-1}) \simeq (\mathrm{B}s)^{-1}(\underline{S}_{p-1})$$

which verifies that diagram (3.0.50) is a pullback.

Then, Lemma (3.0.71) extends to an equivalence of spaces

$$\mathrm{BFun}([p], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}([p], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)))$$

for each $p \geq 0$, since $\mathrm{BFun}^W([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ satsifying the Segal condition means that its values on [0] and [1] determine all of its higher $[p]$ values. This, in particular, implies that $\mathrm{BFun}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)))$ is a complete Segal space, since $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)))$ is a complete Segal space precisely because $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ is a complete Segal space.

□

Finally, we are fully equipped to apply Theorem (3.0.68) to prove Lemma (3.0.63), which we have set up to follow immediately.

Proof. [Lemma (3.0.63)] In the proof of the previous lemma (3.0.101), we showed

$$\mathrm{BFun}^{W_n}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))) \tag{3.0.51}$$

which, in particular means that the hypothesis of Theorem (3.0.68) is satsified. Thus, by (3.0.68),

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))[W_n^{-1}] \simeq \mathrm{BFun}^{W_n}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))).$$

Then, by the equivalence (3.0.51), we have an equivalence of simplicial spaces

$$\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))[W_n^{-1}] \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)))$$

which establishes that $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ localizes on W_n to $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$.

Note that this localization is given by the forgetful functor from $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ to $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ because our identification of $\mathrm{BFun}^{W_n}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ with $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)))$ was induced by the forgetful functor (recall the $p = 1$ case (3.0.98)).

Lastly, we will show this localization is over $\mathrm{Fin}^{\mathrm{op}}$. In (3.0.53), we observed that the forgetful functor from $\mathrm{Exit}(\mathrm{Ran}^{u}(\underline{\mathbb{R}}^n))$ to $\mathrm{Exit}(\mathrm{Ran}^{u}(\mathbb{R}^n))$ is naturally over $\mathrm{Fin}^{\mathrm{op}}$ by just remembering the data of underlying sets at the $\mathbb{R}^n$ level. Then, by the universal property of localization, we have:

$$\begin{array}{ccccc} \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) & \xrightarrow{\mathrm{frgt}} & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)) & \simeq & \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))[W_n^{-1}] \\ & \searrow^{\phi_n} & \downarrow{\scriptstyle \phi} & \swarrow^{\exists !} & \\ & & \mathrm{Fin}^{\mathrm{op}} & & \end{array}$$

The unique existence of such a filler is guaranteed because each morphism in W_n gets carried to isomorphisms in $\mathrm{Fin}^{\mathrm{op}}$ under ϕ_n. Thus, we see that the forgetful functor $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \to \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n))$ yields a localization over $\mathrm{Fin}^{\mathrm{op}}$.

□

To summarize, in Lemma (3.0.57) we showed an equivalence of ∞-categories, from the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ and $\mathbf{\Theta}_n^{\mathrm{act}}$. Then, in Lemma (3.0.63) we showed that the exit-path ∞-category of the fine unital Ran space of $\mathbb{R}^n$ localizes to the exit-path ∞-category of the Ran space of $\mathbb{R}^n$. These two lemmas together imply the main result of the chapter Theorem (3.0.46), namely that $\mathbf{\Theta}_n^{\mathrm{act}}$ localizes to the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$.

A localization of $\mathbf{\Theta}_n^{\mathrm{Exit}}$ to $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$

In this section, we establish a consequence of Theorem (3.0.46). Namely, in Corollary (3.0.103) we show that a subcategory of $\mathbf{\Theta}_n^{\mathrm{act}}$ localizes to $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$ the exit-path ∞-category of the stratified space $\mathrm{Ran}(\mathbb{R}^n)$. This subcategory is defined next. Heuristically, this subcategory consists of healthy trees as its objects and morphisms that induce surjections between the sets of leaves.

Definition 3.0.102. The category $\mathbf{\Theta}_n^{\mathrm{Exit}}$ is the subcategory of $\mathbf{\Theta}_n^{\mathrm{act}}$ defined as the pullback

$$\begin{array}{ccc} \mathbf{\Theta}_n^{\mathrm{Exit}} & \hookrightarrow & \mathbf{\Theta}_n^{\mathrm{act}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ \mathrm{Fun}(\{1 < \cdots < n\}, (\mathrm{Fin}_{\neq\emptyset}^{\mathrm{surj}})^{\mathrm{op}}) & \hookrightarrow & \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}}) \end{array}$$

where we recall the functor τ from Observation (3.0.45) (defined by the truncation functor tr_i and γ_i).

Corollary 3.0.103. *There is a localization*

$$\mathbf{\Theta}_n^{Exit} \to Exit(Ran(\mathbb{R}^n))$$

over $(Fin^{surj})^{op}$ *from the category* $\mathbf{\Theta}_n^{Exit}$ *to the exit-path* ∞*-category of the Ran space of* $\mathbb{R}^n$.

First, we will prove two lemmas that we will use in the proof of (3.0.103). The first lemma, (3.0.104), mainly uses previous work of showing the adjunction between W_n^{healthy} and W_n in Lemma (3.0.77) to show that taking the classifying space of the inclusion of ∞-categories $\text{Fun}^{W_n^{\text{healthy}}}([p], \mathbf{\Theta}_n^{\text{act,healthy}}) \hookrightarrow \text{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\text{act}})$ induces an equivalence of spaces.

Lemma 3.0.104. *For each $p \geq 0$, the inclusion functor between ∞-categories*

$$Fun^{W_n^{healthy}}([p], \mathbf{\Theta}_n^{act,healthy}) \hookrightarrow Fun^{W_n}([p], \mathbf{\Theta}_n^{act})$$

induces an between their classifying spaces

$$BFun^{W_n^{healthy}}([p], \mathbf{\Theta}_n^{act,healthy}) \xrightarrow{\simeq} BFun^{W_n}([p], \mathbf{\Theta}_n^{act}).$$

Proof. First, observe that we can describe the subcategory W_n^{healthy} of W_n as the following pullback of categories over $\mathbf{\Theta}_n^{\text{act,healthy}}$:

$$\begin{array}{ccc} W_n^{\text{healthy}} & \hookrightarrow & W_n \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{\Theta}_n^{\text{act,healthy}} & \hookrightarrow & \mathbf{\Theta}_n^{\text{act}} \end{array}$$

where we recall that W^n consists of all the same objects as $\mathbf{\Theta}_n^{\text{act}}$ and all those morphisms that induce bijections on the sets of leaves, and W_n^{healthy} is the full subcategory consisting of only those trees that are healthy.

In Lemma (3.0.77), we showed that the inclusion functor $W_n^{\text{healthy}} \hookrightarrow W_n$ is a right adjoint. The reader may observe that no where in the proof did we use that the morphisms of W_n^h and W_n induce bijection between their sets of leaves. Thus, Lemma (3.0.77) immediately extends to an adjunction between $\mathbf{\Theta}_n^{\text{act,healthy}}$

and $\mathbf{\Theta}_n^{\mathrm{act}}$ whose right adjoint is given by inclusion. Further, observe that the unit transformation of this adjunction is given by morphisms in W_n. Indeed, for each tree $T \in \mathbf{\Theta}_n^{\mathrm{act}}$, the morphism assigned to T by the unit is $T \xrightarrow{\epsilon_T} P_n(T)$, which, in particular, induces a bijection on the leaves, and is thus in W_n. In identifying that the unit of the right adjoint $\mathbf{\Theta}_n^{\mathrm{act,healthy}} \hookrightarrow \mathbf{\Theta}_n^{\mathrm{act}}$ is given by morphisms in W_n, we may extend this adjunction to an adjunction between $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}})$ and $\mathrm{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act}})$ whose right adjoint is inclusion.

Recall that Corollary 2.1.28 in [28] states that the classifying space of an adjunction is an equivalence of spaces. Thus, upon taking the classifying space of the right adjoint $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}}) \hookrightarrow \mathrm{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act}})$, there results the desired equivalence of between classifyinging spaces. $\square$

For the next lemma, we need the following definitions, largely taken from [5].

Definition 3.0.105.

- A *monomorphism* of spaces is an inclusion of path components, i.e., an injection induced between connected components and an isomorphism induced between all higher homotopy groups, $\pi_{>0}(-)$.

- A *monomorphism* of ∞-categories is functor whose induced map between spaces of objects is a monomorphism, and whose induced map between spaces of morphisms is a monomorphism.

- A *monomorphism* of simplicial spaces is a functor for which the induced map between spaces for each $[p]$ is a monomorphism of spaces.

Definition 3.0.106. A functor $\mathcal{C} \to \mathcal{D}$ between ∞-categories is an *inclusion of a cofactor* if there is an ∞-category $\mathcal{E}$ and an equivalence between ∞-categories under

$\mathcal{C}$:

$$\mathcal{C}\coprod\mathcal{E} \;\cong\; \mathcal{D}\;.$$

Lemma 3.0.107. *A functor $\mathcal{C}\xrightarrow{F}\mathcal{D}$ is an inclusion of a cofactor if and only if F is a monomorphism and for each solid commutative square*

$$\begin{array}{ccc} [0] & \longrightarrow & \mathcal{C} \\ {\scriptstyle\nu}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle F} \\ [1] & \longrightarrow & \mathcal{D} \end{array} \tag{3.0.52}$$

for either $\nu := \langle 0\rangle$ or $\nu := \langle 1\rangle$, there exists a filler.

Proof. First, notice that if F is a monomorphism and diagram (3.0.52) is satsified (with the two possible lifts), then $\mathcal{C}\xrightarrow{F}\mathcal{D}$ is fully faithful. Consider the full ∞-subcategory $\mathcal{E}\subset\mathcal{D}$ consisting of those objects that are not isomorphic to objects in the image of $\mathcal{C}\to\mathcal{D}$. Consider the canonical functor

$$\mathcal{C}\coprod\mathcal{E}\longrightarrow\mathcal{D}\;,$$

which is canonically under $\mathcal{C}$. By design, this functor is essentially surjective, and fully faithful. This established the implication that F being a monomorphism and satsifying diagram (3.0.52) implies $\mathcal{C}\xrightarrow{F}\mathcal{D}$ is an inclusion of a cofactor.

We now show the converse. Suppose there is an ∞-category $\mathcal{E}$ together with an equivalence $\mathcal{C}\coprod\mathcal{E}\simeq\mathcal{D}$ under $\mathcal{C}$. Consider a solid diagram

$$\begin{array}{ccc} [0] & \longrightarrow & \mathcal{C} \\ {\scriptstyle\nu}\downarrow & \overset{\exists}{\nearrow} & \downarrow \\ [1] & \longrightarrow & \mathcal{C}\coprod\mathcal{E} \end{array}$$

The functor $[0]\xrightarrow{\nu}[1]$ has the feature that every object in $[1]$ admits a morphism to or from an object in the image of ν. It follows that there is a unique filler, as desired.

$\square$

The next definition introduces the subcategory of $\mathbf{\Theta}_n^{\mathrm{Exit}}$ upon which we localize to obtain $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$. We impliment a slight abuse of notation as we will denote this subcategory W_n^{healthy}, which is distinct from the subcategory W_n^{healthy} of $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$ (Definition (3.0.76) after passing through the equivalence $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \simeq \mathbf{\Theta}_n^{\mathrm{act,healthy}}$ of Lemma (3.0.57)) consisting of all those healthy trees and morphisms amongst them that induce bijections on their leaves. We will always clarify which category W_n^{healthy} we mean by indicating contextually which category it is a subcategory of, namely, $W_n^{\mathrm{healthy}} \subset \mathbf{\Theta}_n^{\mathrm{Exit}}$ or $W_n^{\mathrm{healthy}} \subset \mathbf{\Theta}_n^{\mathrm{act,healthy}}$.

Definition 3.0.108. The subcategory W_n^{healthy} of $\mathbf{\Theta}_n^{\mathrm{Exit}}$ is defined to be the pullback

$$\begin{array}{ccc} W_n^{\mathrm{healthy}} & \hookrightarrow & W_n^{\mathrm{healthy}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{\Theta}_n^{\mathrm{Exit}} & \hookrightarrow & \mathbf{\Theta}_n^{\mathrm{act,healthy}} \end{array}$$

of categories.

Heuristically then, $W_n^{\mathrm{healthy}} \subset \mathbf{\Theta}_n^{\mathrm{Exit}}$ differs from $W_n^{\mathrm{healthy}} \subset \mathbf{\Theta}_n^{\mathrm{act,healthy}}$ only in that it does not have the empty tree as an object.

The inclusion of the subcategory $\mathbf{\Theta}_n^{\mathrm{Exit}} \hookrightarrow \mathbf{\Theta}_n^{\mathrm{act,healthy}}$ together with the induced inclusion of the respective subcategories $W_n^{\mathrm{healthy}} \hookrightarrow W_n^h$ guarantees that for each $p \geq 0$, the induced map $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \hookrightarrow \mathrm{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}})$ is also an inclusion of a ∞-subcategory. The next lemma articulates a desired trait of the induced map between classifying spaces of this inclusion.

Lemma 3.0.109. *For each $p \geq 0$, the inclusion functor*

$$Fun^{W_n^{healthy}}([p], \mathbf{\Theta}_n^{Exit}) \hookrightarrow Fun^{W_n}([p], \mathbf{\Theta}_n^{act,healthy})$$

induces a monomorphism between classifying spaces

$$BFun^{W_n^{healthy}}([p], \mathbf{\Theta}_n^{Exit}) \hookrightarrow BFun^{W_n}([p], \mathbf{\Theta}_n^{act,healthy}).$$

Proof. First, we will vertify that the functor

$$\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \hookrightarrow \mathrm{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}})$$

is an inclusion of a cofactor; that is, we will show that this functor is a monomorphism and satsifies diagram (3.0.52).

First, note that because $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \hookrightarrow \mathrm{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}})$ is an inclusion of ∞-categories, it is in particular a monomorphism.

Similar to Observation (3.0.100), it is straightforward to verify that for each $p \geq 0$ the following diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) & \longrightarrow & \mathrm{Fun}^{W_n^{\mathrm{healthy}}}(\{1-p<p\}, \mathbf{\Theta}_n^{\mathrm{Exit}}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}^{W_n^{\mathrm{healthy}}}(\{0<\cdots<p-1\}, \mathbf{\Theta}_n^{\mathrm{Exit}}) & \longrightarrow & \mathrm{Fun}^{W_n^{\mathrm{healthy}}}(\{p-1\}, \mathbf{\Theta}_n^{\mathrm{Exit}}). \end{array} \tag{3.0.53}$$

is pullback, as is the diagram obtained by just replacing $\mathbf{\Theta}_n^{\mathrm{Exit}}$ with $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$. Thus, to show that diagram (3.0.52) for our situation, namely

$$\begin{array}{ccc} [0] & \longrightarrow & \mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \\ {\scriptstyle \nu}\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow \\ [1] & \longrightarrow & \mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}}) \end{array} \tag{3.0.54}$$

is satsified for all $p \geq 0$, it suffices to show for the cases for $p = 0, 1$. Indeed, it is straightforward to verify that the cases $p = 0, 1$ imply each $p \geq 0$ case upon applying the universal property of pullback from diagram (3.0.53) and the diagram obtained by replacing $\mathbf{\Theta}_n^{\mathrm{Exit}}$ with $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$ in (3.0.53).

Both cases, $p = 0, 1$, come down to the following observation: Each morphism in W_n^{healthy} between objects in $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$ is a morphism in $\mathbf{\Theta}_n^{\mathrm{Exit}}$. The root reason for this is that surjections enjoy the '2 out of 3' property; that is, for any commutative triangle of morphisms among sets in which two of the morphisms are surjections, the third map is necessairly a surjection as well. For our situation, any morphism $T \xrightarrow{f} T'$ in W_n^{healthy} yields a bijection between the sets of leaves, $\gamma_n(f) : \gamma_n(T') \xrightarrow{\cong} \gamma_n(T)$. For any $1 \leq i \leq n-1$, in applying the natural transformation ϵ from Observation (3.0.43), whose value on T is the natural map between sets of leaves $\gamma_n(T) \xrightarrow{\epsilon_T} \gamma_i(\mathrm{tr}_i(T))$ induced by the structure of T, we obtain the following diagram among sets:

$$\begin{array}{ccc} \gamma_n(T') & \xrightarrow{\cong} & \gamma_n(T) \\ {\scriptstyle \epsilon_{T'}}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_T} \\ \gamma_i(\mathrm{tr}_i(T')) & \xrightarrow{\gamma_i(\mathrm{tr}_i(f))} & \gamma_i(\mathrm{tr}_i(T)). \end{array}$$

Observe that because T and T' are healthy trees, both ϵ_T and $\epsilon_{T'}$ are surjections. Then, by the '2 out of 3' property, $\gamma_i(\mathrm{tr}_i(T')) \xrightarrow{\gamma_i(\mathrm{tr}_i(f))} \gamma_i(\mathrm{tr}_i(T))$ is a surjection. Such a surjection at each i guarantees that the image of f under the functor $\mathbf{\Theta}_n^{\mathrm{act}} \rightarrow \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}})$ lands in $\mathrm{Fun}(\{1 < \cdots < n\}, (\mathrm{Fin}_{\neq\emptyset}^{\mathrm{surj}})^{\mathrm{op}})$ and is thus a morphism in $\mathbf{\Theta}_n^{\mathrm{Exit}}$. Using this observation, we now verify diagram (3.0.54) for the cases $p = 0, 1$:

For $p = 0$, the desired lift in

$$\begin{array}{ccc} [0] & \xrightarrow{\langle T \rangle} & W_n^{\mathrm{healthy}} \\ {\scriptstyle \langle 0 \rangle}\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow \\ [1] & \xrightarrow[\langle T \xrightarrow{f} T' \rangle]{} & W_n^{\mathrm{healthy}} \end{array}$$

is given by selecting out the morphism $T \xrightarrow{f} T'$, which is in $\mathbf{\Theta}_n^{\mathrm{Exit}}$ because each morphism in W_n^{healthy} between objects in $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$ is a morphism in $\mathbf{\Theta}_n^{\mathrm{Exit}}$, as previously discussed. A similar arguement yields a lift for the square whose downward arrow on the left is $\langle 1 \rangle$.

For $p = 1$, the desired lift in

$$\begin{array}{ccc} [0] & \longrightarrow & \mathrm{Fun}^{W_n^{\mathrm{healthy}}}([1], \mathbf{\Theta}_n^{\mathrm{Exit}}) \\ {\scriptstyle \langle 0 \rangle}\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow \\ [1] & \xrightarrow{\alpha} & \mathrm{Fun}^{W_n^{\mathrm{healthy}}}([1], \mathbf{\Theta}_n^{\mathrm{act,healthy}}) \end{array}$$

is again given by α, which is straightforward to check upon applying the fact discussed above, that each morphism in W_n^{healthy} between objects in $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$ is a morphism in $\mathbf{\Theta}_n^{\mathrm{Exit}}$. A similar arguement applies for the square whose downward arrow on the left is $\langle 1 \rangle$.

Thus, $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \hookrightarrow \mathrm{Fun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}})$ is an inclusion of a cofactor, meaning the target is equivalent to a coproduct, one term of which is the source. Thus, because the classifying space respects colimits, the induced map between classifying spaces $\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \hookrightarrow \mathrm{BFun}^{W_n}([p], \mathbf{\Theta}_n^{\mathrm{act,healthy}})$ is, in particular, still a monomorphism. $\square$

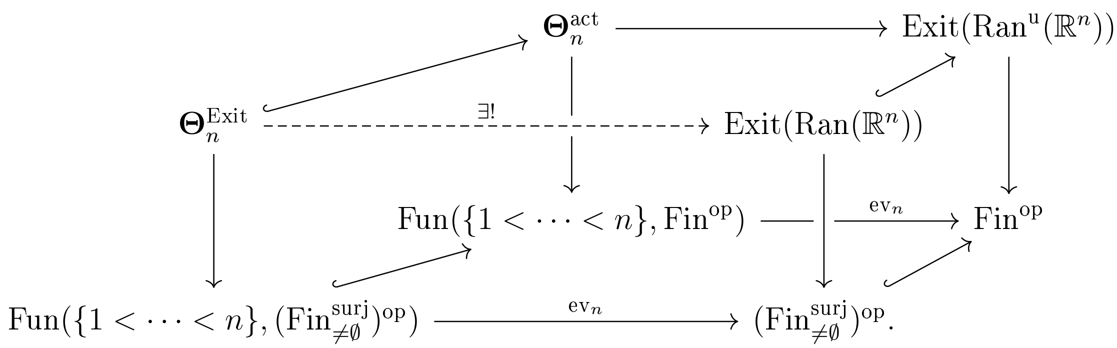
Finally, we are equipped to prove Corollary (3.0.103) by showing that the category $\mathbf{\Theta}_n^{\mathrm{Exit}}$ localizes about W_n^{healthy} to the exit-path ∞-category of the Ran space of $\mathbb{R}^n$.

Proof. (3.0.103) Similar to the proof of Lemma (3.0.63), we will use Theorem (3.0.68) that if the classifying space of $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}})$ is a complete Segal space, then it is equivalent to the localization $\mathbf{\Theta}_n^{\mathrm{Exit}}$ about W_n^{healthy}. First, we will show that there is an equivalence of simplicial spaces from the classifying space of $\mathrm{Fun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}})$ to $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$. To do this, we use the following diagram of simplicial spaces

$$\begin{array}{ccccc} \mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}}) & \hookrightarrow & \mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{act,healthy}}) & \xrightarrow{\simeq} & \mathrm{BFun}^{W_n}([\bullet], \mathbf{\Theta}_n^{\mathrm{act}}) \\ \downarrow & & & \swarrow \simeq & \\ \mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))) & \hookrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))). & & \end{array} \tag{3.0.55}$$

We explain each functor of the diagram next: The top horizontal arrow on the left is a monomorphism by Lemma (3.0.109), wherein we showed a monomophism for each space given by evaluation at $[p]$. We showed the top horizontal arrow on the left to be an equivalence in Lemma (3.0.104). Because $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$ is a ∞-subcategory of $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$, the bottom horizontal arrow is a monomorphism. We showed the downward equivalence in the proof of Lemma (3.0.101).

Lastly, to define the induced downward functor on the left of diagram (3.0.55), first recall Observation (3.0.12), where we witnessed that the exit-path ∞-category of the Ran space of $\mathbb{R}^n$ is naturall a ∞-subcategory of the exit-path ∞-category of the unital Ran space of $\mathbb{R}^n$, given as a pullback over surjective finite sets. Then, the induced downward arrow in (3.0.55) from $\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}})$ to $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$ is induced by the unique (up to a contractible space of choices) functor given by the universal property of pullback in the following diagram of ∞-categories:



(3.0.56)

where we note that the top, back horizontal functor is the localization from Theorem (3.0.46), and the square on the right wall is the pullback that we just observed in diagram (3.0.1). Also note that we apply the universal property of the classifying space to ensure that the unique functor in diagram (3.0.56) from $\mathbf{\Theta}_n^{\mathrm{Exit}}$ to $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$ induces a functor from $\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}})$ to $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$ in diagram (3.0.55).

We wish to show that this induced functor

$$\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}}) \xrightarrow{\kappa} \mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$$

in diagram (3.0.55) is an equivalence. First, observe that monomorphisms enjoy the '2 out of 3' property (by 5.4 in [5]) and thus, κ is a monomorphism.

All that remains to be shown then, in showing that κ is an equivalence of simplicial spaces, is to show that κ induces a surjection on path components between each space given by the value on $[p]$,

$$\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([p], \mathbf{\Theta}_n^{\mathrm{Exit}}) \xrightarrow{\kappa} \mathrm{Hom}_{\mathrm{Cat}_\infty}([p], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))).$$

Recall in the proof of Lemma (3.0.101) where we show that $\mathrm{BFun}^{W_n}([\bullet], \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\mathbb{R}^n)))$ satsifies the Segal condition. Observe that the same arguement applies to $\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \mathbf{\Theta}_n^{\mathrm{Exit}})$ to show that it, too, satsifies the Segal condition. Thus, to show κ is a surjection on path components, it suffices to show it for the cases $p = 0, 1$.

For the case $p = 0$, we wish to show that the map of spaces

$$\mathrm{B}W_n^{\mathrm{healthy}} \xrightarrow{\kappa} \coprod_{r \geq 1} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$$

is a surjection on path components. Indeed, this follows by Corollary (3.0.74), wherein we showed a homotopy equivalence between the classifying space of the subcategory W_n^{healthy} of $\mathbf{\Theta}_n^{\mathrm{act,healthy}}$ and the coproduct $\coprod_{r \geq 0} \mathrm{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$; the only difference here is that $r = 0$ is allowed.

For the case $p = 1$, we wish to show that the map of spaces

$$\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([1], \mathbf{\Theta}_n^{\mathrm{Exit}}) \xrightarrow{\kappa} \mathrm{mor}(\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$$

is a surjection on path components. Let $\mathrm{cylr}(S \xrightarrow{f} T) \overset{E}{\hookrightarrow} \mathbb{R}^n \times \Delta^1$ be a point in the target. Recall that by diagram (3.0.56), κ is determined by $\mathbf{\Theta}_n^{\mathrm{Exit}} \hookrightarrow \mathbf{\Theta}_n^{\mathrm{act}} \simeq \mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ over $\mathrm{Fun}(\{1 < \cdots < n\}, (\mathrm{Fin}_{\neq\emptyset}^{\mathrm{surj}})^{\mathrm{op}})$. Thus, we will identify a point in the fiber over E under κ by identifying a morphism in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ over $(\mathrm{Fin}^{\mathrm{surj}})^{\mathrm{op}}$. Such a morphism is precisely obtained by naming the projection data of E, namely:

$$\begin{array}{ccc}
\operatorname{cylr}(S \xrightarrow{f} T) & \xhookrightarrow{\quad E \quad} & \mathbb{R}^n \times \Delta^1 \\
{\scriptstyle \mathrm{pr}_{<n}}\big\downarrow & & \big\downarrow \\
\operatorname{cylr}(\mathrm{pr}_{<n}(S) \to \mathrm{pr}_{<n}(T)) & \hookrightarrow & \mathbb{R}^{n-1} \times \Delta^1 \\
\big\downarrow & & \big\downarrow \\
\vdots & & \vdots \\
\big\downarrow & & \big\downarrow \\
\operatorname{cylr}(\mathrm{pr}_1(S) \to \mathrm{pr}_1(T)) & \hookrightarrow & \mathbb{R} \times \Delta^1
\end{array}$$

the value of which under the functor $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n)) \to \mathrm{Fun}(\{1 < \cdots < n\}, \mathrm{Fin}^{\mathrm{op}}$ factors through $\mathrm{Fun}(\{1 < \cdots < n\}, (\mathrm{Fin}^{\mathrm{surj}})^{\mathrm{op}})$ precisely because $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$ is naturally over $\mathrm{Fun}(\{1 < \cdots < n\}, (\mathrm{Fin}^{\mathrm{surj}})^{\mathrm{op}})$ as observed from its pullback description in diagram (3.0.1). Thus, this morphism in $\mathrm{Exit}(\mathrm{Ran}^{\mathrm{u}}(\underline{\mathbb{R}}^n))$ defines a morphism in $\boldsymbol{\Theta}_n^{\mathrm{Exit}}$ whose value under κ is E. Thus, κ for the case $p = 1$ is a surjection on path components which, as previously argued, implies that κ is an equivalence of simplicial spaces. The target of κ, $\mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$ is, in particular, a complete Segal space, and hence, $\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \boldsymbol{\Theta}_n^{\mathrm{Exit}})$ is a complete Segal space. As such, the hypothesis of Theorem (3.0.68) is satsified and we establish that $\boldsymbol{\Theta}_n^{\mathrm{Exit}}$ localizes on W_n^{healthy} to the exit-path ∞-category of the Ran space of $\mathbb{R}^n$.

Lastly, to see that this localization is over $(\mathrm{Fin}^{\mathrm{surj}})^{\mathrm{op}}$, we recall that the equivalence $\mathrm{BFun}^{W_n^{\mathrm{healthy}}}([\bullet], \boldsymbol{\Theta}_n^{\mathrm{Exit}}) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}([\bullet], \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n)))$ was induced by the functor $\boldsymbol{\Theta}_n^{\mathrm{Exit}} \to \mathrm{Exit}(\mathrm{Ran}(\mathbb{R}^n))$ from diagram (3.0.56), which is in particular over $(\mathrm{Fin}^{\mathrm{surj}})^{\mathrm{op}}$, which implies that the localization, too, is over $(\mathrm{Fin}^{\mathrm{surj}})^{\mathrm{op}}$.

□

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