ACTIONS OF THE CIRCLE GROUP

ON PRESENTABLE STABLE ∞-CATEGORIES

by

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ABSTRACT

This dissertation seeks to identify actions by the circle group on presentable stable ∞-categories. A circle action on the category of chain complexes over a commutative ring is equivalent with the data of a degree 1 differential map. In the more generalized setting, this characterization of a circle group action is obstructed precisely by the Hopf fibration. This is identified using the theory of stratifications.

STRATIFICATIONS

Notation and Conventions

(1) For an integer $n \geq -1$, define the n-sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

(2) Let X denote an arbitrary topological space, and M denote an arbitrary manifold.

(3) For any space X, and for any $0 \leq k \leq n$, let $X^k \xrightarrow{i} X^n$ be the canonical inclusion into the first k coordinates.

(4) Let $U \subset X$ be a subset. Denote the compliment of U in X as $X \backslash U$.

(5) Let $\mathcal{C}$ denote an arbitrary ∞-category. Furthermore, let $\mathcal{V}$ and $\mathcal{X}$ denote arbitrary presentable stable ∞-categories.

(6) The ∞-category of ***arrows in*** $\mathcal{C}$ is $\mathsf{Ar}(\mathcal{C}) := \mathsf{Fun}([1], \mathcal{C})$.

(7) Let $c \in \mathcal{C}$. We refer to the corresponding object in $\mathcal{C}^{\mathsf{op}}$ as c° to distinguish that this is an object in the opposite ∞-category.

Background on Stratified Spaces

Introduction

The first chapter recalls the basic definitions of stratified spaces and of conically smooth stratified spaces. Stratifications decompose a topological space X into strata. A stratification of a topological space naturally arises from a number of familiar contexts: an closed-open decomposition of the topological space, the skeletal decomposition of a CW structure on the topological space, via the unstable subspaces of a Morse function on the topological space. One instance of how stratifications are used is in [3], where the homotopy type of configuration spaces of points in $\mathbb{R}^n$ is identified by considering a stratification of the configuration space such that each stratum is equivalent to Euclidean space.

When looking at stratified spaces, one quickly realizes that it is natural to consider stratified spaces with added regularity, just as one might consider smooth manifolds instead of ordinary topological spaces. In the Appendix of [10], Lurie introduces conically stratified spaces to understand constructible sheaves on a topological space. Ayala, Francis, and Tanaka then introduce the theory of conically smooth stratified spaces in [5]. Conically smooth stratified spaces have the added benefit of being defined inductively in terms of open covers. In particular, every conically smooth stratified space is an example of a conically stratified space in the sense of Lurie. The added structure on a stratification in order to be a conically smooth stratified space can be thought of as being analogous to the extra structure of a smooth manifold in comparison to a topological space. Indeed, a smooth manifold can canonically be considered as a conically smooth stratified spaces, just as a topological space can canonically be considered as a stratified space by using the trivial stratification.

After the recollection of conically smooth stratified spaces, we seek to show the following theorem.

Theorem (Theorem 140)**.** *Let $X \to \mathsf{P}$ be a closed P-filtration (Definition 114).*

(1) For each $p \in \mathsf{P}$, the restricted stratification

$$X_{\leq p} \to \mathsf{P}_{\leq p}$$

is conically smooth.

(2) The stratification $X \to \mathsf{P}$ is conically stratified.

There are numerous examples of closed P-filtrations. In particular, a sequence of closed embeddings of smooth manifolds determines a closed P-filtration. An example of interest in this work is the sequence of closed embeddings of complex projective spaces

$$\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^n \ .$$

This sequence of closed embeddings induces a canonical stratification of $\mathbb{CP}^n$ which is a conically smooth stratified space by Theorem 140.

A conically smooth stratified space $X \to \mathsf{P}$ determines an ∞-category called an exit path ∞-category $\mathsf{Exit}(\underline{X})$. A closed P-filtration for a non-finite poset P may not be conically smooth, which may cause difficulties when trying to calculate the exit path ∞-category. This is rectified by showing a closed P-filtration is a union of conically smooth stratified spaces, and as such the exit path ∞-category can be defined as the union over $p \in \mathsf{P}$ of the exit path categories $\mathsf{Exit}(X_{\leq p})$. The last part of this chapter gives a calculation of the the exit path ∞-category of the closed P-filtration of $\mathbb{CP}^\infty$.

Theorem (Theorem 165)**.** *The exit path ∞-category of the stratification* $\mathsf{colim}(\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \dots) = \mathbb{CP}^\infty$ *is identified as*

$$\mathsf{Exit}(\underline{\mathbb{CP}^\infty}) \simeq \left(\bigstar_{k \in \mathbb{Z}_{\geq 0}} \mathbb{T} \right)_{/\mathbb{T}} \ .$$

Here the exit path ∞-category $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$ is identified as quotient the categorical join (Definition 121) of the circle group $\mathbb{T}$, where the quotient is induced by the canonical action of $\mathbb{T}$ on the join. This identification of the exit path ∞-category of $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$ is used in the fifth chapter to identify circle actions on a presentable stable ∞-category.

The action of the circle group $\mathbb{T}$ on an object V in a presentable stable ∞-category $\mathcal{V}$

is the data of a functor from the classifying space of $\mathbb{T}$ to the ∞-category $\mathcal{V}$

$$\mathsf{B}\mathbb{T} \to \mathcal{V} .$$

The classifying space of $\mathbb{T}$ is the complex projective space $\mathbb{CP}^\infty$. Therefore, a circle action determines, and is detemined by, a functor

$$\mathbb{CP}^\infty \to \mathcal{V} .$$

The main result of the thesis is the following theorem, which is informally stated here, with the precise statement given in Theorem 310

Theorem (Theorem 310)**.** *A circle action on an object $V \in \mathcal{V}$ determines and is determined by:*

+) An object $V \in \mathcal{V}$.

+) A morphism in $\mathcal{V}$

$$\partial : \Sigma V \to V \ .$$

+) For each $k \in \mathbb{Z}_{\geq 2}$, an identification of $\eta^{k-1}\partial : \Sigma^n V \to V$ with $\partial^n : \Sigma^n V \to V$ compatibly.

In the statement of this theorem, η is the map induced by the Hopf fibration (Definition 217). The difficulty in this is the compatibility requirement, which made preceise in Theorem 310. The problem of trying to handle this higher coherence data is studied by stratifying the ∞-category $\mathsf{Fun}\,(\mathsf{Exit}(\underline{\mathbb{CP}^\infty}), \mathcal{V})$ by considering the closed subcategories $\mathsf{Fun}\,(\mathsf{Exit}\,(\underline{\mathbb{CP}^n})\,, \mathcal{V})$ for each n.

A corollary of this result is as follows.

Corollary 1. *Let $\Bbbk$ be a commutative ring spectrum. If $0 = \eta \in \pi_1(\Bbbk)$, then ta circle action on an object $V \in \mathcal{V}$ determines, and is determined by*

(1) An object $V \in \mathcal{V}$.

(2) A morphism in $\mathcal{V}$

$$\partial : \Sigma V \to V$$

(3) For each $k \in \mathbb{Z}_{\geq 2}$, and identification

$$\left(\Sigma^n V \xrightarrow{0} V\right) \simeq (\partial^n : \Sigma^n V \to V)$$

compatibily.

Here, the lefthand ∞-category is that of $\mathbb{T}$-modules in the stable ∞-category of $\Bbbk$-modules; the righthand ∞-category is that of non-negatively indexed chain complex objects in the stable ∞-category of $\Bbbk$-modules.

What follows is some example and non examples of when the Hopf map η vanishes.

Example 2. Let $\Bbbk$ be a commutative ring spectrum. The element $0 \in \eta \in \pi_1(\Bbbk)$ vanishes in the following cases.

(1) $\Bbbk$ is an (ordinary) commutative ring. For instance, $\Bbbk = \mathbb{Z}$ is the ring of integers or $\Bbbk = \mathbb{Q}$ is the ring of rational numbers or $\Bbbk = \mathbb{F}_q$ is a finite field.

(2) The first homotopy group of the underlying spectrum $\pi_1(\Bbbk)$ has no 2-torsion. For instance,

  (1) $\Bbbk = \mathsf{KU}$ is the complex K-theory spectrum, since $\pi_1(\mathsf{KU}) = 0$.

  (2) $\Bbbk = \mathbb{S}_p$, the p-local sphere spectrum for p an odd prime.

(3) $\Bbbk = \mathsf{K}_p(n)$ or $\Bbbk = \mathsf{E}_p(n)$ is the Morava K-theory and E-theory spectrum for $n > 0$ and p an odd prime.

Example 3. Here are examples of commutative ring spectra for which the element $\eta \in \pi_1(\Bbbk)$ is not zero.

(1) KO, the real K-theory spectrum. Indeed, $\eta \in \pi_1(\mathsf{KO}) \cong \mathbb{Z}_{/2\mathbb{Z}}$ is a generator.

(2) $\mathbb{S}$, the sphere spectrum. Indeed, $\eta \in \pi_1(\mathbb{S}) \cong \mathbb{Z}_{/2\mathbb{Z}}$ is a generator.

We begin by giving a background in stratified spaces. Working towards the definition of a stratified space, we first recall the definition of a posets, as well as different constructions on the category of posets that we will need in our discussion of stratified spaces.

Posets

This subsection introduces the notion of a partially ordered set, or poset for short. Intuitively, a partially ordered set is a set with an ordering on the elements of the set, allowing statements about elements of the set being less than another element. There are many examples of posets, and many of the examples listed in this section will be used throughout this work.

The purpose of introducing posets is to work toward the definition of a stratified space. A stratified space is the data of a continuous map from a topological space to a poset. The main result of this section is to construct a fully faithful embedding of the category of posets into the category of topological spaces. The embedding of posets into topological spaces ensures the notion of a continuous map into a poset.

Definition 4. A ***poset*** $(\mathsf{P}, \leq)$ is:

+) A set P.

+) A binary relation $\leq$, which is a subset of $\mathsf{P} \times \mathsf{P}$.

such that for all $x, y, z \in \mathsf{P}$, the binary relation satisfies:

-) $(x, x) \in \leq$ (***Reflexive***)

-) If $(x, y) \in \leq$ and $(y, x) \in \leq$, then $x = y$ (***Antisymmetric***)

-) If $(x, y) \in \leq$ and $(y, z) \in \leq$, then $(x, z) \in \leq$ (***Transitive***)

Definition 5. A poset P is ***linearly ordered*** if for all $p, q \in \mathsf{P}$, either $p \leq q$ or $q \leq p$.

Notation 6. For a poset $(\mathsf{P}, \leq)$, we will write $x \leq y$ to mean $(x, y) \in \leq$. We will frequently say that x is less than or equal to y if $x \leq y$. Similarly, we say $x < y$ to mean that $(x, y) \in \leq$, and that $x \neq y$. In this case, we will say that x is less than y. Furthermore, we will often just write P instead of $(\mathsf{P}, \leq)$ unless there is more than one binary relation on P involved.

Definition 7. Let $(\mathsf{P}, \leq_{\mathsf{P}})$ and $(\mathsf{Q}, \leq_{\mathsf{Q}})$ be posets. A map between their underlying sets

$$f : \mathsf{P} \to \mathsf{Q}$$

is ***order preserving*** if $a \leq_{\mathsf{P}} b$ implies that $f(a) \leq_{\mathsf{Q}} f(b)$. Equivalently, a map $f : \mathsf{P} \to \mathsf{Q}$ is order preserving if there exists a filler

$$\begin{tikzcd}
\leq_{\mathsf{P}} \arrow[r, dashed, "\exists"] \arrow[d, hook] & \leq_{\mathsf{Q}} \arrow[d, hook] \\
\mathsf{P} \times \mathsf{P} \arrow[r, "f \times f"'] & \mathsf{Q} \times \mathsf{Q}
\end{tikzcd} .$$

A filler between their binary relations is necessarily unique, since the inclusion $\leq_Q \hookrightarrow \mathsf{Q} \times \mathsf{Q}$ is a monomorphism in the category Set.

Notation 8. Define the category Poset to be the category with posets as objects, and order preserving maps as morphisms.

Notation 9. Throughout the remainder of this paper, let P denote an arbitrary poset.

A stratification will consist of a map from a topological space X into a poset P. The goal then is to construct a topology on a poset P. This topology should ensure that continuous maps bewteen posets should be the same as order preserving maps between posets. Order preserving maps corresponding bijectively to continuous maps ensures the category Poset fully faithfully embeds into the category Top. The embedding $\mathsf{Poset} \hookrightarrow \mathsf{Top}$ allows one to consider continuous maps from a topological space to a poset P.

There are several constructions of posets needed in the discussion of determining a topology on a poset P. We begin by recalling these constructions.

Notation 10. Let P be a poset. For each $p \in \mathsf{P}$, define the following four subposets of P to be

$$\mathsf{P}_{\leq p} := \{q \in \mathsf{P} \mid q \leq p\}$$

$$\mathsf{P}_{<p} := \{q \in \mathsf{P} \mid q < p\}$$

$$\mathsf{P}_{\geq p} := \{q \in \mathsf{P} \mid p \leq q\}$$

$$\mathsf{P}_{>p} := \{q \in \mathsf{P} \mid p < q\}.$$

Notation 11. Define the category $\mathsf{P}_{\leq p \cap \leq q}$ to be the pullback

$$\begin{tikzcd} \mathsf{P}_{\leq p \cap \leq q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{P}_{\leq p} \arrow[d] \\ \mathsf{P}_{\leq q} \arrow[r] & \mathsf{P} \end{tikzcd}.$$

This pullback[1] is the intersection of the subposets $\mathsf{P}_{\leq p}$ and $\mathsf{P}_{\leq q}$ in P.

[1]There is a notion of products in a poset P, by canonically regarding P as a category. If the poset P admits products, the poset $\mathsf{P}_{\leq p \cap \leq q}$ agrees with the poset $\mathsf{P}_{\leq p \times q}$

Definition 12. Define the poset $[n]$ to be the linearly ordered poset with $n+1$ elements

$$[n] := \{0 \leq 1 \leq \cdots \leq n\} \ .$$

Example 13. Define the poset $\mathbb{Z}$ to be the integers, with the standard ordering

$$\mathbb{Z} := \{\cdots \leq -2 \leq -1 \leq 0 \leq 1 \leq 2 \leq \dots\}.$$

The poset $\mathbb{Z}$ is an example of a poset which is linearly ordered.

Definition 14. The ***depth*** of a poset P is the maximal $[n]$, if it exists, such that there is an inclusion

$$[n] \to \mathsf{P} \ .$$

The empty poset $\emptyset$ is defined to have depth -1.

Observation 15. Note that two finite linearly ordered posets P and Q are isomorphic if and only if their underlying sets have the same cardinality.[2] Therefore, every finite linearly ordered nonempty poset is isomorphic to $[n]$ for some non-negative integer n.

Notation 16. Let P be a poset. A poset P can be depicted as a directed graph, where there is an arrow $p \to q$ if $p \leq q$. For a triplet $p \leq q \leq r$, we depict as

$$p \to q \to r \ ,$$

and do not depict the arrow from $p \to r$. This is similar to how a category can be depicted as a directed graph.

[2]Moreover, the posets P and Q are uniquely isomorphic.

Example 17. The poset $[n]$ (Definition 12) can be depicted as

$$0 \to 1 \to \cdots \to n-1 \to n \ .$$

Example 18. Let P and Q be posets. The product of sets $\mathsf{P}\times\mathsf{Q}$ inherits a canonical ordering. Let (p,q) and (p',q') be in $\mathsf{P}\times\mathsf{Q}$. Define the ordering on $\mathsf{P}\times\mathsf{Q}$ to be $(p,q) \le (p',q')$ if and only if $p \le p'$ in P and $q \le q'$ in Q.

Example 19. The poset $[1]\times[1]$ is the poset

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \end{tikzcd} .$$

Definition 20. Let S be a set. Define $\mathcal{P}(S) \in \mathsf{Poset}$ to be the power set[3] of S, which we regard as a poset with the ordering given by inclusion. If S is a topological space, define $\mathbf{Cls}(S) \subseteq \mathcal{P}(S)$ to be the full subposet of $\mathcal{P}(S)$ consisting of the closed subsets of S.

Definition 21. An element $p \in \mathsf{P}$ is a ***minimal element*** if $q \le p$ implies $q = p$.

Remark 22. In an arbitrary poset P, a minimal element might not be unique. However, if P is linearly ordered, or if P admits all greatest lower bounds, then a minimal element must be unique, provided the minimal element exists.

There is a canonical way to regard a poset as a category. The set of objects of the category P is the underlying set of P. Given two elements p, p' in P, the set of morphisms

[3]Recall that the power set of a set S is the set of all subsets of S.

$\mathsf{Hom}_{\mathsf{P}}(p, p')$ is defined to be

$$\mathsf{Hom}_{\mathsf{P}}(p, p') := \begin{cases} * & p \leq p' \\ \emptyset & else \end{cases} .$$

The composition rule is unique if it exists, since each hom-set is either $*$ or $\emptyset$. The transitive property of the poset P ensures that a composition rule exists. The reflexive property ensures that there is an identity element for every $p \in \mathsf{P}$. The antisymmetric property of the poset P ensures that this category is skeletal, that is, two objects are isomorphic if and only if they are equal. Note that given two posets P and Q, there is a bijection of sets

$$\mathsf{Hom}_{\mathsf{Poset}}(\mathsf{P}, \mathsf{Q}) \cong \mathsf{Hom}_{\mathsf{Cat}}(\mathsf{P}, \mathsf{Q}) .$$

The bijection between the hom sets ensures the following lemma.

Lemma 23. *The canonical inclusion*

$$\mathsf{Poset} \hookrightarrow \mathsf{Cat}$$

is fully faithful.[4]

The fully faithful embedding $\mathsf{Poset} \hookrightarrow \mathsf{Cat}$ allows one to consider colimits and limits indexed by posets, as well as to recover some familiar concepts of a poset P in category theoretical terms. Namely, a poset P determines a poset P^{op} with the opposite ordering. Regarded as categories, the category P^{op} is the usual opposite category of the category P. Another instance of being able to recover a familiar concept is the greatest lower bound and the least upper bound of a subset of elements of P. The greatest lower bound and least

[4]The essential image of this functor are categories $\mathcal{C}$ such that for each $a, b \in \mathcal{C}$, the set of morphisms $\mathsf{Hom}_{\mathcal{C}}(a, b)$ is either empty or a singleton.

upper bound correspond to products and coproducts in the category P.

Lemma 24. *Let* P *be a poset.*

(1) The product of two elements p, p' *in* P *is the greatest lower bound of* p *and* p' *if it exists.*

(2) The coproduct of two elements of $p, q \in \mathsf{P}$ *is the least upper bound, if it exists.*[5]

(3) The poset P *has a unique minimal element if and only if* P *is not empty and the product* $\prod_{p\in\mathsf{P}} p$ *exists.*

(4) The poset P *has a unique maximal element if and only if* P *is not empty and the coproduct* $\coprod_{p\in\mathsf{P}} p$ *exists.*

Proof. (1) and (2) follow by inspection of the definitions of a greatest lower bound and a least upper bound, as well as recalling that the morphisms of P are determined by the ordering of P. (3) follows since the product of every element of P must have the property that that $\prod_{p\in\mathsf{P}} p \leq p$ for every element of P by examining the projection maps. Finally (4) follows as the dual of (3) □

Notation 25. Define $\boldsymbol{\Delta}$ to be the full subcategory of Poset consisting of the non-empty finite linearly ordered posets. Note that equivalently, this category is the full subcategory on the posets $[n]$ (Definition 12) for each $n \in \mathbb{Z}_{\geq 0}$.

Definition 26. Let P be a poset. Define the ***subdivision of*** P, denoted $\mathsf{sd}(\mathsf{P})$, to be the full subcategory of $\Delta_{/\mathsf{P}}$ consisting of objects $[n] \xrightarrow{\mathsf{inj}} \mathsf{P}$ that are injective.

Observation 27. The subdivision of a poset P is a poset. Indeed, an injective map between posets is a monomorphism in the category of posets. This is two say, that for two objects

[5] One could take this fact as a definition of the greatest lower bound and least upper bound .

$([n] \to \mathsf{P})$ and $([m] \to \mathsf{P})$ of $\mathsf{sd}(\mathsf{P})$, the space of fillers

$$\begin{tikzcd}
{[m]} \arrow[rr, dashed] \arrow[dr] & & {[n]} \arrow[dl] \\
& \mathsf{P} &
\end{tikzcd}$$

is either empty or admits a unique filler.

Example 28. The poset $\mathsf{sd}([2])$ is the poset

$$\begin{tikzcd}
& & \{1\} \arrow[dl] \arrow[d] \arrow[dr] & & \\
& \{0 \leq 1\} \arrow[r] & \{0 \leq 1 \leq 2\} & \{1 \leq 2\} \arrow[l] & \\
\{0\} \arrow[ur] \arrow[rr] & & \{0 \leq 2\} \arrow[u] & & \{2\} \arrow[ul] \arrow[ll]
\end{tikzcd} .$$

Here, each object $[n] \hookrightarrow \mathsf{P}$ is depicted by its image.

Definition 29. Let S be a set. Define $\mathcal{P}(S) \in \mathsf{Poset}$ to be the power set[6] of S, which we regard as a poset with the ordering given by inclusion. If S is a topological space, define $\mathbf{Cls}(S) \subseteq \mathcal{P}(S)$ to be the full subposet of $\mathcal{P}(S)$ consisting of closed subsets of S.

Observation 30. For a finite linearly ordered poset P, there is an equivalence of posets

$$\mathsf{sd}(\mathsf{P}) \cong \mathcal{P}(\mathsf{P}) \backslash \emptyset \ .$$

Finally, we have enough background to define a topology on a poset P.

Definition 31. Let $U \subset \mathsf{P}$. The subset $U \subset \mathsf{P}$ is ***upwards closed*** if $p \leq q$ and $p \in U$ implies that $q \in U$. Similarly, a subset U is ***downwards closed*** if $p \leq q$ and $q \in U$ implies that $p \in U$.

[6]Recall that the power set of a set S is the set of all subsets of S.

Definition 32. The ***upwards closed topology*** on a poset P is the topology such that $U \subset \mathsf{P}$ is open if and only if it satisfies the property of being upwards closed.

Lemma 33. *The upwards closed topology indeed defines a topology on a poset* P*.*

Proof. The subset $\emptyset$ and P are clearly upwards closed subsets.

Let $\{U_\alpha\}_{\alpha \in S}$ be a collection of upwards closed subsets of P indexed by a set S. Let $p \in \bigcup_{\alpha \in S} U_\alpha$, and $q \in \mathsf{P}$ such that $p \leq q$. We seek to show that $q \in \bigcup_{\alpha \in S} U_\alpha$. Since $p \in \bigcup_{\alpha \in S} U_\alpha$, there exists an α such that

$$p \in U_\alpha \ .$$

Since $p \leq q$ and U_α is upwards closed, then

$$q \in \mathsf{P} \ .$$

Therefore $\bigcup_{\alpha \in S} U_\alpha$ is upwards closed.

The intersection condition for the upwards closed topology is shown in Lemma 52. In fact, a stronger condition is shown, namely that the arbitrary intersection of upwards closed subsets is upwards closed. $\square$

Observation 34. Let $p \in \mathsf{P}$. The subposets $\mathsf{P}_{\geq p}$ and $\mathsf{P}_{>p}$ are upwards closed by the transitive property of the ordering on P. Therefore the subposets $\mathsf{P}_{\geq p}$ and $\mathsf{P}_{>p}$ are open in the upwards closed topology on P. The subposets $\mathsf{P}_{<p}$ and $\mathsf{P}_{\leq p}$ are examples of closed subsets in the upwards closed.[7]

Remark 35. The upwards closed topology is sometimes referred to as the Alexandroff topology, or the specialization topology [1] . More generally, an Alexandroff space is one in which the arbitrary union of closed subsets is closed.

[7] Warning: note that the compliment of a poset $\mathsf{P}_{\geq p}$ is not necessarily $\mathsf{P}_{<p}$.

Lemma 36. *Let* P *and* Q *be posets. A map of sets*

$$\mathsf{P} \to \mathsf{Q}$$

is order preserving if and only if the map is continuous with respect to the upwards closed topology.

Proof. Let $f : \mathsf{P} \to \mathsf{Q}$ be an order preserving map of sets, and let U be an upwards closed subset of Q. We seek to show that $f^{-1}(U)$ is an upwards closed subset of P. Let $p \in f^{-1}(U)$, and $p \leq q$ in P. The map of sets $f : \mathsf{P} \to \mathsf{Q}$ is order preserving, so $p \leq q$ implies that

$$(f(p) \leq f(q)) \in \mathsf{P} \ .$$

Since $f(p)$ is in U, and U is upwards closed, then $f(q)$ is an element of U. Therefore

$$q \in f^{-1}(U) \ .$$

Now assume that $f : \mathsf{P} \to \mathsf{Q}$ is continuous. We seek to show that f is order preserving. Let $p \leq q$ in P. The element $f(p)$ determines an upwards closed subset

$$\mathsf{Q}_{\geq f(p)} := \{s \in \mathcal{Q} \mid f(p) \leq s\} \ .$$

Therefore it suffices to show that $f(q) \in \mathsf{Q}_{\geq f(p)}$. Since the map f is continuous, the subset $f^{-1}(\mathsf{Q}_{\geq f(p)})$ is an upwards closed subset of P that contains p. Therefore $q \in f^{-1}(\mathsf{Q}_{\geq f(p)})$, Applying the map f to q gives $f(q) \in f(f^{-1}(\mathsf{Q}_{\geq f(p)})) = \mathsf{Q}_{\geq f(p)}$. Therefore $f(p) \leq f(q)$ and f is order preserving. $\square$

Corollary 37. The upwards closed topology on a poset determines a fully faithful functor

$$\mathsf{Poset} \xhookrightarrow{\text{f.f.}} \mathsf{Top} \ .$$

Stratifications

We are now ready to define stratified spaces. A stratified space is a continuous map from a topological space X to a poset P. A topological space X and a poset P have the same type by Corollary 37, ensuring the notion of a continuous map from a topological space to a poset is well-defined.

Definition 38. A ***stratification*** of a topological space X by a poset P is a continuous map

$$Z : X \to \mathsf{P},$$

where the poset P has the upwards closed topology.

There is also a notion of a map of stratified spaces, as follows.

Definition 39. Let $X \to \mathsf{P}$ and $Y \to \mathsf{Q}$ be stratified spaces. A ***map between stratified spaces*** from the first to the second is a commutative square in Top

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathsf{P} \arrow[r] & \mathsf{Q} \end{tikzcd} .$$

Notation 40. Define the category StTop to be the category whose objects are stratified spaces, and morphisms are maps between stratified spaces.

Notation 41. We will write $\underline{X}$ for a stratification $X \to \mathsf{P}$ if the stratification is clear from context. We will also write $\underline{X} \to \underline{Y}$ for a map of stratified spaces.

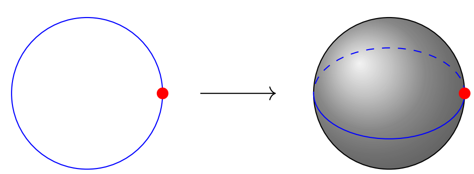
Observation 42. The category StTop fits into a pullback

$$\begin{tikzcd} \mathsf{StTop} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Ar}(\mathsf{Top}) \arrow[d, "t"] \\ \mathsf{Poset} \arrow[r, hook] & \mathsf{Top} \end{tikzcd} .$$

The pullback witnesses the category StTop as a subcategory of $\mathsf{Ar}(\mathsf{Top})$ on those arrows such that the target of the arrow is a poset. Therefore a morphism of topological spaces can be thought of as natural transformation between two arrows in Top. Recall a natural transformation between arrows is a functor from the poset $[1] \times [1]$ into Top.

Let us consider a few examples of stratifications. The examples that follow have a focus on the examples needed throughout this paper.

Example 43. Consider the stratification of $\mathbb{S}^1$ by the poset $[1]$ that sends $(1,0) \in \mathbb{R}^2$ to $0 \in [1]$, and $\mathbb{S}^1 \backslash (1,0)$ to $1 \in [1]$. Consider a stratification of $\mathbb{S}^2$ by the poset $[2]$ that sends the point $(1,0,0)$ to $0 \in [2]$, the remainder of the equator to $1 \in [2]$, and the north and south hemispheres to $2 \in [2]$. The inclusion of the equator



is a map of stratified spaces. Note that the stratification of $\mathbb{S}^2$ determine a sequence of closed embeddings

$$* \hookrightarrow \mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \ .$$

Notation 44. Throughout the remainder of this paper, let $Z : X \to \mathsf{P}$ denote an arbitrary stratified space.

Definition 45. Let $i \in \mathsf{P}$. Define the ***i-stratum*** X_i to be the fiber of Z over i

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "Z"] \\ \{i\} \arrow[r] & \mathsf{P} \end{tikzcd} .$$

The map $X_i \to i$ can be viewed as a stratification of X_i by the poset $\{i\}$, and the inclusion of the i stratum into X is a map of stratified spaces.

Notation 46. Let $Z : X \to \mathsf{P}$ be a stratification, and let Q be a subposet of P. Define X_Q to be the pullback

$$\begin{tikzcd} X_{\mathsf{Q}} \arrow[r, hook] \arrow[d, "Z_{\mathsf{Q}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "Z"] \\ \mathsf{Q} \arrow[r, hook] & \mathsf{P} \end{tikzcd} .$$

Note this pullback is the subspace $Z^{-1}(\mathsf{Q}) \subset X$. The map $Z_{\mathsf{Q}} : X_{\mathsf{Q}} \to \mathsf{Q}$ is an example of a stratification, and the inclusions of $X_{\leq \mathsf{Q}} \hookrightarrow X$ extends to a morphism of stratified spaces.

Notation 47. Dennote the following notation

$$X_{\geq p} := X_{\mathsf{P}_{\geq p}}$$

$$X_{\leq p} := X_{\mathsf{P}_{\leq p}}$$

$$X_{< p} := X_{\mathsf{P}_{< p}}$$

$$X_{> p} := X_{\mathsf{P}_{> p}} .$$

Definition 48. A poset P is ***downwards finite*** if for each $p \in \mathsf{P}$, the poset $\mathsf{P}_{\leq p}$ is finite.

We next look at another way that stratifications arise in practice. In practice, a filtration of a topological space X, that is, a sequence of closed subsets

$$(X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X) \ ,$$

determines a stratification the space X by the poset $[n]$. The next definition abstracts this notion where filtrations are indexed by a general poset P. We will refer to these filtrations as P-filtrations. The next goal is to show that P-filtrations provide a host of examples of stratified spaces. Similarly, a stratified space determines an P-filtration (Lemma 54). P-filtrations are often how stratifications arise in practice.

Definition 49. A P***-filtration*** of a topological space X is:

+) A functor

$$\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(X) \ .$$

such that:

-) The union of the closed subsets determined by the functor $\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(X)$ is the space X

$$\bigcup_{p \in \mathsf{P}} \mathsf{Z}(p) = X \ .$$

-) For each $p, q \in \mathsf{P}$, the union of closed subsets determined by restriction of Z to the poset $\mathsf{P}_{\leq p \cap q}$ is the intersection of $\mathsf{Z}(p)$ and $\mathsf{Z}(q)$

$$\bigcup_{r \in \mathsf{P}_{\leq p \cap q}} Z(r) = \mathsf{Z}(p) \cap \mathsf{Z}(q) \ .$$

This is the ***stratification condition***.

-) For any downwards closed subposet $\mathsf{D} \subset \mathsf{P}$, the union of the closed subsets of X indexed by D is again closed

$$\bigcup_{d \in \mathsf{D}} \mathsf{Z}(d) \in \mathbf{Cls}(X) \ .$$

This is the ***continuity condition***.

Observation 50. Note that the stratification condition is automatically satisfied in the case that P is a linearly ordered poset. Furthermore, the continuity condition is automatically satisfied if the poset P has the property that any proper downwards closed subset is finite, such as when P is finite, or when $\mathsf{P} = \mathbb{Z}_{\geq 0}$.

The proof of P-filtrations providing the same data as stratifications relies on a few facts about the upwards closed topology on P, which we now show.

Lemma 51. *Let P be a poset. Then the downwards closed subsets of P are the closed subsets with respect to the upwards closed topology on P*

$$\mathsf{Down}(\mathsf{P}) := \{\mathsf{D} \subset \mathsf{P} | \mathsf{D} \textit{ is downwards closed.}^{8}\} = \mathbf{Cls}(\mathsf{P}) \ .$$

Proof. First, we seek to show that every closed subset is downwards closed. Let $\mathsf{U} \subset \mathsf{P}$ be upwards closed, and let D be the compliment of U in P. We seek to show that $\mathsf{D} \in \mathsf{Down}(\mathsf{P})$. Assume that D is not downwards closed, that is, there exists a $d \in \mathsf{D}$, and a $p \in \mathsf{U}$ such that $p \leq d$. The poset U is upwards closed, which implies that $d \in U$. This contradicts that $d \in \mathsf{D} = \mathsf{P} \backslash \mathsf{U}$. Therefore D is downwards closed as desired, and

$$\mathbf{Cls}(\mathsf{P}) \subset \mathsf{Down}(\mathsf{P}) \ . \tag{1.1}$$

Next assume that D is downwards closed, and let U be the compliment of D in P. We seek to show that U is an open subset of P. We begin by noting

$$\mathsf{Down}(\mathsf{P}) \xrightarrow{\cong} \mathbf{Open}(\mathsf{P}^{\mathsf{op}}) \ . \tag{1.2}$$

The poset D is a downwards closed subset of P, which implies D° is an open subset in the opposite poset P^{op}

$$\mathsf{D}^\circ \in \mathbf{Open}(\mathsf{P}^{\mathsf{op}})$$

by (1.2). By taking compliments, the subset $\mathcal{U}^\circ$ is a closed subset of P^{op}

$$\mathsf{U}^\circ \in \mathbf{Cls}(\mathsf{P}^\circ) \ .$$

Since $\mathbf{Cls}(\mathsf{P}^\circ) \subset \mathsf{Down}(\mathsf{P}^\circ)$ by (1.1), the compliment U° of D° in P^{op} is downwards closed in P°

$$\mathsf{U}^\circ \in \mathsf{Down}(\mathsf{P}^{\mathsf{op}}).$$

Finally, again by (1.2), the subset U is open in P

$$\mathsf{U} \in \mathbf{Open}(\mathsf{P}) \ .$$

□

Lemma 52. *The downward closed subsets of* P *are closed under arbitrary unions. That is, if* $\{\mathsf{D}_\alpha\}_\alpha$ *is a collection of down-closed subsets of* P*, then* $\bigcup\limits_\alpha \mathsf{D}_\alpha$ *is a downwards closed subset.*

Proof. Let $\{\mathsf{D}_\alpha\}_{\alpha\in A}$ be a collection of downwards closed subsets of P indexed by a set A. Let d be an arbitrary element in $\bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$. The poset $\bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$ is downwards closed if $p \in \bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$ and $q \leq p$ implies that $q \in \bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$. Since $p \in \bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$, there exists an α such that $p \in \mathsf{D}_\alpha$. Since D_α is downwards closed, then $q \in \mathsf{D}_\alpha$, and $q \in \bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$. Therefore $\bigcup\limits_{\alpha\in A} \mathsf{D}_\alpha$ is downwards closed. This implies that the arbitrary union of closed subsets of a poset is a closed subset. □

Lemma 53. *Let* $\mathsf{D} \subset \mathsf{P}$ *be a downwards closed poset. Then there is an equivalence of posets*

$$\mathsf{D} = \bigcup_{d\in\mathsf{D}} \mathsf{P}_{\leq d} \ .$$

Proof. Clearly D is a subset of $\bigcup_{d\in\mathsf{D}} \mathsf{P}_{\leq d}$. Therefore we seek to show

$$\bigcup_{d\in\mathsf{D}} P_{\leq d} \subset \mathsf{D}$$

Let $x \in \bigcup_{d\in\mathsf{D}} \mathsf{P}_{\leq d}$. Then there exits a $d \in \mathsf{D}$ such that $x \in \mathsf{P}_{\leq d}$. The element x is in $\mathsf{P}_{\leq d}$ if and only if $x \leq d$. Finally, since D is downwards closed and $d \in \mathsf{D}$, then $x \in \mathsf{D}$. □

Lemma 54. *Let P be a downwards finite poset. The following data is equivalent:*

(1) A stratification $Z : X \to \mathsf{P}$.

(2) A P-filtration $\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(X)$ of a space X.

Proof. Let $Z : X \to \mathsf{P}$ be a stratification. Define a functor

$$\begin{aligned} \mathsf{Z} : \mathsf{P} &\longrightarrow \mathbf{Cls}(X) \\ p &\longmapsto X_{\leq p} \end{aligned} .$$

The continuity of the map $Z : X \to \mathsf{P}$ ensures that the preimage of the closed subset $\mathsf{P}_{\leq p}$, which is $X_{\leq q}$, is closed. Note that for $p \leq q$, there is an inclusion $X_{\leq p} \hookrightarrow X_{\leq q}$ by the properties of pullbacks. Therefore this functor is well-defined, and we next check that it is a P-filtration.

We first seek to check that

$$\bigcup_{p\in\mathsf{P}} \mathsf{Z}(p) = \bigcup_{p\in\mathsf{P}} X_{\leq p} = X \ .$$

Clearly $\bigcup_{p\in\mathsf{P}} X_{\leq p} \subset X$ since each $X_{\leq p} \subset X$. Therefore we seek to show

$$X \subset \bigcup_{p\in\mathsf{P}} X_{\leq p} \ .$$

Let x be an element of X. The map $Z : X \to \mathsf{P}$ assigns a value $Z(x) \in \mathsf{P}$. Therefore the element x is in $X_{\leq Z(p)}$, which implies that $x \in X_{\leq Z(x)} \subset \bigcup_{p \in \mathsf{P}} X_{\leq p}$. The element x was arbitrary, and therefore

$$\bigcup_{p \in \mathsf{P}} \mathsf{Z}(x) = \bigcup_{p \in \mathsf{P}} X_{\leq p} = X \ .$$

Next, we seek to show that that Z satisfies the stratification condition. That is, for each $p, q \in \mathsf{P}$, there is an equality

$$\bigcup_{r \in \mathsf{P}_{\leq p \cap q}} \mathsf{Z}(r) = \mathsf{Z}(p) \cap \mathsf{Z}(q) = X_{\leq p} \cap X_{\leq q} \ .$$

First let $x \in X_{\leq p} \cap X_{\leq q}$. Then $x \in X_{\leq p}$ and $x \in X_{\leq q}$. Define $k := Z(x)$. By definition of $X_{\leq p}$ and $X_{\leq q}$, the element k is less than or equal to p and q. This implies the element x is an element of

$$x \in X_{\leq k} \subset \bigcup_{r \in \mathsf{P}_{\leq p \cap q}} X_{\leq r} \ .$$

Next assume that $x \in \bigcup_{r \in \mathsf{P}_{\leq p \cap q}} X_{\leq r}$. Then there exists an r such that $x \in X_{\leq r}$. Furthermore this $r \leq p$ and $r \leq q$. By the properties of pullbacks there are canonical inclusions

$$X_{\leq p} \hookleftarrow X_{\leq r} \hookrightarrow X_{\leq q}$$

Therefore $x \in X_{\leq p}$ and $X_{\leq q}$, which implies that x is in their intersection. Therefore the stratification condition is satisfied

$$\bigcup_{r \in \mathsf{P}_{\leq p \cap q}} X_{\leq r} = X_{\leq p} \cap X_{\leq q} = \mathsf{Z}(q) \cap \mathsf{Z}(p) \ .$$

Next, we seek to show the continuity condition. Let D be a downwards closed subposet.

The poset D is equivalent with the poset

$$\mathsf{D} = \bigcup_{d \in \mathsf{D}} \mathsf{P}_{\leq d}$$

by Lemma 53. Taking preimages gives

$$X_{\mathsf{D}} = Z^{-1}(\mathsf{D}) = \bigcup_{d \in \mathsf{D}} Z^{-1}(\mathsf{P}_{\leq d}) = \bigcup_{d \in \mathsf{D}} X_{\leq d} \ .$$

Since the map Z is continuous, $Z^{-1}(\mathsf{D}) \in \mathbf{Cls}(X)$, which implies

$$\bigcup_{d \in \mathsf{D}} X_{\leq d} \in \mathbf{Cls}(X) \ .$$

Therefore the functor $\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(X)$ is a P-filtration.

Next, we seek to show that a P-filtration $\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(X)$ determines a stratification $Z : X \to \mathsf{P}$. Define the map of sets

$$\begin{aligned} Z : X &\longrightarrow \mathsf{P} \\ x &\longrightarrow \min\{p \mid x \in X_{\leq p}\} \end{aligned} \ .$$

We seek to show that this is well-defined. Towards this goal, define the poset $\mathsf{P}_x \subset \mathsf{P}$ to be

$$\mathsf{P}_x := \{p \in \mathsf{P} \mid x \in \mathsf{Z}(p)\} \ .$$

The map $Z : X \to \mathsf{P}$ is well-defined if and only if the poset P_x has a unique minimal element to it.

The condition that $\bigcup_{p \in \mathsf{P}} \mathsf{Z}(p) = X$ ensures that for each $x \in X$, there exists a $p \in \mathsf{P}$ such that $x \in \mathsf{Z}(p)$, which shows that P_x is nonempty. Furthermore, note that a subset of a downwards finite poset is also downwards finite. Therefore $\mathsf{P}_x \subset \mathsf{P}$ is a downwards finite poset. Assume that there is no minimal element to the poset P_x. As P_x is nonempty, there exists an element $p \in \mathsf{P}_x$. Since p is not a minimal element by assumption, there exists an element $p_0 < p$ in P_x. Now assume there are elements $\{p, p_0, p_1, \ldots, p_{n-1}, p_n\} \subset \mathsf{P}_x$ such that

$$\{p_n < p_{n-1} < \cdots < p_0 < p\}$$

Since the element p_n is not a minimal element, there always exists a $p_{n+1} \in \mathsf{P}_x$ such that $p_{n+1} < p_n$, and therefore by the transitive property of the relation on P,

$$\{p_{n+1} < p_n < p_{n-1} < \cdots < p_1 < p_0 < p\} \ .$$

Therefore there exists an infinite sequence of elements that are less than p in P_x, which is to say $(\mathsf{P}_x)_{\leq p}$ is infinite. This is a contradiction to the fact that the poset P_x is downwards finite. Therefore it must be that P_x has a minimal element.

Next, we seek to show that the minimal element is unique. Assume that p, q are distinct minimal elements of P_x. The stratification condition implies that

$$\bigcup_{r \in \mathsf{P}_{\leq p \cap q}} \mathsf{Z}(r) = \mathsf{Z}(p) \cap \mathsf{Z}(q) \ .$$

Note that the right-hand side is nonempty, as the point $x \in X$ is in both $\mathsf{Z}(p)$ and $\mathsf{Z}(q)$ by definition of P_x. Therefore the set

$$\mathsf{P}_{\leq p \cap \leq q} \neq \emptyset \ .$$

Moreover, there exists an element $l \in \mathsf{P}_{\leq p \cap \leq q}$ such that $x \in \mathsf{Z}(l)$, which implies that $l \in \mathsf{P}_x$. This contradicts that p, q are distinct minimal elements of P_x, since $l \leq p$ and $l \leq q$. Therefore there is a unique minimal element of P_x. This verifies that the map of sets $Z : X \to \mathsf{P}$ is well defined.

Last, we seek to show that the map Z is continuous. The map Z is continuous if the preimage of every downward closed set D is a closed subset of X. By Lemma 53

$$\mathsf{D} = \bigcup_{d \in \mathsf{D}} \mathsf{P}_{\leq d} \ .$$

Taking preimages gives

$$Z^{-1}(\mathsf{D}) = \bigcup_{d \in \mathsf{D}} Z^{-1}(\mathsf{P}_{\leq d}) \ .$$

Assuming that $Z^{-1}(\mathsf{P}_{\leq p}) = X_{\leq p} = \mathsf{Z}(p)$, then

$$Z^{-1}(\mathsf{D}) = \bigcup_{d \in \mathsf{D}} Z^{-1}(\mathsf{P}_{\leq d}) = \bigcup_{d \in \mathsf{D}} X_{\leq d} = \bigcup_{d \in \mathsf{D}} \mathsf{Z}(d) \ .$$

By the continuity condition, the last term is closed, and therefore $Z^{-1}(\mathsf{D})$ is closed.

Therefore, what remains to be shown is that

$$Z^{-1}(\mathsf{P}_{\leq p}) = X_{\leq p} \ .$$

Let $x \in Z^{-1}(\mathsf{P}_{\leq p})$. Applying the map Z gives

$$Z(x) = \mathsf{min}\{r \in \mathsf{P} | x \in \mathsf{Z}(r)\} \in Z \circ Z^{-1}(\mathsf{P}_{\leq p}) = \mathsf{P}_{\leq p} \ .$$

Therefore since $Z(x) = r \leq p$, applying the functor Z gives

$$x \in \mathsf{Z}(r) \hookrightarrow \mathsf{Z}(p) \ .$$

Therefore $x \in \mathsf{Z}(p)$.

Next assume that $x \in \mathsf{Z}(p)$. We seek to show that $x \in Z^{-1}(\mathsf{P}_{\leq p})$. Applying the continuous function Z to x reports the minimum $r \in \mathsf{P}$ such that $x \in \mathsf{Z}(r)$. Again, consider the poset

$$\mathsf{P}_x = \{s \in \mathsf{P} \mid x \in \mathsf{Z}(s)\} \ .$$

The element r is the unique minimal element of this poset, and $p \in \mathsf{P}_x$. Therefore $r \leq p$ since p is also in this poset by assumption. Therefore $x \in Z^{-1}(\mathsf{P}_x) \subset Z^{-1}(\mathsf{P}_{\leq p})$. $\square$

Examples of Stratifications

Example 55. Recall that by Lemma 54, a stratification

$$Z : X \to [n]$$

determines, and is determined by a [n]-filtration

$$\begin{aligned} \mathsf{Z} : \mathsf{P} &\longrightarrow \mathbf{Cls}(X) \\ p &\longrightarrow X_{\leq p} \end{aligned} \ .$$

Note that for a linearly ordered poset, the stratification condition and the continuity condition on a functor $\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(X)$ is automatically satisfied. Furthermore, the poset $[n]$ has a terminal element, and as such $\bigcup_{k \in [n]} \mathsf{Z}(k) = \mathsf{Z}(n) = X$. Therefore, a functor $\mathsf{F} : [n-1] \to \mathbf{Cls}(X)$ uniquely determines a $[n]$-filtration by extending the functor F to $[n]$, by sending the element $n \in [n]$ to $X \in \mathbf{Cls}(X)$. In the case of $n = 1$, this is to say that a [1]-filtration of a topological space X is the data of a closed subset of X.

Example 56. Consider a sequence of closed embeddings

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X := X_n \ .$$

This gives a canonical functor

$$\begin{aligned} [n] &\longrightarrow \mathbf{Cls}(X) \\ i &\longmapsto X_i \end{aligned}$$

which is the same data as a stratification

$$\begin{aligned} Z : X &\longrightarrow [n] \\ x &\longmapsto \mathsf{max}\{i \mid x \in X_i\} \end{aligned}$$

by Lemma 54.

Convention 57. Frequently in what follows, we will name a sequence of closed embeddings into a space X that are indexed by a poset P in order to define a P-filtration or stratification just as in Example 56 . An explicit example of this is the next example.

Example 58. Define $\mathsf{O}(1) := \{-1, 1\}$ to be the group of units of the ring $\mathbb{Z}$. Define the action of $\mathsf{O}(1)$ on $\mathbb{S}^n$ to be the antipodal group action

$$\begin{aligned} \mathsf{O}(1) \times \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ (\lambda, x_1, \ldots, x_{n+1}) &\longrightarrow (\lambda x_1, \ldots, \lambda x_{n+1}) \end{aligned}$$

Define the ***real projective space*** $\mathbb{RP}^n$ to be the quotient of $\mathbb{S}^n$ by this group action

$$\mathbb{RP}^n := (\mathbb{S}^n)_{/\mathsf{O}(1)} \ .$$

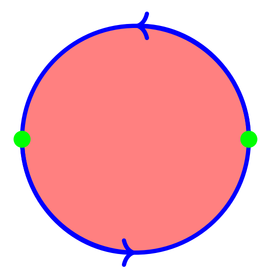
The canonical $[n]$-filtration

$$\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^n$$

respects this $\mathsf{O}(1)$-action. Therefore it induces a $[n]$-filtration on the quotient spaces

$$\mathbb{RP}^0 \hookrightarrow \mathbb{RP}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{RP}^n \ .$$

In the case of $n = 2$, we get a stratification of $\mathbb{RP}^2$ pictured below. To visualize this stratification, restrict $\mathbb{S}^2$ to the northern hemisphere, so $\mathbb{RP}^2$ is pictured as a disk, with antipodal points on the boundary identified



.

The green points are identified and are the 0-stratum, which is the the space $\mathbb{RP}^0 \simeq *$. The blue lines are identified as indicated to get the 1-stratum, which is the space to get the space $\mathbb{RP}^1 \backslash \mathbb{RP}^0$. The red disk is the 2-strataum, which is the space $\mathbb{RP}^2 \backslash \mathbb{RP}^1$. Note that this decomposes the space $\mathbb{RP}^2$ into the union of contractible strata.

Example 59. Let $Z_X : X \to \mathsf{P}$ and $Z_Y : Y \to \mathsf{Q}$ be stratifications of topological spaces. Taking the products in Top induces a ***product stratification***

$$Z_X \times Z_Y : X \times Y \to \mathsf{P} \times \mathsf{Q} \ .$$

A particular case of interest is with $Q = *$ which induces a product stratification

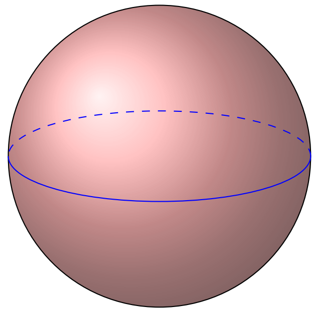
$$X \times Y \to \mathsf{P} \times * \simeq \mathsf{P} \ .$$

Here the stratification map can be factored as the projection onto the X coordinate, composed with the stratification map Z_X.

Example 60. There is a $[n]$-filtration of the odd dimensional spheres

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^{2n+1} \ .$$

given by inclusion into the first coordinates. The case of $n = 1$ determines a [1]-filtration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3$ pictured as follows



The sequence of closed embeddings $(\mathbb{S}^1 \to \mathbb{S}^3 \to \cdots \to \mathbb{S}^{2n+1})$ determines a $[n]$-filtration. Therefore by Lemma 54, this determines a stratification of the (2n+1)-sphere by the poset $[n]$. Throughout this paper, define $\underline{\mathbb{S}^{2n+1}} \in \mathsf{StTop}$ to be this stratification.

Definition 61. Define the ***infinite sphere*** $\mathbb{S}^\infty$ to be the union of spheres

$$\mathbb{S}^\infty := \mathsf{colim}\left(\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \hookrightarrow \dots\right) = \bigcup_{n \in \mathbb{S}^n} \mathbb{S}^{2n+1} .$$

The stratifications $\underline{\mathbb{S}^{2n+1}}$ extends to a stratification

$$\mathsf{colim}(\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \hookrightarrow \dots) = \mathbb{S}^\infty \to \mathsf{colim}([0] \hookrightarrow [1] \hookrightarrow \dots) = \mathbb{Z}_{\geq 0}$$

by taking colimits. Define $\underline{\mathbb{S}^\infty}$ to be this stratification of the infinite sphere.

Example 62. This example is the main example of a stratification used in this paper. Each sphere $\mathbb{S}^{2n+1}$ is canonically considered as a subset of $\mathbb{C}^{n+1}$ by the identification of the spaces $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$. Consider the diagonal action of $\mathbb{T}$ on $\mathbb{C}^{i+1}$

$$\begin{array}{ccccc} \mathbb{T} \times \mathbb{C}^{i+1} & \hookrightarrow & \mathbb{C} \times \mathbb{C}^{i+1} & \xrightarrow{\mathsf{scale}} & \mathbb{C} \\ (p, (x_1, \dots, x_i)) & \longmapsto & (p, (x_1, \dots, x_i)) & \longrightarrow & (px_1, \dots, px_i) \end{array} .$$

The scale map is the canonical scaling map that defines a complex vector space structure on

$\mathbb{C}^i$. This action restricts to an action on $\mathbb{S}^{2i+1}$. Define the ***complex projective space*** $\mathbb{CP}^i$ to be the quotient

$$\mathbb{CP}^i := \mathbb{S}^{2i+1}{}_{/\mathbb{T}}$$

by this $\mathbb{T}$ action. Furthermore, let Span be the quotient map

$$\mathsf{Span} : \mathbb{S}^{2i+1} \longrightarrow \mathbb{CP}^i \quad .$$

The inclusion maps $\mathbb{S}^{2i+1} \hookrightarrow \mathbb{S}^{2j+1}$ for $i \leq j$ are $\mathbb{T}$-equivariant with respect to the $\mathbb{T}$-action, which induces a sequence of closed embeddings of the quotient spaces

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, hook] \arrow[d] & \mathbb{S}^3 \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] \arrow[d] & \mathbb{S}^{2n+1} \arrow[d] \\
\mathbb{CP}^0 \arrow[r, hook] & \mathbb{CP}^1 \arrow[r, hook] & \dots \arrow[r, hook] & \mathbb{CP}^n
\end{tikzcd} .$$

Again by Example 56, this determines a stratification of the topological space $\mathbb{CP}^n$ by the poset $[n]$. Define $\underline{\mathbb{CP}^n}$ to be this stratification of $\mathbb{CP}^n$. Explicitly, the stratification is given by the map of topological spaces

$$\begin{aligned} Z : \mathbb{CP}^i &\longrightarrow [i] \\ \mathsf{Span}(t_0, \dots, t_i) &\longmapsto \mathsf{max}\{k \mid t_k \neq 0\} \end{aligned} .$$

Just as in Example 60, the stratification of $\mathbb{CP}^i$ extends to a stratification

$$Z : \mathbb{CP}^\infty := \mathsf{colim}\left(\mathbb{CP}^0 \to \mathbb{CP}^1 \to \dots\right) \to \mathsf{colim}\left([0] \to [1] \to \dots\right) = \mathbb{Z}_{\geq 0}$$

upon taking colimits. Throughout this paper, $\underline{\mathbb{CP}^\infty}$ will refer to this stratified space.

Cones of Stratified Spaces

There are particular constructions in the category StTop that will be used frequently throughout this paper. One in particular that is prevalent in the theory of conically stratified spaces of [10] and conically smooth stratified spaces [5] is that of the cone of a stratified space. We introduce this construction now.

Notation 63. Define the ***open cone*** of a topological space X as the pushout

$$\begin{tikzcd} X \times \{0\} \arrow[r, hook] \arrow[d, "!"'] & X \times \mathbb{R}_{\geq 0} \arrow[d] \\ \{0\} \arrow[r] & \mathsf{C}(X) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} .$$

Define the ***closed cone*** of a topological space X as the pushout

$$\begin{tikzcd} X \times \{0\} \arrow[r, hook] \arrow[d, "!"'] & X \times [0,1] \arrow[d] \\ \{0\} \arrow[r] & \overline{\mathsf{C}}(X) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} .$$

Observation 64. A map of topological spaces $f : X \to Y$ determines a canonical map $\mathsf{C}(X) \to \mathsf{C}(Y)$ induced by the map

$$\mathbb{R}_{\geq 0} \times X \xrightarrow{\mathsf{id} \times f} \mathbb{R}_{\geq 0} \times Y \ .$$

Therefore the cone can be considered as a functor

$$\mathsf{C}(-) : \mathsf{Top} \to \mathsf{Top} \ .$$

Similarly, the closed cone defines a functor

$$\overline{\mathsf{C}}(-) : \mathsf{Top} \to \mathsf{Top} \ .$$

Notation 65. Define $\mathsf{C}^n(X)$ to be the topological space obtained by applying the cone functor n times. Similarly, define $\overline{\mathsf{C}^n}(X)$ to be the topological space obtained by applying the closed cone functor n times.

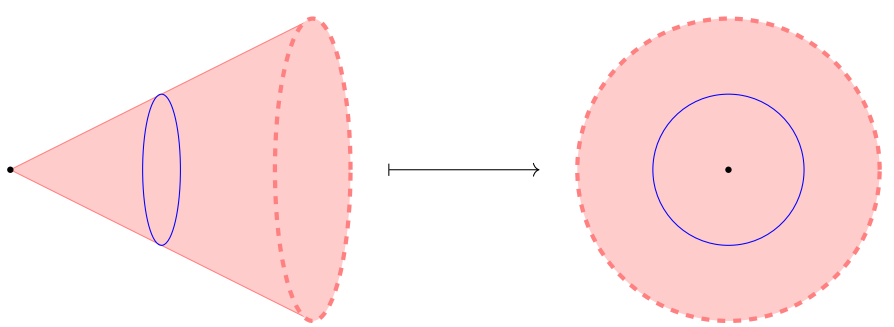
Example 66. Note the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^{n-1} \times \{0\} \arrow[r, "!"] \arrow[d, hook] & * \arrow[d, hook, "0"] \\
\mathbb{S}^{n-1} \times \mathbb{R}_{\geq 0} \arrow[r, "\mathsf{scale}"'] & \mathbb{R}^n
\end{tikzcd},$$

where the scale map is given by canonically including $\mathbb{S}^{n-1} \times \mathbb{R}_{\geq 0}$ into $\mathbb{R}^n \times \mathbb{R}$, and then scaling using the canonical vector space structure on $\mathbb{R}^n$. This induces a homeomorphism from the cone on $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

$$\begin{aligned}
\mathsf{C}(\mathbb{S}^{n-1}) &\xrightarrow{\qquad\simeq\qquad} \mathbb{R}^n \\
[(x_1, \ldots, x_n), t] &\longmapsto (tx_1, \ldots, tx_n)
\end{aligned}.$$

Here is a picture of the identification $\mathsf{C}(\mathbb{S}^1) \simeq \mathbb{R}^2$



The blue circle is the inclusion of $\mathbb{S}^1$ into the cone and into $\mathbb{R}^2$, and the black point is the cone point mapping to $(0,0) \in \mathbb{R}^2$.

Notation 67. Let P be a poset. Define the ***left cone*** $\mathsf{P}^{\triangleleft}$ to be the poset determined by adjoining a minimal element $-$ to P. The category $\mathsf{P}^{\triangleleft}$ is presented by the pushout

$$\begin{tikzcd}
\mathsf{P} \times \{0\} \arrow[r] \arrow[d] & \mathsf{P} \times [1] \arrow[d] \\
\ast \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathsf{P}^{\triangleleft}
\end{tikzcd}$$

in Poset.

Observation 68. Consider the stratification of $\mathbb{R}_{\geq 0}$ induced via Lemma 54 by the $[1]$-filtration

$$\{0\} \hookrightarrow \mathbb{R}_{\geq 0} \ .$$

For a stratification $Z : X \to \mathsf{P}$ there is a canonical stratification of $\mathsf{C}(X)$

$$\begin{tikzcd}
X \times \{0\} \arrow[rr] \arrow[dr] \arrow[dd] & & X \times \mathbb{R}_{\geq 0} \arrow[dr] \arrow[dd] & \\
& \mathsf{P} \times \ast \arrow[rr] \arrow[dd] & & \mathsf{P} \times [1] \arrow[dd] \\
\{0\} \arrow[rr] \arrow[dr] & & \mathsf{C}(X) \arrow[dr, dashed] & \\
& \ast \arrow[rr] & & \mathsf{P}^{\triangleleft}
\end{tikzcd}$$

induced by the product stratification $X \times \mathbb{R}_{\geq 0} \to \mathsf{P} \times [1]$ as in Example 59. This stratification is refered to as the ***conical stratification***. Conical stratifications define a functor

$$\mathsf{C}(-) : \mathsf{StTop} \to \mathsf{StTop} \ .$$

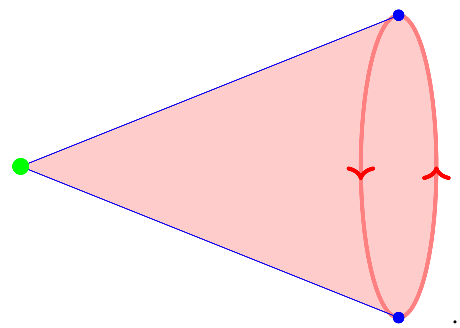
Similarly the closed cone defines a functor

$$\overline{\mathsf{C}}(-) : \mathsf{StTop} \to \mathsf{StTop} \ .$$

Example 69. Consider the stratification $\mathbb{RP}^1 \to [1]$ in Example 58, where the 0-strata is given by the subspace $\mathbb{RP}^0 \cong *$. The conical stratification of the stratification on $\mathbb{RP}^1$ gives a stratification

$$\mathsf{C}(\mathbb{RP}^1) \to [1]^{\triangleleft} \simeq [2]$$

that is depicted in the following diagram



Here the green point is the cone point, which is the 0-stratum, the blue line[9] is the 1-stratum, and the red is the preimage of 2-stratum.

Cosimplicial Stratified Spaces

We construct the standard cosimplicial stratified space and extended standard cosimplicial stratified space functors in this section. These two functors will be used in the definitions of the ∞-category of conically smooth stratified spaces and in the construction of the exit path ∞-category.

Definition 70. The ***k-simplex*** is the topological space

$$\Delta^k := \left\{ \left(\{0, \ldots, k\} \xrightarrow{\alpha} [0,1] \middle| \sum_{i \in \{0,\ldots,n\}} \alpha(i) = 1 \right) \right\}$$

[9]Note the two blue lines in the picture are identified in the space $\mathsf{C}(\mathbb{RP}^1)$.

Given a map $[k] \xrightarrow{\varphi} [j]$ between posets, define the map

$$\begin{array}{rcl} \varphi_\Delta : \Delta^k & \longrightarrow & \Delta^j \\ \left(\{0,\dots,k\} \xrightarrow{\alpha} [0,1]\right) & \longmapsto & \left(\{0,\dots,j\} \xrightarrow{\varphi_\alpha} [0,1]\right) \\ & & l \longmapsto \begin{cases} \sum_{i\in\varphi^{-1}(l)} \alpha(i) & \varphi^{-1}(l) \neq \emptyset \\ 0 & \varphi^{-1}(l) = \emptyset \end{cases} \end{array} .$$

Definition 71. Define the ***standard cosimplicial topological space*** to be the functor

$$\begin{array}{rcl} \Delta & \longrightarrow & \mathsf{Top} \\ [k] & \longmapsto & \Delta^k \\ \left([k] \xrightarrow{\varphi} [j]\right) & \longmapsto & \left(\Delta^k \xrightarrow{\varphi_\Delta} \Delta^j\right) \end{array} .$$

Observation 72. There is a canonical inclusion of the $(n-1)$-simplex into the n-simplex

$$\begin{array}{rcl} \Delta^{n-1} & \hookrightarrow & \Delta^n \\ (\{0,\dots,n-1\} \xrightarrow{\alpha} [0,1]) & \longmapsto & (\{0,\dots,n-1\} \coprod \{n\} \xrightarrow{\alpha \coprod 0} [0,1]) \end{array}$$

that extends a map $\{0,\dots,n-1\}$ to the set $\{0,\dots,n\}$ by sending n to $0 \in [0,1]$. The extension by zero map is the unique extension that satisfies the property of being an element of Δ^n:

$$\begin{tikzcd} \{0,\dots,n-1\} \arrow[rr, hook] \arrow[dr, "\alpha"'] & & \{0,\dots,n\} \arrow[dl, dashed, "\mathsf{ext}_0"] \\ & {[0,1]} & \end{tikzcd}$$

The inclusions $\Delta^{i-1} \to \Delta^i$ induces a $[n]$-filtration of Δ^n

$$\Delta^0 \hookrightarrow \Delta^1 \hookrightarrow \cdots \hookrightarrow \Delta^n .$$

The explicit map for this stratification is

$$\begin{array}{rcl} \mathsf{st}(n) : \Delta^n & \longrightarrow & [n] \\ \Big(\{0,\ldots,n\} \xrightarrow{\alpha} [0,1]\Big) & \longmapsto & \mathsf{max}\{i|\alpha(i) \neq 0\} \end{array}.$$

Furthermore, for each $\phi : [k] \to [j]$, the cosimplicial maps $\phi_\Delta : \Delta^k \to \Delta^j$ are stratified maps.

Notation 73. Let $\underline{\Delta^k}$ be the stratified space $\mathsf{st}(k) = \big(\Delta^k \to [k]\big)$.

Observation 74. There is an homeomorphism of spaces $\overline{\mathsf{C}}(*) \cong \Delta^1$ given by

$$\begin{array}{rcl} \overline{\mathsf{C}}(*) = [0,1] \times * & \longrightarrow & \Delta^1 \\ (t,*) & \longmapsto & (1-t,t) \end{array}.$$

Note that we use the canonical identification of $\overline{\mathsf{C}}(*) \cong [0,1] \times *$ by inspecting the pushout diagram of the cone. Furthermore, there is a homeomorphism of topological spaces between $\overline{\mathsf{C}}(\underline{\Delta^{n-1}}) \cong \underline{\Delta^n}$ implimented by the map

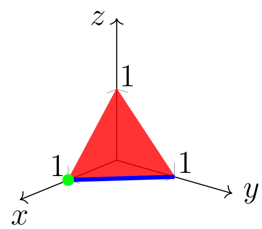
$$\begin{array}{rcl} \overline{\mathsf{C}}(\Delta^{n-1}) & \longrightarrow & \Delta^n \\ [((\alpha_1,\ldots,\alpha_n),t)] & \longmapsto & (1-t, t\alpha_1,\ldots,t\alpha_n) \end{array}.$$

Together, these two isomorphisms imply the equivalence

$$\overline{\mathsf{C}^n}(*) \cong \underline{\Delta^n}$$

in StTop.

Example 75. The stratified space $\underline{\Delta^2}$ is



.

Definition 76. The ***standard cosimplicial stratified space*** is the functor

$$\begin{aligned} \mathsf{st} : \mathbf{\Delta} &\longrightarrow \mathsf{StTop} \\ [k] &\longmapsto \underline{\Delta^k} \\ \left([k] \xrightarrow{\varphi} [j]\right) &\longmapsto \left(\underline{\Delta^k} \xrightarrow{\varphi_\Delta} \underline{\Delta^j}\right) \end{aligned} .$$

Furthermore there is a homeomorphism of topological spaces.

Definition 77. The ***extended k-simplex*** Δ_e^k is defined to be the topological space

$$\Delta_e^k := \left\{ \left(\{0, \ldots, k\} \xrightarrow{\alpha} \mathbb{R} \middle| \sum_{i \in \{0,\ldots,n\}} \alpha(i) = 1 \right) \right\}$$

This can be thought of as a thickening of the $k-$simplex.

Definition 78. The ***extended cosimplicial topological space*** is the functor

$$\begin{aligned} \Delta &\longrightarrow \mathsf{Top} \\ [k] &\longmapsto \Delta_e^k \\ \left([k] \xrightarrow{\varphi} [j]\right) &\longmapsto \left(\Delta_e^k \xrightarrow{\varphi_{\Delta_e}} \Delta_e^j\right) \end{aligned} .$$

Definition 79. There is a canonical fully faithful functor

$$\mathsf{Top} \to \mathsf{StTop}$$

by sending a topological space X to the constant stratification $X \to *$. The extended cosimplicial stratified space together with the trivial stratification determines a functor

$$\begin{aligned} \mathsf{st}_e : \mathbf{\Delta} &\longrightarrow \mathsf{Top} \longrightarrow \mathsf{StTop} \\ k &\longmapsto \Delta^k_e \longmapsto \left(\Delta^k_e \to *\right) \end{aligned}$$

Define the stratfied extended k-simplex as

$$\underline{\Delta^k_e} := \mathsf{st}_e(k) = (\Delta^k_e \to *) \ .$$

Conically Smooth Stratified Spaces

The theory of conically smooth stratified spaces was developed in [5]. One benefit of conically smooth stratified spaces is that they admit an ∞-category of exit paths. Exit-path categories are developed in [4]. We give a brief recollection of these topics here, with a special focus on the results that will be used in the remainder of this paper.

C^0 Stratified Spaces

Definition 80. Define the category of C^0-stratified spaces to be the smallest full subcategory of StTop such that

(1) The empty set $\emptyset$ is a C^0 stratified space, stratified by the empty poset.

(2) If $X \to \mathsf{P}$ is a C^0 stratified space and X and P are compact, then the conical stratification

$$\mathsf{C}(X) \to \mathsf{P}^{\triangleleft}$$

is a C^0 stratified space.

(3) If $X \to \mathsf{P}$ and $Y \to \mathsf{Q}$ are C^0 stratified spaces, then the product stratification

$$X \times Y \to \mathsf{P} \times \mathsf{Q}$$

is a C^0 stratified space.

(4) If $X \to \mathsf{P}$ is a C^0 stratified space, and there is an open embedding of stratified spaces

$$\begin{tikzcd} U \arrow[r, hook, "\mathsf{open}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \mathsf{Q} \arrow[r, hook] & \mathsf{P} \end{tikzcd} ,$$

(5) If $X \to \mathsf{P}$ is a stratified topological space admitting an open cover by C^0 stratified spaces, then $X \to \mathsf{P}$ is a C^0 stratified space.

Example 81. The stratified space

$$\mathbb{R}_{\geq 0} \longrightarrow [1] , \qquad x \mapsto \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases} ,$$

is a C^0 stratified space, since $\mathsf{C}(*) \cong \mathbb{R}_{\geq 0}$.

Example 82. Consider the 1-stratum of the previous example

$$\begin{tikzcd} \mathbb{R}_{>0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}_{\geq 0} \arrow[d] \\ \{1\} \arrow[r] & [1] \end{tikzcd} .$$

Since $\mathbb{R}_{>0} \hookrightarrow \mathbb{R}_{\geq 0}$ is an open embedding, the stratification $\mathbb{R}_{>0} \to \{1\}$ is a C^0-stratification. By identifying $\mathbb{R}_{>0} \cong \mathbb{R}$, then $\mathbb{R} \to *$ is a C^0 stratification. By taking products, the product stratification

$$\mathbb{R}^n \to *$$

is a C^0-stratification.

Example 83. A C^0-manifold M is stratified by a point is an example of a C^0-stratified space, since it admits an open cover by Euclidean spaces.

Definition 84. A C^0***-basic*** is a C^0 stratified space of the form $\mathbb{R}^n \times \mathsf{C}(L) \to \mathsf{P}^{\triangleleft}$ where $n \geq 0$, $L \to \mathsf{P}$ is a compact C^0 stratified space, and the stratification of $\mathbb{R}^n \times \mathsf{C}(L) \to \mathsf{P}^{\triangleleft}$ is the product of the trivial stratification $\mathbb{R}^n \to *$ and the conical stratification $\mathsf{C}(L) \to \mathsf{P}^{\triangleleft}$.

Definition 85. The ***depth of a stratified space*** $X \to \mathsf{P}$ is defined to be the depth of its image in the stratifying poset P (Definition 14).

Lemma 86. *Let $X \to \mathsf{P}$ be a C^0-stratified space. The p-stratum X_p is a topological manifold.*

Proof. This is Corollary 2.3.5 of [5]. $\square$

Definition 87. Let $Z : X \to \mathsf{P}$ be a nonempty C^0-stratified space. The ***local dimension*** of $x \in X$ is the Lebesgue covering dimension of X at $x \in X$. Denote this value as $\dim_x(X)$. The ***dimension*** of $Z : X \to \mathsf{P}$ is the supremum of the local dimensions

$$\dim(X) = \sup_{x \in X} \dim_x(X) \ .$$

Definition 88. Let $Z : X \to \mathsf{P}$ be a C^0-stratified space. The ***local topological depth*** at $x \in X$ is defined to be the difference in local dimensions

$$\mathsf{depth}_x(X) = \dim_x(X) - \dim_x(X_{Z(x)})$$

The ***topological depth*** of $Z : X \to \mathsf{P}$ is defined to be the supremum of the local depths

$$\mathsf{depth}(X) := \sup_{x \in X} \mathsf{depth}_x(X) \ .$$

Example 89. Let $\mathbb{R}^n \times \mathsf{C}(Z)$ be a C^0-basic. Then

$$\dim(\mathbb{R}^n \times \mathsf{C}(Z)) = \dim(Z) + 1 + n$$

and

$$\mathsf{depth}(\mathbb{R}^n \times \mathsf{C}(Z)) = \mathsf{depth}(Z) + 1 \ .$$

Conically Smooth Stratified Spaces

We now endow the notion of C^0 stratified spaces with more regularity: that of conically smooth stratified space. We recall the definition of a conically smooth stratified space, and then give some explanation of the terms involved. See [5] for a thorough explanation of conically smooth stratified spaces.

Definition 90 (Heuristic (see [5] for details))**.** We simultaneously define the notion of a ***conically smooth stratified space*** and of a ***conically smooth map*** between such.

(1) A ***conically smooth stratified space*** is

- A C^0-stratified space $X \to \mathsf{P}$.
- A ***conically smooth atlas***, which is a collection

$$\mathcal{U} := \left\{ \left(n_\alpha, Z_\alpha, \mathbb{R}^{n_\alpha} \times \mathsf{C}(Z_\alpha) \xrightarrow{\varphi_\alpha} X \right) \right\}_\alpha$$

  in which each $n_\alpha \geq -1$ is an integer, each Z_α is a compact conically smooth stratified space, and each φ_α is an open embedding between stratified spaces.

These data are required to satisfy three conditions:

- The collection $\mathcal{U}$ is an open cover of X.
- The transition maps in $\mathcal{U}$ are ***conically smooth***.
- $\mathcal{A}$ is maximal with respect to the above two conditions.

A conically smooth stratified space $(X \to \mathsf{P}, \mathcal{U})$ is often simply denoted as its underlying topological space X if the other data are understood.

(2) Let X and Y be conically smooth stratified spaces. A map between stratified spaces $X \xrightarrow{f} Y$ is ***conically smooth*** if it is with respect to each member of the atlases of X and of Y. In turn, a map between basics $\mathbb{R}^n \times \mathsf{C}(I) \xrightarrow{f} \mathbb{R}^m \times \mathsf{C}(J)$ is conically smooth if it is away from cone loci and the limit of the difference quotient along each cone locus exists and is conically smooth.

What may appear as a circular definition is, in fact, an inductive definition: induction on depth. See [5] for all details.

Example 91. The notion of a conically smooth stratified space is inductive. The base case of a conically smooth stratified space is the C^0-stratification

$$\emptyset \to \emptyset \ .$$

This space is defined to have a unique conically smooth atlas.

Example 92. Let M be a smooth manifold. M is canonically regarded as a stratified space by taking the trivial stratification

$$M \to * \ .$$

The stratification $M \to *$ has a canonical conically smooth atlas by taking the smooth atlas of M.

The regularity of a conically smooth stratified space X is precisely to enable the definition of a ***link***. Its construction is a theorem from [5], which we state informally below, though first make the following.

Observation 93. Let $n \geq -1$ be an integer. Let L be a compact conically smooth stratified space. There is a diagram among conically smooth stratified spaces:

$$\begin{tikzcd} \mathbb{R}^n \times L \arrow[r, "\text{inclusion}_0"] \arrow[d, "\mathsf{pr}"'] & \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times L \arrow[d, "\text{quotient}"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^n \times \mathsf{C}(L) \ . \end{tikzcd} \tag{1.3}$$

The next theorem from [5] references the notion of a ***conically smooth stratified space with boundary***, and ***proper constructible bundles***, also defined in [5].

Theorem 94 ([5])**.** *Let $X_0 \subseteq X$ be a closed stratum in a conically smooth stratified space. The collection of diagrams (1.3) indexed by the basics in the atlas for X that intersect X_0 patch together as a diagram*

$$\begin{tikzcd} \mathsf{Link}_{X_0}(X) \arrow[r, "\text{inclusion}"] \arrow[d, "\pi"'] & \mathsf{Bl}_{X_0}(X) \arrow[d] \\ X_0 \arrow[r] & X, \end{tikzcd} \tag{1.4}$$

in which $\mathsf{Bl}_{X_0}(X)$ is a conically smooth stratified space with boundary $\mathsf{Link}_{X_0}(X)$, and the downward maps are proper constructible bundles

Example 95. Let $W \hookrightarrow M$ be a properly embedded smooth submanifold of a smooth manifold. Consider the stratification

$$(W \subseteq M) \ := \left(\begin{array}{ccc} M & \longrightarrow & [1] \\ & & \\ x & \longmapsto & \begin{cases} 0 & x \in W \\ 1 & x \neq W \end{cases} \end{array} \right).$$

The smooth structure of M conically supplies a conically smooth structure on this stratified space $(W \subseteq M)$. As so, $W \subseteq (W \subseteq M)$ is a closed stratum. In this case, the diagram (1.4) can be identified as the diagram

$$\begin{tikzcd} \mathbb{S}(\mathsf{N}_{W \subseteq M}) \arrow[r, "\text{inclusion}"] \arrow[d] & \mathsf{Bl}_W(M) \arrow[d] \\ W \arrow[r] & (W \subseteq M), \end{tikzcd}$$

in which the top right term is the (real) blow-up of M along W, and the top left term is the unit normal bundle of the embedding $W \hookrightarrow M$.

Definition 96. Define the category Strat to be the category of conically smooth stratified spaces, with morphisms consisting of conically smooth maps.

Definition 97. The category Strat can be regarded as a Kan-enriched category $\underline{\mathsf{Strat}}$ as follows.

+) An object of $\underline{\mathsf{Strat}}$ is defined to be an object of Strat, that is, a conically smooth stratified space.

+) The Kan complex between two stratified spaces $X \to Y$ and $Y \to \mathsf{Q}$ is defined to be

$$\mathsf{Hom}_{\underline{\mathsf{Strat}}}(X, Y) = \mathsf{Hom}_{\mathsf{Strat}}(X \times \Delta^{\bullet}_e, Y)$$

Remark 98. The simplicial set $\mathsf{Hom}_{\mathsf{Strat}}(X \times \Delta_e^\bullet, Y)$ is indeed a Kan complex by Lemma 4.1.4 of [5].

Definition 99. Define the ∞-category $\mathcal{S}\mathsf{trat}$ as the simplicial nerve of $\underline{\mathsf{Strat}}$.

Definition 100. A conically smooth map $X \xrightarrow{f} Y$ is a stratified homotopy equivalence if and only if it fits into a diagram in Strat

$$\begin{tikzcd}
 & X \arrow[rr, "\mathsf{id}_X"] \arrow[d, "\mathsf{id} \times \mathsf{const}_0"'] & & X \arrow[d, "\mathsf{id} \times \mathsf{const}_1"] \\
 & X \times \mathbb{R} \arrow[rr, "H"] & & X \times \mathbb{R} \\
Y \arrow[r, "g"] \arrow[d, "\mathsf{id} \times \mathsf{const}_0"'] & X \arrow[r, "f"] \arrow[u, "\mathsf{id} \times \mathsf{const}_1"] & Y \arrow[r, "g"] \arrow[d, "\mathsf{id} \times \mathsf{const}_0"] & X \arrow[u, "\mathsf{id} \times \mathsf{const}_0"'] \\
Y \times \mathbb{R} \arrow[rr, "H'"] & & Y \times \mathbb{R} & \\
Y \arrow[rr, "\mathsf{id}_Y"] \arrow[u, "\mathsf{id} \times \mathsf{const}_1"] & & Y \arrow[u, "\mathsf{id} \times \mathsf{const}_1"'] &
\end{tikzcd}$$

In which both H and H' lie over $\mathbb{R}$. That is, H and H' fit into the following diagrams

$$\begin{tikzcd}
X \times \mathbb{R} \arrow[rr, "H"] \arrow[dr, "\pi_2"'] & & Y \times \mathbb{R} \arrow[dl, "\pi_2"] \\
 & \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times \mathbb{R} \arrow[rr, "H'"] \arrow[dr, "\pi_2"'] & & X \times \mathbb{R} \arrow[dl, "\pi_2"] \\
 & \mathbb{R} &
\end{tikzcd}.$$

Lemma 101. *The natural functor* $\mathsf{Strat} \to \mathcal{S}\mathsf{trat}$ *induces an equivalence of* $\infty-$*categories*

$$\begin{tikzcd}
\mathsf{Strat} \arrow[r] \arrow[d] & \mathcal{S}\mathsf{trat} \\
\mathsf{Strat}[H^{-1}] \arrow[ur, dashed, "\simeq"'] &
\end{tikzcd}$$

from the localization of the ordinary category Strat *with respect to stratified homotopy equivalences and the* ∞*-category associated to the Kan-enriched category* Strat.

Proof. This is Theorem 2.4.5 in [4]. $\square$

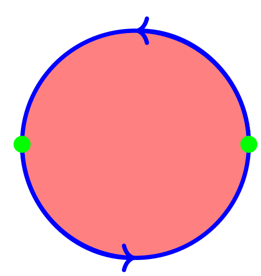
Observation 102. Let $\mathbb{R}^{n_1} \times \mathsf{C}(\mathbb{S}^{d_1-1}) \hookrightarrow \mathbb{R}^{n_2} \times \mathsf{C}(\mathbb{S}^{d_2-1})$ be a map between basics, such that the canonical map $\mathbb{R}^{n_1+d_1} \hookrightarrow \mathbb{R}^{n_2+d_2}$ induced by the scaling map is smooth. Then the transition map between basics is conically smooth.

Exit Path ∞-Categories

This section introduces the exit path ∞-category of a conically stratified topological space. Before stating formally what the exit path ∞-category is, we work through an informal example to give some geometric intuition.

Given a space X, the fundamental groupoid $\Pi_1(X)$ is the category whose objects are points in a space X, morphisms are paths in X up to homotopy (rel end-points), and composition is given by concatenating paths. After fixing a stratification on a space X , we can consider a subcategory of $\Pi_1(X)$, consisting of those paths that "exit" lower-dimensional strata into a higher-dimensional strata. The justification of considering the exit path category is seen by Theorem 112, which states that the ∞-category of constructible sheaves of spaces (also known as constructible ∞-stacks) is free on the exit-path ∞-category – here, a sheaf is constructible if its restriction to each stratum is locally constant. Furthermore, given a conically smooth staratified space $X \to \mathsf{P}$, we can recover the underlying space X from its exit path ∞-category by formally inverting all of the morphisms in the exit path ∞-category (See Theorem 3.3.12 of [4]).

Recall the stratification $\mathbb{RP}^2 \to [2]$ of Example 58



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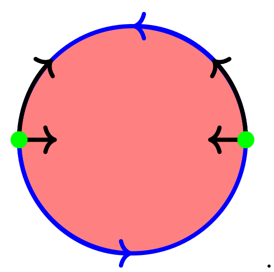
To get some intuition for the exit path ∞-category, we informally seek to identify the ∞-category $\mathsf{Exit}(\underline{\mathbb{RP}^2})$. We begin by noting that each stratum is contractible, since the i-stratum is an i-dimensional disk for each $i \in [2]$. Therefore

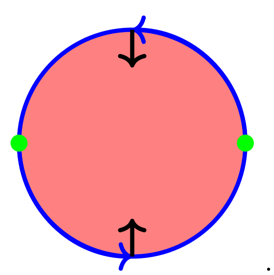
$$\mathsf{Obj}(\mathsf{Exit}(\underline{\mathbb{RP}^2})) \simeq \{0, 1, 2\} \ .$$

Starting at the 0-stratum. There are two ways to exit into both the 1-stratum and the 2-stratum picture in black



.

There is also essential two ways to exit from the 1-stratum to the 2-stratum up to homotopy, which again are pictured in black



.

Through careful identification of the exiting paths, the exit path category of $\mathbb{RP}^2$ can be identified as

$$\mathsf{Exit}(\underline{\mathbb{RP}^2}) = \begin{tikzcd} & 1 \arrow[dr, "O(1)"] & \\ 0 \arrow[ur, "O(1)"] \arrow[rr, "O(1)"'] & & 2 \end{tikzcd}$$

where the composition is given by group operation on $O(1)$. This discusion introducing the exit path ∞-category was meant to be informal, and give some geometric intuition for the exit path ∞-category. We now proceed to give a formal introduction to the exit path ∞-category.

Definition 103. The ***exit path ∞-category*** functor is the restricted yoneda functor

$$\mathsf{Exit} : \mathcal{S}\mathsf{trat} \xrightarrow{\mathsf{yoneda}} \mathsf{PShv}(\mathcal{S}\mathsf{trat}) \xrightarrow{\mathsf{st}^*} \mathsf{PShv}(\Delta) .$$

where st is the standard cosimplicial stratified space functor of Definition 76.

The following lemma identifies the spaces of 0-simplicies and the space of 1-simplices of the simplicial space $\mathsf{Exit}(X)$ for a conically smooth stratified space $X \to \mathsf{P}$.

Lemma 104. *Let $X \to \mathsf{P}$ be a conically smooth stratified space. The space of 0-simplices of $\mathsf{Exit}(X)$ is canonically identified*

$$\mathsf{Exit}(X)([0]) = \coprod_{p \in \mathsf{P}} X_p$$

as the coproduct of the underlying spaces of the strata of X. For each $p, q \in \mathsf{P}$, the space of 1-simplices from X_p to X_q is canonically identified

$$(X_p \times X_q) \underset{\mathsf{Exit}(X)(\{0\} \amalg \{1\})}{\times} \mathsf{Exit}(X)([1]) \simeq \mathsf{Link}_{X_p}(X)_q$$

as the underlying space of p'-stratum of the link of the p-stratum.

Proof. This is Lemma 3.3.5 of [4] . □

Corollary 105. The functor $\mathsf{Exit}(\mathcal{S}\mathsf{trat}) \to \mathsf{PShv}(\Delta)$ takes values in complete Segal spaces, and therefore presents an ∞-category.

Proof. This is Corollary 3.3.6 of [4]. □

Observation 106. Let $X \to \mathsf{P}$ be a conically smooth stratification. There is a canonical functor between ∞-categories $\mathsf{Exit}(\underline{X}) \to \mathsf{P}$ presented by

$$\begin{aligned} \mathsf{Exit}(\underline{X})([n]) := \mathsf{Hom}_{\mathcal{S}\mathsf{trat}}\left(\underline{\Delta}^n, \underline{X}\right) &\longrightarrow \mathsf{Hom}_{\mathsf{Poset}}\left([n], \mathsf{P}\right) \\ ((\Delta^n \to [n]) \to (X \to \mathsf{P})) &\longmapsto ([n] \to \mathsf{P}) \end{aligned}$$

Corollary 107. Let $X \to [n]$ be a closed $[n]$-filtration. The simplicial space $\mathsf{Exit}(\underline{X})$ is a complete Segal space, and therefore presents an ∞-category.

<u>Constructible Sheaves</u>

We recall here one of the main theorems of [4] and [11] regarding the relation of the exit path ∞-category to constructible sheaves (Theorem 112) . We first work up to stating what a constructible sheaf is, before giving the theorem.

Definition 108. Consider the functor

$$\begin{aligned} \mathcal{S}\mathsf{paces} &\longrightarrow \mathsf{Shv}(X) \\ Z &\longmapsto \mathsf{Maps}(-, Z) \end{aligned} .$$

The essential image of this functor is the ***constant sheaves*** on X.

Definition 109. The ***locally constant sheaves*** on a space X is the full ∞-subcategory of sheaves on X

$$\mathsf{Shv}^{\mathsf{loc}}(X) \subset \mathsf{Shv}(X)$$

on those presheaves F such that there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for all $\alpha \in A$

$$\mathsf{F}_{|U_\alpha} \in \mathsf{Shv}(U_\alpha)$$

is constant.

Definition 110. For a stratified space $X \to \mathsf{P}$, the ***constructible sheaves*** on X is the full ∞-subcategory of sheaves on X

$$\mathsf{Shv}^{\mathsf{cbl}}(X) \subset \mathsf{Shv}(X)$$

on those sheaves F such that for all $p \in \mathsf{P}$, the restriction of F to the p-stratum

$$\mathsf{F}_{|X_p} \in \mathsf{Shv}(X_p)$$

is a locally constant sheaf.

Remark 111. See Lemma 2.2.1 of [4] for equivalent conditions of a sheaf being constructible.

Theorem 112. *Let $X \to \mathsf{P}$ be a conically stratified space. Then there is an equivalence of ∞-categories*

$$\mathsf{Fun}(\mathsf{Exit}(\underline{X}), \mathcal{S}\mathsf{paces}) \simeq \mathsf{Shv}^{\mathsf{cbl}}(X)$$

between the ∞-category of copresheaves on $\mathsf{Exit}(\underline{X})$ and those sheaves on X which are constructible with respect to the stratification $X \to \mathsf{P}$.

Proof. See Lemma 3.3.9 of [4], Section $A.9$ of [11]. $\square$

Observation 113. Theorem 112 only requires a stratification $X \to \mathsf{P}$ to be conical, instead of conically smooth. However, the exit path ∞-category of a conically smooth stratified space is often much easier to identify, as seen by Lemma 104. This can be seen in particular examples such as in Proposition 162 and Theorem 165 which identify the exit path categories of conically smooth stratifications of $\mathbb{S}^\infty$ and $\mathbb{CP}^\infty$ respectively.

Closed P-filtrations are Conical

Recall the $\mathbb{Z}_{\geq 0}$-filtration of Example 62

$$\mathsf{colim}(\mathbb{CP}^0 \to \mathbb{CP}^1 \to \dots) = \mathbb{CP}^\infty \ .$$

More generally consider the following class of stratifications.

Definition 114. A P-filtration $\mathsf{Z} : \mathsf{P} \to \mathbf{Cls}(M)$ is a ***closed*** P***-filtration*** if:

-) M is a smooth manifold.

-) The poset P is downwards closed and linearly ordered.

-) For each $p \leq q \in \mathsf{P}$, the inclusion

$$\mathsf{Z}(p) \hookrightarrow \mathsf{Z}(q)$$

is a closed embedding of smooth manifolds.

This section seeks to show that there is a notion of an exit path ∞-category of a closed P-filtration $X \to \mathsf{P}$. The proof relies on induction on the depth of the poset P. In the case that P is finite, we show that the stratification $X \to \mathsf{P}$ is conically smooth. In [4], it is shown that the exit path ∞-category exists for a conically smooth stratified space. Consider however the P-filtration of $\mathbb{CP}^\infty$

$$\left(\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \dots\right) = \mathbb{CP}^\infty \ .$$

Conically smooth stratified spaces have the property that the space is locally finite dimensional, which fails in the case of $\mathbb{CP}^\infty$. Therefore this space is not conically smooth.

However, the closed P-filtration and functorality of $\mathsf{Exit}(-)$ gives a sequence

$$\left(\mathsf{Exit}(\underline{\mathbb{CP}^0}) \hookrightarrow \mathsf{Exit}(\underline{\mathbb{CP}^1}) \hookrightarrow \dots\right) \ .$$

Therefore one can consider the colimit of these exit path ∞-categories in order to define $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$. Therefore the main goal is a well defined notion of an exit path ∞-category for closed P-filtrations.

A piece of this work involves showing that finite depth closed P-filtrations are conically smooth. When constructing a basic about a given point in a stratification $X \to \mathsf{P}$, we will end up needing to contemplate a stratified space of the form $\mathsf{C}(W) \times \mathsf{C}(V)$. We construct a space $W \between V$, so that there is an equivalence of stratified spaces

$$\mathsf{C}(W) \times \mathsf{C}(V) \cong \mathsf{C}(W \between V) \ . \tag{1.5}$$

A feature about this stratification on the left is that it will not arise as the product of conical stratifications with W and V. However the right side of (1.5) arises as a conical stratification from a stratification on $W \between V$. Therefore we introduce the notion of joins before proving finite depth closed P-filtrations are conically smooth.

<u>Join Construction for Topological Spaces and Posets</u>

Definition 115. The ***categorical join*** of two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is the ∞-category $\mathcal{C}\bigstar\mathcal{D}$ fitting into a colimit diagram in $\mathsf{Cat}_{(\infty,1)}$:

$$\begin{tikzcd}
 & \mathcal{C} \times \mathcal{D} \times \{1\} \arrow[r] \arrow[d] & \mathcal{D} \arrow[dd] \\
\mathcal{C} \times \mathcal{D} \times \{0\} \arrow[r] \arrow[d] & \mathcal{C} \times \mathcal{D} \times [1] \arrow[dr] & \\
\mathcal{C} \arrow[rr] & & \mathcal{C}\bigstar\mathcal{D}
\end{tikzcd} \ .$$

Observation 116. Let P and Q be posets regarded as ∞-categories. Their join $\mathsf{P}\bigstar\mathsf{Q}$ is also a poset, which is readily seen by inspecting the colimit diagram in Definition 115 .

Observation 117. There is a canonical inclusion $\mathcal{C}\amalg\mathcal{D} \hookrightarrow \mathcal{C}\bigstar\mathcal{D}$ that is fully faithful. Also, for each $c \in \mathcal{C}$ and $d \in \mathcal{D}$, the space of morphisms is contractible

$$\mathsf{Hom}_{\mathcal{C}\bigstar\mathcal{D}}(c,d) \simeq * \ .$$

The join can in fact be characterized as the smallest ∞-category satisfying these two properties.

Example 118. The join of $[n]$ and $[m]$ is isomorphic to the poset $[n+m+1]$.

$$[n]\bigstar[m] = \{0_n \leq 1_n \leq n \leq 0_m \leq 1_m \leq 2_m \leq \ldots m\} \ .$$

Here the subscripts keep track the image of the inclusion $[n]\coprod[m] \hookrightarrow [n]\bigstar[m]$.

Example 119. Consider the poset

$$\mathcal{Q} := \quad \begin{tikzcd} & + \arrow[dr] & \\ 0 \arrow[ur] \arrow[dr] & & 1 \\ & - \arrow[ur] & \end{tikzcd} \ .$$

The join $\mathcal{Q}\bigstar\mathcal{Q}$ is the poset

$$\begin{tikzcd} & \color{green}{+} \arrow[dr, green] & & & \color{blue}{+} \arrow[dr, blue] & \\ \color{green}{0} \arrow[ur, green] \arrow[dr, green] & & \color{green}{1} \arrow[r] & \color{blue}{0} \arrow[ur, blue] \arrow[dr, blue] & & \color{blue}{1} \\ & \color{green}{-} \arrow[ur, green] & & & \color{blue}{-} \arrow[ur, blue] & \end{tikzcd} \ .$$

Here the colors reflect the two fully faithful embeddings of Q.

Observation 120. The categorical join $\mathcal{C}_0 \bigstar \mathcal{C}_1$ of two ∞-categories $\mathcal{C}_0$ and $\mathcal{C}_1$ is equivalent to the coend (Definition 333) of the two functors

$$\begin{aligned} \mathsf{L} : \mathsf{sd}([1]) &\longrightarrow \mathsf{Cat}_{(\infty,1)} \\ \mathsf{I} \subset [1] &\longmapsto \mathsf{I} \end{aligned}$$

$$\begin{aligned} \mathsf{R} : \mathsf{sd}([1])^{\mathsf{op}} &\longrightarrow \mathsf{Cat}_{(\infty,1)} \\ \mathsf{I} \subset [1] &\longmapsto \prod_{i \in \mathsf{I}} \mathcal{C}_i \end{aligned} .$$

This allows for a more general categorical join.

Definition 121. We define a more general version of the join as follows. Consider a collection of ∞-categories $\{\mathcal{C}_\alpha\}_{\alpha \in \mathsf{P}}$ indexed by a poset P. The ***categorical join*** is the ∞-category $\bigstar_{k \in \mathsf{P}} \mathcal{C}_k$ defined to be the coend of the two functors

$$\begin{aligned} \mathsf{L} : \mathsf{sd}(\mathsf{P}) &\longrightarrow \mathsf{Cat}_{(\infty,1)} \\ \{p_0 \leq \cdots \leq p_n\} \subset \mathsf{P} &\longmapsto (p_0 \to \cdots \to p_n) \end{aligned}$$

$$\begin{aligned} \mathsf{R} : \mathsf{sd}(\mathsf{P})^{\mathsf{op}} &\longrightarrow \mathsf{Cat}_{(\infty,1)} \\ \{p_0 \leq \cdots \leq p_n\} &\longmapsto \prod_{i \in \{p_0, \ldots, p_n\}} \mathcal{C}_i \end{aligned} .$$

Note in the case that $\mathsf{P} = [1]$ this recovers the usual definition of the categorical join.

Definition 122. Let X and Y be topological spaces. The topological join $X \between Y$ is the colimit in Top:

$$\begin{tikzcd}
 & X \times Y \times \{1\} \arrow[r, "\pi_Y"] \arrow[d] & Y \arrow[dd] \\
X \times Y \times \{0\} \arrow[r] \arrow[d, "\pi_X"'] & X \times Y \times [0,1] \arrow[dr] & \\
X \arrow[rr] & & X \between Y
\end{tikzcd} .$$

Observation 123. Let $[0,1]$ be stratified via Lemma 54 using the $[1]$-filtration

$$\{0\} \hookrightarrow [0,1] \ .$$

Let $X \to \mathsf{P}$ and $Y \to \mathsf{Q}$ be stratified spaces. The canonical map from the topological join diagram to the categorical join diagram

$$\begin{tikzcd}
 & X \times Y \times \{1\} \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[dr] & \\
X \times Y \times \{0\} \arrow[r] \arrow[d] \arrow[dr] & X \times Y \times [0,1] \arrow[dr] & \mathsf{P} \times \mathsf{Q} \arrow[r] \arrow[d] & \mathsf{Q} \\
X \arrow[dr] & \mathsf{P} \times \mathsf{Q} \times \{0\} \arrow[r] \arrow[d] & \mathsf{P} \times \mathsf{Q} \times [1] & \\
 & \mathsf{P} & &
\end{tikzcd}$$

induces a canonical stratification

$$X \between Y \to \mathsf{P} \bigstar \mathsf{Q} \ .$$

Remark 124. While this is the canonical stratification on joins, this is not the stratification used frequently in this paper on the joins.

Observation 125. The join $X_0 \between X_1$ is homeomorphic to the coend (Definition 333) of the two functors

$$\begin{aligned} \mathsf{L} : \mathsf{sd}([1]) &\longrightarrow \mathsf{Top} \\ I \subset [1] &\longmapsto \Delta^I \end{aligned}$$

$$\begin{aligned} \mathsf{R} : \mathsf{sd}([1])^{\mathsf{op}} &\longrightarrow \mathsf{Top} \\ I \subset [1] &\longmapsto \textstyle\prod_{i \in I} X_i \end{aligned} .$$

This homeomorphism is the canonical map induced by the map of topological spaces

$$\begin{aligned} X \times Y \times \Delta^1 &\longrightarrow X \times Y \times [0,1] \\ (x, y, (\alpha_0, \alpha_1)) &\longmapsto (x, y, \alpha_1) \end{aligned} .$$

Definition 126. We define a more general version of the join as follows. Consider a collection of topological spaces $\{X_\alpha\}_{\alpha \in \mathsf{P}}$ indexed by a poset P The ***join*** is the topological space $\underset{k \in \mathsf{P}}{\Large\between} X_k$ defined to be the coend of the two functors

$$\begin{aligned} \mathsf{L} : \mathsf{sd}(\mathsf{P}) &\longrightarrow \mathsf{Top} \\ \{p_0 \leq \cdots \leq p_n\} \subset \mathsf{P} &\longmapsto \Delta^{\{p_0, \ldots, p_n\}} \end{aligned}$$

$$\begin{aligned} \mathsf{R} : \mathsf{sd}(\mathsf{P})^{\mathsf{op}} &\longrightarrow \mathsf{Top} \\ \{p_0 \leq \cdots \leq p_n\} \subset \mathsf{P} &\longmapsto \prod_{i \in \{p_0, \ldots, p_n\}} X_i \end{aligned} .$$

Note in the case that $\mathsf{P} = [1]$ this recovers the usual definition of the join.

Observation 127. Let $\{X_i\}_{i \in [n]}$ be a collection of non empty topological spaces. The coend construction identifies the join $\underset{i \in [n]}{\Large\between} X_i$ as the quotient of the space $\Delta^n \times \prod_{i \in [n]} X_i$ by the equivalence relation generated by

$$(t_0, \ldots, t_i = 0, \ldots t_n, x_0, \ldots, x_i, \ldots x_n) \ \sim \ (t_0, \ldots, t_i = 0, \ldots, t_n, x_0, \ldots, x_i', \ldots, x_n) \ .$$

A point $(t_0, \ldots, t_n, x_0, \ldots, x_n)$ is sometimes written (See [8]) as a sum

$$\sum_{i \in [n]} t_i x_i := (t_0, \ldots, t_n, x_0, \ldots, x_n)$$

where the equivalence relation on $\Delta^n \times \prod_{i \in [n]} X_i$ states that if $t_i = 0$, the term $t_i x_i$ may be omitted from the sum.

Example 128. The join $\between_{i \in [n]} *$ is the n-simplex

$$\between_{i \in [n]} * \cong \Delta^n \ .$$

Observation 129. Consider the join of $n+1$ topological spaces $\between_{i \in [n]} X_i$. There is a canonical map

$$\between_{i \in [n]} X_i \to \between_{i \in [n]} * = \Delta^n \ .$$

For any non-empty subset $\mathsf{I} \subset [n]$, there is a pullback

$$\begin{tikzcd}
\between_{i \in \mathsf{I}} X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \between_{i \in [n]} X_i \arrow[d] \\
\Delta^{\mathsf{I}} \arrow[r] & \Delta^n
\end{tikzcd} \ .$$

Notation 130. The closed embeddings

$$\between_{i \in I} X_i \hookrightarrow \between_{i \in [n]} X_i$$

induce a canonical $[n]$-filtration of $\between_{i \in [n]} X_i$. These closed embeddings give a canonical stratification of $\between_{i \in [n]} X_i$ by considering the $[n]$-filtration

$$X_0 \hookrightarrow X_0 \between X_1 \hookrightarrow \cdots \hookrightarrow \between_{i \in [n]} X_i \ .$$

Denote the stratification induced by this $[n]$-filtration as

$$\underline{\between_{i\in[n]} X_i} := \between_{i\in[n]} X_i \to [n]$$

Example 131. The join of two spheres $\mathbb{S}^i$ and $\mathbb{S}^j$ is a sphere of dimension $i+j+1$

$$\begin{aligned} \mathbb{S}^i \between \mathbb{S}^j &\xrightarrow{\quad\simeq\quad} \mathbb{S}^{i+j+1} \\ ((t_0,t_1),x,y) &\longmapsto \tfrac{(t_0x,t_1y)}{\|(t_0x,t_1y)\|} \end{aligned} .$$

Definition 132. Let

$$\begin{aligned} \{0,\dots,n\} &\longrightarrow \mathsf{Top}^{*/} \\ i &\longmapsto (*_i \in X_i) \end{aligned}$$

be a selection of $n+1$ pointed topological spaces. There is a ***pointed stratification*** on the product

$$\begin{aligned} \prod_{k\in[n]} X_k &\longrightarrow [n] \\ (x_0,\dots,x_n) &\longmapsto \{\mathsf{max}(i) | x_i \neq *_i\} \end{aligned} .$$

Example 133. The cone on a topological space X can be canonically considered as a pointed set by taking the cone point as the selected basepoint. Therefore given a collection of topological spaces $\{X_i\}_{i\in[n]}$ there exists a pointed stratification

$$\prod_{i\in[n]} \mathsf{C}(X_i) \to [n] \ .$$

Define $\underline{\prod_{i\in[n]} \mathsf{C}(X_i)}$ to be the pointed stratification.

Example 134. For any group G, the identity element is a canonical choice of a basepoint, which gives a forgetful functor from the category of groups to the category of pointed topological spaces.

$$\mathsf{Groups} \to \mathsf{Top}^{*/}$$

In particular, we take $0 \in \mathbb{R}^m$ to be the canonical basepoint. For a $[n]$-indexed collection of Euclidean spaces $\{\mathbb{R}^{m_i}\}_{i\in[n]}$, denote $\underline{\prod_{i\in[n]} \mathbb{R}^{m_i}}$ as the pointed stratification of the product of the Euclidean spaces.

Lemma 135. *Let X and Y be compact topological spaces. The map of spaces*

$$\begin{aligned}(X \times \mathbb{R}_{>0}) \times (Y \times \mathbb{R}_{>0}) &\longrightarrow (X \times Y \times [0,1]) \times \mathbb{R}_{>0}\\ ((x,t_x),(y,t_y)) &\longrightarrow ((x,y,\tfrac{t_y}{t_x+t_y}), t_x+t_y)\end{aligned}$$

induces a map on the stratified spaces

$$\begin{aligned}J : \underline{\mathsf{C}(X) \times \mathsf{C}(Y)} &\longrightarrow \mathsf{C}(\underline{X \between Y})\\ ([x,t_x],[y,t_y]) &\longrightarrow [(x,y,\tfrac{t_y}{t_x+t_y}), t_x+t_y]\end{aligned}$$

which is an isomorphism in StTop.

Proof. First consider the map on underlying topological spaces. Corollary 3.4.10 of [5] gives that this map is a homeomorphism of topological spaces, therefore it suffices to check this map is a map of stratified spaces. The stratification on the left hand side is represented by the [2]-filtration

$$(*_X, *_Y) := ([x_0],[y_0]) \hookrightarrow \mathsf{C}(X) \times *_Y \to \mathsf{C}(X) \times \mathsf{C}(Y)\ .$$

The stratification on the right hand side is represented by taking the conical stratification of the [1]-filtration

$$X \hookrightarrow X \between Y.$$

J by definition sends the 0-stratum of $\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}$, which is the point $([x,0],[y,0])$, to the cone point $[(x,y,0),0]$ by definition. The 1-stratum of the space $\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}$ is identified as

$$\Big(\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}\Big)_1 := \{([x,t_x],[y,t_y]) \in \underline{\mathsf{C}(X) \times \mathsf{C}(Y)} | t_y = 0 \text{ and } t_x \neq 0\}$$

by Definition 132. The map J restricted to the 1-stratum of $\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}$ is

$$\begin{array}{ccc} \Big(\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}\Big)_1 & \longrightarrow & \mathsf{C}(\underline{X \between Y}) \\ ([x,t_x],[y,0]) & \longmapsto & [(x,y,\frac{0}{t_x+0}),t_x] \end{array} .$$

The image of this map clearly is $\mathsf{C}(X)$ without the cone point, which is the 1-stratum of $\mathsf{C}(\underline{X \between Y})$. Similarly, for points in the 2-stratum of $\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}$

$$\Big(\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}\Big)_2 := \{([x,t_x],[y,t_y]) \in \underline{\mathsf{C}(X) \times \mathsf{C}(Y)} | t_y \neq 0\}$$

the map J restricts to a map

$$\begin{array}{ccc} \Big(\underline{\mathsf{C}(X) \times \mathsf{C}(Y)}\Big)_2 & \longrightarrow & \mathsf{C}(\underline{X \between Y}) \\ ([x,t_x],[y,t_y]) & \longrightarrow & [(x,y,\frac{t_y}{t_x+t_y}),t_x+t_y] \end{array}$$

that lands in the 2-stratum of $\mathsf{C}(\underline{X \between Y})$ since $\frac{t_y}{t_x+t_y} \neq 0$ and $t_x + t_y \neq 0$ since $t_y > 0$. $\square$

Closed P-filtrations are Conically Smooth

Lemma 136. *Let V be a k dimensional vector space, and let $V_0 \hookrightarrow V$ be the inclusion of a k_0 dimension vector subspace.*

(1) The inclusion of $V_0 \hookrightarrow V$ is isomorphic with the canonical inclusion of vector spaces

$$\mathbb{R}^{k_0} \xrightarrow{i} \mathbb{R}^k .$$

(2) About each point $x \in V_0$, there exists a basic of the form

$$\mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k-k_0-1}) \ .$$

(3) About each point $x \in V_1$, there exists a basic of the form

$$\mathbb{R}^{k} \times \mathsf{C}(\emptyset) \hookrightarrow V_1$$

(4) The canonical stratification $V \to [1]$ induced by Lemma 54 by the $[1]$-filtration $V_0 \hookrightarrow V$ is conically smooth.

Proof. We begin by proving (1). Choose an inner product on the vector space V. Using this inner product, select an orthonormal basis of the subspace V_0, which is the data of a linear isometry

$$\phi_0 : V_0 \xrightarrow{\cong} \mathbb{R}^{k_0} \ .$$

Extend this orthonormal basis of V_0 to an orthonormal basis of V, which extends the linear isometry of $V_0 \simeq \mathbb{R}^{k_0}$ to a linear isometry

$$\phi : V \xrightarrow{\cong} \mathbb{R}^{k}$$

Define $k_1 := k - k_0$. The linear isometry $\phi : V \to \mathbb{R}^k$ provides a canonical linear isometry

$$\phi_1 : V_0^{\perp} \xrightarrow{\cong} \mathbb{R}^{k_1} \ .$$

Together this compiles into a diagram

$$\begin{tikzcd}
V_0 \arrow[r] \arrow[d, "\phi_0"'] & V_0 \oplus {V_0}^{\perp} \arrow[r, "\cong"] \arrow[d, "\phi_0\oplus\phi_1"'] & \mathbb{R}^k \arrow[d, dashed, "\exists!\cong"] \\
\mathbb{R}^{k_0} \arrow[r, "i_0"'] & \mathbb{R}^{k_0} \oplus \mathbb{R}^{k_1} \arrow[r, "\cong"'] & \mathbb{R}^k
\end{tikzcd} .$$

Here the isomorphisms $\mathbb{R}^{k_0} \oplus \mathbb{R}^{k_1} \to \mathbb{R}^k$ is the canonical isomorphism that includes $\mathbb{R}^{k_0}$ into the first k_0 coordinates and $\mathbb{R}^{k_1}$ into the last $k - k_0 = k_1$ coordinates. This verifies claim (1).

Next we seek to show about each $x \in V_0$, there exists a basic of the form $\mathbb{R}^k \times \mathsf{C}(\mathbb{S}^{k-k_0-1})$. Note that the vector space V is isomorphic to $V_0 \oplus \mathbb{R}^{k_1}$, by considering the splitting $V_0 \oplus {V_0}^{\perp}$ of the vector space V, and noting there is an isomorphism

$$V_0 \oplus {V_0}^{\perp} \xrightarrow{\mathsf{id} \oplus \phi_1} V_0 \oplus \mathbb{R}^{k_1} .$$

Finally, using that $\mathsf{C}(\mathbb{S}^{k_1-1}) \cong \mathbb{R}^{k_1}$ using the canonical scaling map $[0, \infty) \times \mathbb{S}^{k_1-1} \to \mathbb{R}^{k_1}$ gives the result

$$V_0 \times \mathsf{C}(\mathbb{S}^{k_1-1}) = V_0 \times \mathsf{C}(\mathbb{S}^{k-k_0-1}) .$$

The stratification

$$V_0 \oplus \mathsf{C}(\mathbb{S}^{k-k_0-1}) \xrightarrow{\cong} V \to [1]$$

agrees with the canonical stratification on $V_0 \oplus \mathsf{C}(\mathbb{S}^{k-k_0-1})$ induced by considering the conical stratification determined by the trivial stratification $\mathbb{S}^{k-k_0-1} \to \{1\}$. Moreover, this stratification $\mathbb{S}^{k-k_0-1} \to \{1\}$ is conically smooth, since $\mathbb{S}^{k-k_0-1}$ has a canonical structure of a smooth manifold. Therefore, the identification of V with $\mathbb{R}^k \oplus \mathsf{C}(\mathbb{S}^{k-k_0-1}) \xrightarrow{\cong}$ witnesses a basic about the point $x \in V_0$.

We next seek to show that about each point $x \in V_1$, there exists a basic about x of the form $\mathbb{R}^k \times \mathsf{C}(\emptyset)$. The subset V_1 identified as an open subset $\mathbb{R}^k$ through the isomorphism

$\phi : V \to \mathbb{R}^k$. Therefore, since an open subset of $\mathbb{R}^k$ canonically inherits the structure of a smooth manifold, there exists a smooth embedding of $\mathbb{R}^k \hookrightarrow V_1$ such that x is in the image. Finally noting that $\mathbb{R}^k \cong \mathbb{R}^k \times \mathsf{C}(\emptyset)$ gives the desired result.

Finally, we seek to show $V \to [1]$ is conically smooth. By (2) and (3), there exists an atlas of V by basics. Moreover, this atlas is a smooth atlas, since the transition maps are clearly smooth, which implies that this is a conically smooth atlas. This conically smooth atlas extends to a maximal conically smooth atlas on V in the same way that a smooth atlas on a manifold extends to a maximal smooth atlas. Therefore $V \to [1]$ is a conically smooth stratified space. $\square$

Notation 137. Let $M_0 \hookrightarrow M_1$ be an closed embedding of smooth manifolds. Define $\mathsf{N}(M_0 \hookrightarrow M_1)$ to be the total space of th normal bundle of the embedding $M_0 \hookrightarrow M_1$. Note there is a canonical section

$$M_0 \hookrightarrow \mathsf{N}(M_0 \hookrightarrow M_1)$$

which is the zero section.

Lemma 138. *Let $M \to [1]$ be a closed P-filtration of a smooth manifold M.*

(1) Let $x \in M_0$, and $k_0 := \dim(M_{\leq 0})$. There exists a basic about $x \in M$ of the form

$$\mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1-1}) \hookrightarrow M$$

such that the inclusion is a smooth embedding with respect to the canonical smooth structure on $\mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1-1}) \simeq \mathbb{R}^{k_0+k_1}$. Here k_0 is the dimension of the connected component of M_0 containing x, and k_1 is the dimension of the connected component of $M_{\leq 1}$ containing x.

(2) Let $x \in M_1$, there exists a basic of the form

$$\mathbb{R}^k \times \mathsf{C}(\emptyset) \hookrightarrow M$$

such that the inclusion is a smooth embedding. Here k is the dimension of the connected component of M_1 containing x.

(3) The stratification $M \to [1]$ is conically smooth.

Proof. We begin by proving (1). Let $x \in M_0$. Since M_0 is a smooth manifold, there exists a smooth open embedding

$$\mathbb{R}^{k_0} \hookrightarrow M_0$$

about the point x. By the tubular neighborhood theorem, there exists a smooth extension to the total space of the normal bundle

$$\begin{tikzcd} \mathbb{R}^{k_0} \arrow[r] \arrow[d] & M \\ \mathsf{N}(\mathbb{R}^{k_0} \to M) \arrow[ur, dashed] & \end{tikzcd} .$$

Furthermore, since the base space of the normal bundle is contractible, the total space normal bundle is isomorphic to the trivial bundle of rank k_1

$$\mathsf{N}(\mathbb{R}^{k_0} \to M) \cong \mathbb{R}^{k_0} \oplus \mathbb{R}^{k_1} .$$

Note $\mathbb{R}^{k_1} \cong \mathsf{C}(\mathbb{S}^{k_1-1})$. Moreover the stratification of $\mathbb{R}^{k_1} \times \mathsf{C}(\mathbb{S}^{k_1-1})$ induced by the stratification of $\mathbb{S}^{k_1} \to \{1\}$ agrees with the stratification

$$\mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1-1}) \hookrightarrow M \to [1] .$$

Therefore $\mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1} - 1)$ is a basic about x that smoothly embeds into M.

Next, we seek to show there exists a basic of the form $\mathbb{R}^k \times \mathsf{C}(\emptyset)$ about a point $x \in M_1$, such that $\mathbb{R}^k \times \mathsf{C}(\emptyset)$ smoothly embeds into M. The 1-stratum M_1 is an open subset of M. Therefore M_1 has the structure of a smooth submanifold of M of dimension k. Therefore there exists a sequence of smooth embeddings

$$\mathbb{R}^k \hookrightarrow M_1 \hookrightarrow M$$

such that x is in the image of the smooth embedding. By considering $\mathbb{R}^k \cong \mathbb{R}^k \times \mathsf{C}(\emptyset)$, we can realize the embedding $\mathbb{R}^k \hookrightarrow M$ as a basic about the point x that smoothly embeds about the point $x \in M_1$.

Finally, we seek to show that $M \to [1]$ is conically smooth. The collection of basics in (1) and (2)

$$\{\mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1-1}) \hookrightarrow M\}_{x \in M_0} \amalg \{\mathbb{R}^{k_1} \times \mathsf{C}(\emptyset) \hookrightarrow M\}_{x \in M_1}$$

form a smooth atlas of the manifold M, and therefore the collection is also a conically smooth atlas. This atlas extends to a unique maximal conically smooth atlas. Therefore the stratification $M \to [1]$ is conically smooth.

$\square$

Lemma 139. *Let $M \to [n]$ be a closed $[n]$-filtration.*

(1) Let $p \in \{1, \ldots, n\}$. For each point $x \in M_p$, there exists a basic of the form

$$\mathbb{R}^{k_p} \times \mathsf{C}\left(\underline{\bigstar_{i \in \{p+1 \leq \cdots \leq n\}} \mathbb{S}^{k_i - 1}}\right) \hookrightarrow M \ .$$

This basic is a smooth embedding into M.

(2) For each point $x \in M_0$, there exists a basic of the form

$$\mathbb{R}^{k_0} \times \mathsf{C}\left(\underline{\mathop{\between}\limits_{i \in \{1 \leq \cdots \leq n\}} \mathbb{S}^{k_i - 1}} \right) \hookrightarrow M \ .$$

This basic is a smooth embedding into M .

(3) The stratification $M \to [n]$ is conically smooth.

Proof. We seek to show this by induction on $[n]$. The base case is $[1]$, which is Lemma 138 . Therefore, assume Lemma 139 is true for closed $[n-1]$-filtrations. The induction step will be used twice. Once to determine the basic in the statement of (1), and once to show that $\underline{\mathop{\between}\limits_{i \in \{1,\ldots,n\}} \mathbb{S}^{n_i}}$ is conically smooth.

We begin by proving (1). Let $p \in \{1, \ldots, n\}$, and let $x \in M_p$. This p defines a restriction of the stratification $M \to [n]$ to the submanifold $M_{\geq p}$

$$M_{\geq p} \to \{p \leq \cdots \leq n\} \ .$$

The depth of this stratification $n - p$, which is stricly less than n. Therefore the restricted stratification is conically smooth by the inductive step. In particular, there is a basic centered at $x \in M_p$ of the form

$$\mathbb{R}^{k_p} \times \mathsf{C}\left(\underline{\mathop{\between}\limits_{i \in \{p+1 \ldots n\}} \mathbb{S}^{k_i - 1}} \right) \hookrightarrow M_{\geq p} \ .$$

This embedding extends to an open embedding into the manifold M

$$\mathbb{R}^{k_p} \times \mathsf{C}\left(\underline{\mathop{\between}\limits_{i \in \{p+1 \ldots n\}} \mathbb{S}^{k_i - 1}} \right) \hookrightarrow M_{\geq p} \hookrightarrow M.$$

This embedding realizes $\mathbb{R}^{k_p} \times \mathsf{C}\left(\underline{\mathop{\between}\limits_{i \in \{p+1 \ldots n\}} \mathbb{S}^{k_i - 1}} \right) \hookrightarrow M$ as a basic about the point $x \in M_p$.

Let $x \in M_0$. Since M_0 is a smooth manifold, there exists a smooth embedding $\mathbb{R}^{k_0} \hookrightarrow M_0$ such that x is in the image of this smooth embedding. Consider the composition

$$\mathbb{R}^{k_0} \hookrightarrow M_0 \hookrightarrow M_{\leq 1} \ .$$

The tubular neighborhood theorem gives a smooth embedding of the normal bundle $\mathsf{N}(\mathbb{R}^{k_0} \to M_{\leq 1})$. Since this is a vector bundle over a contractible space $\mathbb{R}^{k_0}$, the total space $\mathsf{N}(\mathbb{R}^{k_0} \hookrightarrow M_{\leq 1})$ is trivial. Continuing in this way, we get a diagram by considering the restricted stratification to the poset $\{0 \leq 1 \leq 2\}$

$$\begin{tikzcd}
\mathbb{R}^{k_0} \arrow[r] \arrow[d] & M_{\leq 1} \arrow[r] & M_{\leq 2} & \\
\mathsf{N}(\mathbb{R}^{k_0} \to M_{\leq 1}) \arrow[r] \arrow[ur, dashed] & \mathsf{N}(\mathbb{R}^{k_0} \times \mathbb{R}^{k_1} \to M_{\leq 2}) \arrow[r] \arrow[ur, dashed] & \mathsf{N}(\mathbb{R}^{k_0} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \hookrightarrow M_{\leq 3}) & . \\
\underline{\mathbb{R}^{k_0} \times \mathbb{R}^{k_1}} \arrow[r, "i"] \arrow[u, "\cong"] & \underline{\mathbb{R}^{k_0} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}} \arrow[r, "i"] \arrow[u, "\cong"] & \underline{\mathbb{R}^{k_0} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{k_3}} \arrow[u, "\cong"] &
\end{tikzcd}$$

Here the dashed arrows are the open embeddings guaranteed by the tubular neighborhood theorem, and the bottom isomorphisms are the identifications of the normal bundles being trivial. Continuing in this way, we end up with an open embedding

$$\begin{tikzcd}
\mathbb{R}^{k_0} \arrow[r] \arrow[d] & M \\
\mathsf{N}(\mathbb{R}^{k_0} \to M_{\leq 1}) \arrow[ur, dashed] & . \\
\underline{\prod_{i \in [n]} \mathbb{R}^{k_i}} \arrow[u, "\cong"] &
\end{tikzcd}$$

Recall the marked stratification on $\mathbb{R}^{k_0} \times \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$ is the continuous function to the poset $[n]$

$$\begin{array}{ccc} \underline{\prod_{i\in[n]} \mathbb{R}^{k_i}} & \longrightarrow & [n] \\ (x_0,\ldots,x_n) & \longmapsto & \mathsf{max}\{i|x_i \neq 0\} \end{array} .$$

Here it is understood the value of $0 \in \underline{\prod_{i\in[n]} \mathbb{R}^{k_i}}$ is also sent to $0 \in [0]$, as in this case there is no maximum value x_i that is non-zero. The scaling maps on $\mathbb{R}^{k_i}$ provide isomorphisms

$$\mathsf{C}(\mathbb{S}^{k_i-1}) \to \mathbb{R}^{k_i} ,$$

which in turn provide an isomorphism

$$\mathbb{R}^{k_0} \times \prod_{i\in\{1\leq\cdots\leq n\}} \mathsf{C}(\mathbb{S}^{k_i-1}) \xrightarrow{\cong} \prod_{i\in[n]} \mathbb{R}^{k_i} .$$

Denote the cone point of $\mathsf{C}(\mathbb{S}^{k_i-1})$ as $*_i$. Note that since the scaling map sends $*_i \in \mathsf{C}(\mathbb{S}^{k_i-1})$ to $0 \in \mathbb{R}^{k_0}$, the homeomorphism of topological spaces $\mathbb{R}^{k_0} \times \prod_{i\in\{1\leq\cdots\leq n\}} \mathsf{C}(\mathbb{S}^{k_i-1}) \xrightarrow{\cong} \prod_{i\in[n]} \mathbb{R}^{k_i}$ is also an isomorphism in StTop between the marked stratifications

$$\mathbb{R}^{k_0} \times \underline{\prod_{i\in\{1\leq\cdots\leq n\}} \mathsf{C}(\mathbb{S}^{k_i-1})} \cong \mathbb{R}^{k_0} \times \underline{\prod_{i\in\{1\leq\cdots\leq n\}} \mathbb{R}^{k_i}}$$

By Lemma 135, there is a isomorphism of stratified spaces

$$\mathbb{R}^{k_0} \times \mathsf{C}(\between_{i\in\{1\leq\cdots\leq n\}} \mathbb{S}^{k_i-1}) \xrightarrow{\cong} \mathbb{R}^{k_0} \hookrightarrow \mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1-1}) \hookrightarrow \cdots \hookrightarrow \mathbb{R}^{k_0} \times \mathsf{C}(\mathbb{S}^{k_1}-1) \times \cdots \times \mathsf{C}(\mathbb{S}^{k_n-1}) .$$

Here the stratification on $\mathbb{R}^{k_0} \times \mathsf{C}\left(\underline{\between_{i\in[n]} \mathbb{S}^{k_i-1}}\right)$ is the canonical stratification induced by

$[n-1]$ filtration

$$\begin{tikzcd}
\mathbb{S}^{k_1-1} \arrow[r] \arrow[d, "\cong"] & \mathbb{S}^{k_1-1} \between \mathbb{S}^{k_2-1} \arrow[r] \arrow[d, "\cong"] & \dots \arrow[r] \arrow[d, "\cong"] & \between_{i\in\{1,\dots,n\}} \mathbb{S}^{k_i-1} \arrow[d, "\cong"] \\
\mathbb{S}^{k_1-1} \arrow[r] & \mathbb{S}^{k_1+k_2-1} \arrow[r] & \dots \arrow[r] & \mathbb{S}^{k_1+\dots+k_n-1}
\end{tikzcd} .$$

□

Theorem 140. *Let $M \to \mathsf{P}$ be a closed P-filtration.*

(1) For each $p \in \mathsf{P}$, the stratified space

$$M_{\leq p} \to \mathsf{P}_{\leq p}$$

is conically smooth.

(2) The stratification $M \to \mathsf{P}$ is conically stratified.

Proof. (1) follows from Lemma 139 after the observation that $\mathsf{P}_{\leq p}$ is finite and linearly ordered since P is downwards finite and linearly ordered.

We seek to show that $M \to \mathsf{P}$ is conically stratified. Note the closed P-filtration ensures that X is the union of smooth manifolds

$$\mathsf{colim}(M_{\leq 0} \hookrightarrow M_{\leq 1} \hookrightarrow M_{\leq 2} \hookrightarrow \dots) = M \ .$$

Note that there is no requirement that M is a smooth manifold. Let $x \in M$. Then there exists a minimal $p \in \mathsf{P}$ such that $x \in M_{\leq p}$. Since M_p is an open subset of $M_{\leq p}$, there exists an open embedding of $\mathbb{R}^{k_p}$ for some positive integer k_p about the point $x \in M_p$. Note that x is in the p-stratum by the assumption that we are in the minimal $M_{\leq p}$ that contains x. By the tubular neighborhood theorem, there exists an extension to the normal bundle of the inclusion $\mathbb{R}^{k_p} \hookrightarrow M_{\leq p+1}$

$$\begin{tikzcd}
\mathbb{R}^{k_p} \arrow[r] \arrow[d] & M_{\leq p+1} \\
\mathsf{N}(\mathbb{R}^{k_p} \hookrightarrow M_{\leq p+1}) \arrow[ur] & \\
\mathbb{R}^{k_p} \times \mathbb{R}^{k_{p+1}} \arrow[u, "\cong"] &
\end{tikzcd}$$

where k_{p+1} is the unique positive integer such that $k_p + k_{p+1} = \dim(M_{\leq p+1})$. Continuing in this way we get a sequence of smooth embeddings.

$$\begin{tikzcd}
\mathbb{R}^{k_p} \arrow[r] \arrow[d] & M_{\leq p+1} \arrow[r] & M_{\leq p+1} \arrow[r] & \dots \\
\mathsf{N}(\mathbb{R}^{k_p} \hookrightarrow M_{\leq p+1}) \arrow[r] \arrow[ur, dashed] & \mathsf{N}(\mathbb{R}^{k_p} \times \mathbb{R}^{k_{p+1}} \hookrightarrow M_{\leq p+2}) \arrow[r] \arrow[ur, dashed] & \dots & \\
\underline{\mathbb{R}^{k_p} \times \mathbb{R}^{k_{p+1}}} \arrow[u, "\cong"] & \underline{\mathbb{R}^{k_p} \times \mathbb{R}^{k_{p+1}} \times \mathbb{R}^{k_{p+2}}} \arrow[u, "\cong"] & &
\end{tikzcd} .$$

This constructs an stratified open embedding

$$\mathbb{R}^{k_p} \times \mathsf{C}(\mathbb{S}^\infty) \hookrightarrow M \ .$$

Here the stratification on $\mathbb{S}^\infty$ is the stratification induced by the P-filtration

$$(\mathbb{S}^{k_{p+1}-1} \hookrightarrow \mathbb{S}^{k_{p+1}+p_{p+2}-1} \hookrightarrow \dots) = \mathbb{S}^\infty \ .$$

For $q \in \mathsf{P}$ such that $p \leq q$, by construction there is a canonical restriction of stratified spaces

$$\begin{tikzcd}
\mathbb{R}^{k_p} \times \mathsf{C}(\underline{\mathbb{S}^{k_{p+1}+\dots+k_{q-1}}}) \arrow[r, hook] \arrow[d, hook] & \mathbb{R}^{k_p} \times \mathsf{C}(\mathbb{S}^\infty) \arrow[d, hook] \\
M_{\leq q} \arrow[r, hook] & X
\end{tikzcd}$$

where each of the maps are the canonical maps.

□

Exit Path ∞-Category of a Closed P-filtration

A P-filtration for P finite is indeed a conically smooth stratified space by Lemma 139. However, for P not finite, such as the poset $\mathsf{P}_{\geq 0}$, a P-filtration need not be conically smooth. We seek to show in this subsection that in the case of P not finite that there still admits an exit path ∞-category that can be identified in a similar manor to Lemma 104.

Definition 141. Let $\mathcal{K}$ be an ∞-category. The ∞-category $\mathcal{K}$ is filtered if for any finite ∞-category $\mathcal{D}$, and functor

$$\mathcal{D} \to \mathcal{K},$$

there exists an extension to $\mathcal{D}^{\triangleright}$

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{K} \\ \mathcal{D}^{\triangleright} \arrow[ur, dashed, bend right] & \end{tikzcd} .$$

Lemma 142. *Let $\mathcal{K}$ be a filtered ∞-category. Let*

$$\mathcal{K} \to \mathsf{CSS} \hookrightarrow \mathsf{PShv}(\Delta)\ .$$

be a functor. Then the colimit

$$\mathsf{colim}\,(\mathcal{K} \to \mathsf{PShv}(\Delta))$$

takes values in complete Segal spaces.

Proof. Denote the functor $\mathcal{K} \xrightarrow{F} \mathsf{CSS}$. Denote the colimit $X := \mathsf{colim}\,(\mathcal{K} \to \mathsf{PShv}(\Delta)) \in \mathsf{PShv}(\boldsymbol{\Delta})$.

Observe that the Segal and the univalence-completeness conditions on a simplicial space Z state that Z carries certain finite diagrams in $\boldsymbol{\Delta}^{\mathsf{op}}$ to limit diagrams in $\mathcal{S}\mathsf{paces}$. So let

$\mathcal{J} \xrightarrow{D} \mathbf{\Delta}^{\mathsf{op}}$ be such a certain diagram in which $\mathcal{J}$ is a finite ∞-category. We must prove the the canonical morphism between simplicial spaces

$$\underset{k\in\mathcal{K}}{\mathsf{colim}}\left(\underset{i\in\mathcal{J}}{\mathsf{lim}} F\circ D(i)\right) \longrightarrow \underset{i\in\mathcal{J}}{\mathsf{lim}}\left(\underset{k\in\mathcal{K}}{\mathsf{colim}} F\circ D(i)\right)$$

is an equivalence. This is so if and only if this morphism is value-wise an equivalence between spaces. Using that limits and colimits in $\mathsf{PShv}(\mathbf{\Delta})$ are computed value-wise, we are reduced to observing that, for each $[n]\in\mathbf{\Delta}$, the canonical map between spaces

$$\underset{k\in\mathcal{K}}{\mathsf{colim}}\left(\underset{i\in\mathcal{J}}{\mathsf{lim}} F\circ D(i)([n])\right) \longrightarrow \underset{i\in\mathcal{J}}{\mathsf{lim}}\left(\underset{k\in\mathcal{K}}{\mathsf{colim}} F\circ D(i)([n])\right)$$

is an equivalence. Now, a characteristic feature of filtered colimits is that they commute with finite limits. In other words, this canonical morphism is an equivalence, as desired.

□

Lemma 143. *Let $X\to\mathbb{Z}_{\geq 0}$ be a closed $\mathbb{Z}_{\geq 0}$-filtration. Define the presheaf $\mathsf{Exit}(\underline{X})$ to be*

$$\mathsf{Exit}(\underline{X}) := \mathsf{colim}\left(\mathsf{Exit}\left(\underline{M_0}\right)\hookrightarrow \mathsf{Exit}\left(\underline{M_1}\right)\hookrightarrow\dots\right) \ .$$

(1) The simplicial space $\mathsf{Exit}(\underline{X})$ is a complete Segal spaces, and therefore presents an $\infty-$category.

(2) The space of objects of $\mathsf{Exit}(\underline{X})$ is

$$\mathsf{Obj}(\mathsf{Exit}(\underline{X})) = \coprod_{p\in\mathbb{Z}_{\geq 0}} M_p \ .$$

(3) Let x and y be objects of $\mathsf{Exit}(\underline{X})$ with x in the p stratum and y in the q stratum. The

space of morphisms in $\mathsf{Exit}(\underline{X})$ *from* x *to* y *is identified*

$$\mathsf{Hom}_{\mathsf{Exit}(X)}\left(x,y\right) = \mathsf{Hom}_{\mathsf{Exit}(M_{\leq q})}\left(x,y\right) = \mathsf{Link}_{M_p}(X)_q \ .$$

Proof. (1) follows from Lemma 142 after noting that the poset P is filtered. P is filtered since every finite subset of P admits an upper bound (Example 5.3.1.8 of [10]).

The second statement follows from the fact that colimits of presheaves are computed pointwise. That is, the space of objects of $\mathsf{Exit}(\underline{X})$ is computed as the colimit of spaces

$$\mathsf{Obj}\left(\mathsf{Exit}(\underline{X})\right) := \mathsf{colim}\left(\mathsf{Obj}\left(\mathsf{Exit}\left(\underline{M_0}\right)\right) \hookrightarrow \mathsf{Obj}\left(\mathsf{Exit}\left(\underline{M_1}\right)\right) \hookrightarrow \dots\right) \ .$$

By Lemma 104, the objects of $\mathsf{Exit}\left(\underline{M_{\leq p}}\right)$ is the disjoint union of strata

$$\mathsf{Obj}\left(\mathsf{Exit}\left(\underline{M_{\leq p}}\right)\right) = \coprod_{i\in\{0\leq\cdots\leq p\}} M_i \ .$$

Therefore the colimit defining the objects of $\mathsf{Exit}(\underline{X})$ is

$$\mathsf{Obj}\left(\mathsf{Exit}\left(\underline{X}\right)\right) = \left(\coprod_{i\in\{0\}} M_i \hookrightarrow \coprod_{i\in\{0\leq 1\}} M_i \hookrightarrow \coprod_{i\in\{0\leq 1\leq 2\}} M_i \hookrightarrow \dots\right) = \coprod_{i\in\mathbb{Z}_{\geq 0}} M_i \ .$$

Statement (3) follows in a similar fashion of (2), since the space of morphisms is the colimit of the spaces of morphisms of $\mathsf{Exit}(\underline{M_{\leq p}})$, and that the space of morphisms from p to q is given by the link by Lemma 104. $\square$

EXIT PATH ∞-CATEGORY OF $\mathbb{CP}^\infty$

Recall the $\mathbb{Z}_{\geq 0}$-filtration of $\mathbb{CP}^\infty$

$$(\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \dots) = \mathbb{CP}^\infty \ .$$

Lemma 143 defines the exit path ∞-category $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$, as well as states how to identify the space of objects and morphisms of the exit path ∞-category $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$. This section works out this identification, and Theorem 165 is the explicit identification of this exit path ∞-category. We arrive at this theorem by recognizing $\mathbb{CP}^\infty$ as the quotient of $\mathbb{S}^\infty$, and identify the exit path ∞-category of the closed $\mathbb{Z}_{\geq 0}$-filtration

$$(\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \hookrightarrow \dots) = \mathbb{S}^\infty \ .$$

The identification of $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ from this closed $\mathbb{Z}_{\geq 0}$-filtration is simplier to identify, ultimately because the stratification is canonically embedded into a vector space.

Example 144. Recall that for each $n \in \mathbb{Z}_{\geq 0}$ the map of spaces

$$\mathbb{S}^{2n+1} \to \mathbb{CP}^n$$

is a principal $\mathbb{T}$ bundle. This viewpoint is useful in identifying $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$, as is seen by the following proposition.

Proposition 145. *Let G be a group, and let $\tilde{X} \to X$ be the quotient of a free G-action on $\tilde{X}$, and let $X \to \mathsf{P}$ be a closed P-filtration. The quotient map together with the P-filtration gives a stratification of $\tilde{X}$*

$$\tilde{X} \to X \to \mathsf{P} \ .$$

Then the following statements are true.

(1) *The action of G on $\tilde{X}$ determines an action of G on the exit path ∞-category* $\mathsf{Exit}(\underline{\tilde{X}})$.

(2) *The functor* $\mathsf{Exit}(\underline{\tilde{X}}) \to \mathsf{Exit}(\underline{X})$ *is G-invariant.*

(3) *The resulting functor* $\mathsf{Exit}(\underline{\tilde{X}})/G \to \mathsf{Exit}(\underline{X})$ *is an equivalence of ∞-categories.*

Proof. An action of G on $\mathsf{Exit}(\tilde{X})$ is defined to be a functor

$$\mathcal{B}G \to \mathsf{Cat}_{(\infty,1)}$$

that on the level of objects selects out $\mathsf{Exit}(\tilde{X})$, which we will show is canonically determined by the action of G on $\tilde{X}$. The data of the action on $\tilde{X}$ as a topological space gives an action on $\tilde{X}$ in Strat

$$\mathcal{B}G \to \mathsf{Strat}$$

that on the level of objects selects out the stratified space $\tilde{X} \to \mathsf{P}$. We can post compose this functor with the inclusion functor of Strat into $\mathsf{Psh}(\Delta)$

$$\mathcal{B}G \to \mathsf{Strat} \to \mathcal{S}\mathsf{trat} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_{(\infty,1)} \hookrightarrow \mathsf{Psh}(\Delta)\ ,$$

which gives an action on $\mathsf{Exit}(\tilde{X}) \in \mathsf{Psh}(\Delta)$. The functor $\mathsf{Exit}(\tilde{X}) \to \mathsf{Exit}(X)$ is G invariant, since the map of spaces $\tilde{X} \to X$ is G invariant, so we get a functor

$$\mathcal{B}G^{\triangleright} \to \mathsf{Cat}_{(\infty,1)}\ .$$

The colimit of the functor $\mathcal{B}G \to \mathsf{Cat}_{(\infty,1)}$ is the left Kan extension of this functor along $\mathcal{B}G^{\triangleright}$, which is the quotient of of $\mathsf{Exit}(\tilde{X})$ by G, which will be denoted $\mathsf{Exit}(\underline{\tilde{X}})/G$. The value of $\mathsf{Exit}(\tilde{X})/G$ is the following colimit

$$\mathsf{Exit}(\underline{\tilde{X}})/G := \mathsf{colim}\left(\mathcal{B}G_{/+\infty} \to \mathcal{B}G \to \mathsf{Cat}_{(\infty,1)}\right) \ .$$

We lastly need to check that the map

$$\mathsf{colim}(\mathcal{B}G_{/+\infty} \to \mathcal{B}G \to \mathsf{Cat}_{(\infty,1)}) \to \mathsf{Exit}(\underline{X})$$

is an equivalence.

We first reduce to the case that $\tilde{X} \to X$ is a trivial bundle using covering sieves. Define

$$\mathcal{U} := \{(A \xrightarrow{f} X) \in \mathsf{Strat}_{/X} \mid f^*\tilde{X} \to A \text{ is trivializable}\} \subset \mathsf{Strat}_{/X}$$

Similarly, define

$$\tilde{\mathcal{U}} := \{(f^*A \to \tilde{X}) \in \mathsf{Strat}_{/\tilde{X}} \mid (A \to X) \in \mathcal{U}\} \subset \mathsf{Strat}_{/\tilde{X}} \ .$$

$\mathcal{U}$ and $\tilde{\mathcal{U}}$ both contain an open cover of X and $\tilde{X}$ respectively, so $\mathcal{U}$ and $\mathcal{U}^*$ are covering sieves. Covering sieves have the property that

$$\mathsf{colim}(\mathcal{U} \to \mathsf{Strat}) \simeq X$$

$$\mathsf{colim}(\tilde{\mathcal{U}} \to \mathsf{Strat}) \simeq \tilde{X} \ .$$

By Proposition 3.3.8 of [4], the functor

$$\mathcal{U}^{\triangleright} \to \mathsf{Strat} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_{(\infty,1)}$$

preserves colimits of covering sieves. Therefore, there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathsf{Exit}(\underline{\tilde{X}}) \arrow[dd] \arrow[dr] & & \underset{(f^*A\to\tilde{X})\in\tilde{\mathfrak{U}}}{\mathsf{colim}} \big(\mathsf{Exit}(\underline{f^*A})\big) \arrow[ll, "\simeq"'] \arrow[dd] \arrow[dr] & \\
& \mathsf{Exit}(\underline{X}) & & \underset{(A\to X)\in\mathfrak{U}}{\mathsf{colim}} (\mathsf{Exit}(\underline{A})) \arrow[ll, "\simeq"'] \\
\mathsf{Exit}(\tilde{X})/G \arrow[ur] & & \left(\underset{(f^*A\to\tilde{X})\in\tilde{\mathfrak{U}}}{\mathsf{colim}} (\mathsf{Exit}(f^*\underline{A}))\right)/G \arrow[ll, "\simeq"'] \arrow[ur] &
\end{tikzcd} .$$

Therefore if the morphisms

$$\left(\underset{(f^*A\to\tilde{X})\in\tilde{\mathfrak{U}}}{\mathsf{colim}} \big(\mathsf{Exit}(\underline{f^*A})\big)\right)/G \longrightarrow \underset{(A\to X)\in\mathfrak{U}}{\mathsf{colim}}(\mathsf{Exit}(\underline{A}))$$

is an equivalence, then the morphism

$$\mathsf{Exit}(\underline{\tilde{X}})/G \longrightarrow \mathsf{Exit}(\underline{X})$$

is an equivalence. Since the quotient by the action of G is a colimit and using the fact that colimits commute, there is an equivalence

$$\left(\underset{(f^*A\to\tilde{X})\in\tilde{\mathfrak{U}}}{\mathsf{colim}} \big(\mathsf{Exit}(\underline{f^*A})/G\big)\right) \longrightarrow \left(\underset{(f^*A\to\tilde{X})\in\tilde{\mathfrak{U}}}{\mathsf{colim}} \big(\mathsf{Exit}(\underline{f^*A})\big)\right)/G\ .$$

Therefore we have reduced the problem to showing that

$$\mathsf{Exit}(\underline{f^*A})/G \to \mathsf{Exit}(\underline{A})$$

is an equivalence. By definition of $\tilde{\mathfrak{U}}$, $f^*A \to A$ is a trivial G bundle. Therefore there is a isomorphism of bundles

$$\begin{tikzcd} A \times G \arrow[r, "\simeq"] \arrow[d] & f^*A \arrow[d, "\pi_A"] \\ A \arrow[r] & A \end{tikzcd}$$

where the left vertical map is given by projection. Therefore the stratification map $A \times G \to \mathsf{P}$ factors through $\mathsf{P} \times *$.

$$\begin{tikzcd} A \times G \arrow[r, "\pi_A"] \arrow[d, "\mathcal{Z}_{|A} \times !"'] & A \arrow[d, "\mathcal{Z}_{|A}"] \\ \mathsf{P} \times * \arrow[r, "\pi_{\mathsf{P}}"'] & \mathsf{P} \end{tikzcd}$$

By Observation 3.3.3 of [4], there is an equivalence of ∞-categories,

$$\mathsf{Exit}(A \times G) \simeq \mathsf{Exit}(A) \times \mathsf{Exit}(G)$$

Since G is stratified by a point, then

$$\mathsf{Exit}(G) \simeq G$$

Therefore, we have an equivalence between $(\infty, 1)$-categories

$$\mathsf{Exit}(A \times G) \simeq \mathsf{Exit}(A) \times G\ .\ .$$

Furthermore, this equivalence witnesses $\mathsf{Exit}(A \times G)$ as a free (right) G-module on the ∞-category $\mathsf{Exit}(A)$, since the action of G on $A \times G$ is given by

$$\begin{array}{ccc} (A \times G) \times G & \longrightarrow & A \times G \\ & & . \\ (a, g_1, g_2) & \longmapsto & (a, g_1 g_2) \end{array}$$

The action of G on $\mathsf{Exit}(\underline{A}) \times \mathsf{Exit}(\underline{G})$ only acts on $\mathsf{Exit}(\underline{G})$, so there is an equivalence

$$\left(\mathsf{Exit}(\underline{A}) \times \mathsf{Exit}(\underline{G})\right) /G \simeq \mathsf{Exit}(\underline{A}) \times \left(\mathsf{Exit}(\underline{G})/G\right) \ .$$

Altogether we have the following equivalences

$$\mathsf{Exit}(\underline{A \times G})/G \xrightarrow{\simeq} \left(\mathsf{Exit}(\underline{A}) \times \mathsf{Exit}(\underline{G})\right) /G \xrightarrow{\simeq} \mathsf{Exit}(\underline{A}) \times \left(\mathsf{Exit}(\underline{G})/G\right) \xrightarrow{\simeq} \mathsf{Exit}(\underline{A}) \times (G/G)$$

$$\mathsf{Exit}(\underline{A}) \times (G/G) \xrightarrow{\simeq} \mathsf{Exit}(\underline{A})$$

Therefore

$$\left(\underset{(f^*A \to \tilde{X}) \in \tilde{\mathfrak{U}}}{\mathsf{colim}} \left(\mathsf{Exit}(\underline{f^*A})\right)\right) /G \longrightarrow \underset{(A \to X) \in \mathfrak{U}}{\mathsf{colim}} \left(\mathsf{Exit}(\underline{A})\right)$$

is an equivalence, which shows that

$$\mathsf{Exit}(\underline{\tilde{X}})/G \to \mathsf{Exit}(\underline{X})$$

is an equivalence. $\square$

Therefore the case of identifying the exit path ∞-category of the stratification $\underline{\mathbb{CP}^\infty}$ is reduced to calculating the exit path ∞-category of $\underline{\mathbb{S}^\infty}$. The next section recalls a theorem of [7], called the Décollage Theorem, which we will use to identify the ∞-category $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$.

Given a conically smooth stratified space $X \to \mathsf{P}$, there exists an ∞-category $\mathsf{Exit}(\underline{X})$ consisting of those paths that "exit" into a higher strata. There is a canonical functor $\mathsf{Exit}(\underline{X}) \to \mathsf{P}$ induced by inclusion of the exiting paths together with the stratificaiton $X \to \mathsf{P}$. The functor $\mathsf{Exit}(\underline{X}) \to \mathsf{P}$ has the following property.

Definition 146. A functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is ***conservative*** if it reflects isomorphisms. That is, for a morphisms $f : c \to c'$ in $\mathcal{C}$, if $\mathsf{F}(f) : \mathsf{F}(c) \to \mathsf{F}(c')$ is an isomorphims then $f : c \to c'$ is an isomorphism.

Lemma 147. *Let $X \to \mathsf{P}$ be a conically smooth stratified space. Then the canonical map*

$$\mathsf{Exit}(\underline{X}) \to \mathsf{P}$$

is conservative.

Proof. Recall the canonical map from Observation 106. Consider a morphism in $\mathsf{Exit}\,(\underline{X})$

$$\begin{tikzcd} \Delta^1 \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ {[1]} \arrow[r, "\overline{\alpha}"'] & \mathsf{P} \end{tikzcd} .$$

This morphisms in $\mathsf{Exit}\,(\underline{X})$ gets sent to the bottom arrow of the diagram, $[1] \to \mathsf{P}$. This map is an isomorphism if and only if it is the constant map at an element of P, since P is skeletal by the antisymmetric property of the poset P. Assume then the image of $\overline{\alpha}$ is the element $p \in \mathsf{P}$. Then Δ^1 must factor through the fiber of i in X, which is the i-stratum X_i. Therefore taking the inverse path of α in X_i gives the inverse morphism of α. $\square$

Definition 148. Define the ∞-category

$$\left(\mathsf{Cat}_{(\infty,1)}{}_{/\mathsf{P}}\right)^{\mathsf{cons}} := \{\mathsf{F} : \mathcal{C} \to \mathsf{P} \in \mathsf{Cat}_{(\infty,1)}{}_{/\mathsf{P}} | \mathsf{F} \text{ is conservative in } \mathsf{Cat}_{(\infty,1)}\} \ .$$

to be the ∞-category of conservative functors over P .

Definition 149. Let $\mathcal{K} \to \mathsf{P}$ be a conservative functor in $\left(\mathsf{Cat}_{(\infty,1)}{}_{/\mathsf{P}}\right)^{\mathsf{cons}}$, and I a subposet of P. We define $\mathsf{Hom}_{/\mathsf{P}}(\mathsf{I}, \mathcal{K})$ to be the following pullback in $\mathsf{Cat}_{(\infty,1)}$.

$$\begin{tikzcd}
\mathsf{Hom}_{/\mathsf{P}}(\mathsf{I}, \mathcal{K}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}(\mathsf{I}, \mathcal{K}) \arrow[d] \\
* \arrow[r, "\langle \mathsf{I} \hookrightarrow \mathsf{P} \rangle"] & \mathsf{Hom}(\mathsf{I}, \mathsf{P})
\end{tikzcd} .$$

For a conservative functor $\mathcal{K} \to \mathsf{P}$, the fibers over each $p \in \mathsf{P}$ are a space. Therefore the ∞-category $\mathsf{Fun}_{\mathsf{P}}(\mathsf{I}, \mathcal{K})$ is a space.

Definition 150. A presheaf $\mathsf{F} : \mathsf{sd}(\mathsf{P})^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ on the poset $\mathsf{sd}(\mathsf{P})$ (Definition 26) is a ***spatial décollage*** if the canonical map of spaces

$$F(\{p_0 \leq p_1 \leq \cdots \leq p_n) \to F(\{p_0 \leq p_1\}) \underset{F(p_1)}{\times} F(\{p_1 \leq p_2\}) \underset{F(p_2)}{\times} \dots) \times \underset{F(p_{n-1})}{\times} F(\{p_{n-1} \leq p_n\})$$

is an equivalence.

Definition 151. The ∞-category $\mathsf{Déc}_{\mathsf{P}}(\mathcal{S}\mathsf{paces})$ is the full ∞-subcategory of $\mathsf{Psh}(\mathsf{sd}(\mathsf{P}))$ consisting of those presheaves that are also spatial décollages.

Observation 152. There is a functor

$$\Gamma : \left(\mathsf{Cat}_{(\infty,1)}{}_{/\mathsf{P}}\right)^{\mathsf{cons}} \to \mathsf{Déc}_{\mathsf{P}}(\mathcal{S}\mathsf{paces})$$

that sends a conservative functor $X \to \mathsf{P}$ to the presheaf

$$\begin{aligned} \Gamma(X \to \mathsf{P}) : \mathsf{sd}(\mathsf{P})^{\mathsf{op}} &\longrightarrow \mathcal{S}\mathsf{paces} \\ \mathsf{I} &\longmapsto \mathsf{Hom}_{/\mathsf{P}}(\mathsf{I}, X) \end{aligned}$$

In [7], it is shown that this presheaf is indeed a spatial décollage.

Notation 153. Let $i_{\mathsf{sd}(\mathsf{P})} : \mathsf{sd}(\mathsf{P}) \to \mathsf{Cat}_{(\infty,1)}$ be the canonical functor that regards a subposet of P as a category, and the inclusion maps as an order preserving map between them.

Observation 154. The functor $i_{\mathsf{sd}(\mathsf{P})} : \mathsf{sd}(\mathsf{P}) \to \mathsf{Cat}_{(\infty,1)}$ induces a functor

$$\begin{aligned} - \underset{\mathsf{Cat}_{(\infty,1)}}{\otimes} i_{\mathsf{sd}(\mathsf{P})} : \mathsf{D\acute{e}c}_{\mathsf{P}}(\mathcal{S}\mathsf{paces}) &\longrightarrow \left(\mathsf{Cat}_{(\infty,1)}{}_{/\mathsf{P}}\right)^{\mathsf{cons}} \\ \acute{S} &\longmapsto \left(\acute{S} \underset{\mathsf{Cat}_{(\infty,1)}}{\otimes} i_{\mathsf{sd}(\mathsf{P})} \to \mathsf{P}\right) \end{aligned},$$

where $\acute{S} \underset{\mathsf{Cat}_{(\infty,1)}}{\otimes} i_{\mathsf{sd}(\mathsf{P})}$ is the coend (Definition 334) of functors $\acute{S}$ and $i_{\mathsf{sd}(\mathsf{P})}$.

Theorem 155 (Décollage Theorem)**.** *The pair of functors*

$$\left(\mathsf{Cat}_{(\infty,1)}{}_{/\mathsf{P}}\right)^{\mathsf{cons}} \underset{- \underset{\mathsf{Cat}_{(\infty,1)}}{\otimes} i_{\mathsf{sd}(\mathsf{P})}}{\overset{\Gamma}{\rightleftarrows}} \mathsf{D\acute{e}c}_{\mathsf{P}}(\mathcal{S}\mathsf{paces})$$

is an equivalence of ∞-categories.

Proof. This is the statement of Theorem 2.7.4 of [7]. □

<u>Exit Path ∞-Category of $\underline{\mathbb{S}^\infty}$</u>

This section seeks to identify the the exit path ∞-category of $\mathbb{S}^\infty$. We identify the ∞-category $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ by identifying its associated spacial décollage

$$\begin{aligned} \Gamma\left(\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right) : \mathsf{sd}(\mathbb{Z}_{\geq 0}) &\longrightarrow \mathcal{S}\mathsf{paces} \\ \mathsf{I} \subset \mathbb{Z}_{\geq 0} &\longmapsto \mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}\left(\mathsf{I}, \mathsf{Exit}(\underline{\mathbb{S}^\infty})\right) \end{aligned}.$$

Therefore the majority of this section is to provide identifications of the space $\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\mathsf{I}, \mathsf{Exit}(\underline{\mathbb{S}^\infty}))$. The following homotopy equivalence will be used in this goal of identifying the spaces named in the spacial d'ecollage of $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$.

Lemma 156. *Let $i \in \mathbb{Z}_{\geq 1}$. The map of spaces*

$$\begin{aligned} \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} &\longrightarrow \mathbb{T} \\ (z_1, \dots, z_i) &\longrightarrow \tfrac{z_i}{\|z_i\|} \end{aligned}$$

is a homotopy equivalence.

Proof. Note first that the map $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \to \mathbb{T}$ is well defined since the last coordinate z_i must be nonzero.

The homotopy inverse map is given by the inclusion of $\mathbb{T}$ into the last coordinate

$$\begin{aligned} \mathbb{T} &\longrightarrow \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \\ z &\longrightarrow (0, 0, \dots, 0, z) \end{aligned}$$

The composite map defined on $\mathbb{T}$ is given by

$$\begin{array}{ccccc} \mathbb{T} & \longrightarrow & \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} & \longrightarrow & \mathbb{T} \\ z & \longrightarrow & (0, 0, \dots, 0, z) & \longrightarrow & \frac{z}{\|z\|} \end{array},$$

which is the identity since $\|z\| = 1$.

The second composite defined on $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}$ is

$$\begin{array}{ccccc} \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \\ \mathbf{z} := (z_1, \dots, z_i) & \longrightarrow & \frac{z_i}{\|z_i\|} & \longrightarrow & (0, 0, \dots, 0, \frac{z_i}{\|z_i\|}) \end{array}$$

We next define our homotopy between this composition and the identity map

$$\begin{array}{ccc} \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}\times I & \longrightarrow & \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \\ (\mathbf{z},t) & \longrightarrow & \frac{t\mathbf{z}+(1-t)(0,\ldots,\frac{z_i}{\|z_i\|})}{\|t\mathbf{z}+(1-t)(0,\ldots,\frac{z_i}{\|z_i\|})\|} \end{array}.$$

We need to check that for all $t \in I$, that $t\mathbf{z} + (1-t)(0,\ldots,\frac{z_i}{\|z_i\|}) \neq 0$. Note that

$$t\mathbf{z} + (1-t)(0,\ldots,\frac{z_i}{\|z_i\|}) = (tz_1,\ldots,tz_{i-1},tz_i + (1-t)\frac{z_i}{\|z_i\|})$$

Assume that the point $(tz_1,\ldots,tz_{i-1},tz_i + (1-t)\frac{z_i}{\|z_i\|}) = 0$. Then either $t = 0$ or $(z_1,\ldots,z_{i-1}) = 0$. If $t = 0$ then the last coordinate is

$$0 * z_i + (1-0)\frac{z_i}{\|z_i\|} = \frac{z_i}{\|z_i\|}$$

which is nonzero since $\frac{z_i}{\|z_i\|} \in \mathbb{S}^1$. If $(z_1,\ldots,z_{i-1}) = 0$, then $\|z_i\| = 1$. Therefore since $\|z_i\| = 1$, the last coordinate becomes

$$tz_i + (1-t)\frac{z_i}{\|z_i\|} = tz_i + (1-t)z_i = z_i$$

Therefore we have that $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \simeq \mathbb{T}$.

$\square$

Lemma 157. *The spacial décollage associated to the* ∞*-category* $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ *evaluates on* $\{i\} \subset \mathbb{Z}_{\geq 0}$ *as* $\mathbb{T} \in \mathcal{S}\mathsf{paces}$ *.*

$$\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))(\{i\}) = \mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i\},\mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \mathbb{T}$$

Proof. The space of objects of $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ is the disjoint union of all of the strata, by Lemma 104. By the Yoneda Lemma, and Lemma 104 which gives the spaces of objects are identified with the coproducts of strata,

$$\mathsf{Hom}(\{i\}, \mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \coprod_{i \in \mathsf{P}} \mathbb{S}^{2i+1} \backslash \mathbb{S}^{2i-1} \ .$$

Similarly, we have that

$$\mathsf{Hom}(\{i\}, \mathbb{Z}_{\geq 0}) \simeq \coprod_{i \in \mathbb{Z}_{\geq 0}} \{i\} \ .$$

Therefore our pullback is given by

$$\begin{tikzcd}
\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i\}, \underline{\mathbb{S}^\infty}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_{i \in \mathsf{P}} \mathbb{S}^{2i+1} \backslash \mathbb{S}^{2i-1} \arrow[d] \\
\{i\} \arrow[r] & \coprod_{i \in \mathsf{P}} \{i\}
\end{tikzcd} ,$$

which is the fiber over $\{i\}$. The fiber over $\{i\}$ is just the i-stratum, which is the space $\mathbb{S}^{2i+1} \backslash \mathbb{S}^{2i-1}$. Finally, the homotopy equivalence of Lemma 156 gives

$$\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i\}, \mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \mathbb{T} \ .$$

$\square$

Lemma 158. *The spacial décollage associated to the ∞-category $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ evaluates on $\{i \leq j\} \subset \mathbb{Z}_{\geq 0}$ as $\mathbb{T} \times \mathbb{T} \in \mathcal{S}\mathsf{paces}$.*

$$\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))(\{i \leq j\}) = \mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i \leq j\}, \mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \mathbb{T} \times \mathbb{T}$$

Proof. The space $\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i \leq j\}, \mathsf{Exit}(\underline{\mathbb{S}^\infty}))$ is the space of morphisms from the ith strata to the jth strata. Therefore by Lemma 104, the space of morphisms is given by

$$\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i \leq j\}, \underline{\mathbb{S}^\infty}) \simeq \mathsf{Mor}(\mathsf{Exit}(\underline{\mathbb{S}^\infty})^i_j \simeq \mathsf{Link}_{\mathbb{S}^{2i+1} \backslash \mathbb{S}^{2i-1}}(\underline{\mathbb{S}^\infty})_j \ .$$

The link $\mathsf{Link}_{/\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}}(\underline{\mathbb{S}^\infty})_j$ is defined to be

$$\mathsf{Link}_{\mathbb{S}^{2i+1}/\mathbb{S}^{2i-1}}(\underline{\mathbb{S}^\infty})_j \simeq \mathbb{S}^{\mathsf{fib}}\left(\mathsf{N}(\mathbb{S}^{2(i)+1} \hookrightarrow \mathbb{S}^{2(j)+1})\right)_{|\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}} .$$

Over a given point $p \in \mathbb{S}^{2i+1}$, the normal bundle $\mathsf{N}(\mathbb{S}^{2i+1} \hookrightarrow \mathbb{S}^{2j+1})$ is the quotient of the the tangent space $T\mathbb{S}^{2j+1}$ by the image of the tangent bundle $T\mathbb{S}^{2i+1}$

$$\left(\mathsf{N}(\mathbb{S}^{2(i)+1} \hookrightarrow \mathbb{S}^{2(j)+1})\right)_p = \left(\mathsf{T}\mathbb{S}^{2j+1}/\mathsf{T}\mathbb{S}^{2i+1}\right)_p .$$

The tangent bundle of a sphere $\mathbb{S}^{2i+1}$ over a given point $p \in \mathbb{S}^{2i+1}$ is given by

$$\mathsf{T}(\mathbb{S}^{2i+1})_p = (p, \mathsf{Hom}^{\mathsf{lin}}\left(\mathsf{Span}\{p\}, \mathbb{C}^i \perp p\right)) .$$

where $\mathsf{Hom}^{\mathsf{lin}}(\mathsf{Span}\{p\}, \mathbb{C}^i \perp p)$ is all $\mathbb{R}$-linear maps from the $\mathsf{Span}p$ to $\mathbb{C}^i \perp p$. Therefore the normal bundle over a point p is the quotient

$$\mathsf{N}(\mathbb{S}^{2i+1} \hookrightarrow \mathbb{S}^{2j+1})_p = \frac{(p, \mathsf{Hom}^{\mathsf{lin}}(\mathsf{Span}\{p\}, \mathbb{C}^j \perp p))}{(p, \mathsf{Hom}^{\mathsf{lin}}(\mathsf{Span}\{p\}, \mathbb{C}^i \perp p))} .$$

Using the isomorphism $\mathsf{Hom}^{\mathsf{lin}}(\mathsf{Span}\{p\}, V) \simeq \mathsf{Hom}^{\mathsf{lin}}(\mathbb{R}, V) \simeq V$, the quotient space is given by

$$= \frac{(p, \mathbb{C}^{j-i} \oplus \mathbb{C}^i \perp p)}{(\mathbb{C}^i \perp p)},$$

$$= (p, \mathbb{C}^{j-i})$$

Therefore the fiber over p does not depend on the point p. This gives an vector bundle isomorphism of the normal bundle with the trivial bundle over $\mathbb{S}^{2i+1}$. Therefore when we restrict the normal bundle over $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}$, there is an isomorphism

$$\mathbb{S}^{\mathsf{fib}}\left(\mathsf{N}(\mathbb{S}^{2i+1}\hookrightarrow\mathbb{S}^{2j+1})\right)_{|_{\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}}}\simeq\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}\times\mathbb{T}^{j-i}$$

Finally the homotopy equivalence of Lemma 156 gives the desired result

$$\mathbb{S}^{\mathsf{fib}}\left(\mathsf{N}(\mathbb{S}^{2i+1}\hookrightarrow\mathbb{S}^{2j+1})\right)_{|_{\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}}}\simeq\mathbb{T}\times\mathbb{T}$$

$\square$

Lemma 159. *Through the equivalences of Lemma 157 and Lemma 158, the source map and target maps for* $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ *are given as follows.*

(1) The source map $\mathsf{Exit}(\underline{\mathbb{S}^\infty})(\{i\}\to\{i\leq j\})$ *is given by projection onto* $\mathbb{T}$

$$\begin{array}{ccc}\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}\left(\{i\leq j\},\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right)\simeq\mathbb{T}\times\mathbb{T} & \longrightarrow & \mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}\left(\{i\},\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right)\simeq\mathbb{T} \\ (x,p) & \longmapsto & x\end{array}.$$

(2) The target map $\mathsf{Exit}(\underline{\mathbb{S}^\infty})(\{j\}\to\{i\leq j\})$ *is given by projection onto the last coordinate*

$$\begin{array}{ccc}\mathsf{Hom}_{\mathbb{Z}_{\geq 0}}\left(\{i\leq j\},\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right)\simeq\mathbb{T}\times\mathbb{T} & \longrightarrow & \mathsf{Hom}_{\mathbb{Z}_{\geq 0}}\left(\{j\},\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right)\simeq\mathbb{T} \\ (x,p) & \longmapsto & p\end{array}.$$

Proof. The source map is given by the restriction of the bundle map

$$\mathsf{N}(\mathbb{S}^{2i+1}\hookrightarrow\mathbb{S}^{2j+1})\to\mathbb{S}^{2i+1}$$

to the link $\mathsf{Link}_{\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}}(\mathbb{S}^{2i+1})_j$ over $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}$. By Lemma 158, the normal bundle is trivial. Therefore the source map through our homotopy equivalence of 156 is given by

$$\begin{tikzcd}
\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{T} \arrow[r] \arrow[d, "\pi_{\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}}"'] & \mathbb{T}\times\mathbb{T} \arrow[d, "\pi_1"] \\
\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \arrow[r] & \mathbb{T}
\end{tikzcd}.$$

The target map is given by restricting a tubular neighborhood to the sphere bundle. Consider the map

$$\begin{aligned}
\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i+1} \times \mathbb{C}^{j-i} &\longrightarrow \mathbb{C}^{j+1} \\
(s,p) &\longmapsto \tfrac{\sqrt{2}}{2}(s,p)
\end{aligned}$$

The link $\mathsf{Link}_{\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}} \simeq \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{T}$ includes into $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{C}^{j-i}$, by the canonical inclusion of $\mathbb{S}^{2i+1}\mathbb{S}^{2i-1}$, and the inclusion of $\mathbb{T}$ into the last copy of $\mathbb{C}$ in $\mathbb{C}^{j-i}$.

$$\begin{array}{ccccc}
\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{T} & \longrightarrow & \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{C}^{j-i} & \longrightarrow & \mathbb{C}^{j+1} \\
(x,p) & \longmapsto & (x,0,\ldots,0,p) & \longmapsto & \tfrac{\sqrt{2}}{2}(x,0,\ldots,p)
\end{array}$$

Note that the composite map from $\mathsf{Link}_{\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1}}(\mathbb{S}^\infty)_j$ to $\mathbb{C}^{j+1}$ factors through $\mathbb{S}^{2j+1}\backslash\mathbb{S}^{2j-1}$ since

$$\|\frac{\sqrt{2}}{2}(x,0\ldots,0,p)\| = \frac{1}{2}(\sqrt{\|x\|}+\sqrt{\|p\|}) = \frac{1}{2}(1+1) = 1$$

and also the last coordinate p is nonzero, since $p \in \mathbb{T}$. Therefore since the space of embeddings of the normal bundle $\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{C}^{j-i}$ into $\mathbb{S}^{2j+1}\backslash\mathbb{S}^{2j-1}$ is contractible, this embedding gives the target map

$$\begin{aligned}
\mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{T} &\longrightarrow \mathbb{S}^{2j+1}\backslash\mathbb{S}^{2i-1} \\
(x,p) &\longmapsto \tfrac{\sqrt{2}}{2}(x,0,\ldots 0,p)
\end{aligned}.$$

The next step is to identify the target map through our homotopy equivalence. The target map from $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is the filler

$$\begin{tikzcd} \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{T} \arrow[r] \arrow[d] & \mathbb{S}^{2j+1}\backslash\mathbb{S}^{2j-1} \arrow[d] \\ \mathbb{T} \times \mathbb{T} \arrow[r, dashed] & \mathbb{T} \end{tikzcd} .$$

Let $(x, p) \in \mathbb{S}^{2i+1}\backslash\mathbb{S}^{2i-1} \times \mathbb{T} \subset \mathbb{C}^{i+1} \times \mathbb{T}$ and x_{i+1} be the last coordinate of $x \in \mathbb{C}^{i+1}$. Then applying the target maps and homotopy equivalences, the point (x, p) is sent to the following

$$\begin{tikzcd} (x,p) \arrow[r, mapsto] \arrow[d, mapsto] & \frac{\sqrt{2}}{2}(x, 0\ldots, 0, p) \arrow[d, mapsto] \\ (\frac{x_i}{\|x_i\|}, p) & \frac{\frac{\sqrt{2}}{2}p}{\|\frac{\sqrt{2}}{2}p\|} = p \end{tikzcd} .$$

Therefore the filler map should send $\left(\frac{x_i}{\|x_i\|}, p\right)$ to the point p. Therefore the target map after applying the homotopy equivalence in Lemma 156 is the projection onto the second coordinate

$$\pi_2 : \mathbb{T} \times \mathbb{T} \to \mathbb{T} .$$

$\square$

Lemma 160. *The spacial décollage associated to the ∞-category $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ evaluates on $\mathsf{I} = \{i_0, \ldots, i_n\} \subset \mathbb{Z}_{\geq 0}$ as $\mathbb{T}^{\mathsf{I}} \in \mathcal{S}\mathsf{paces}$.*

$$\mathsf{Hom}_{\mathbb{Z}_{\geq 0}}(\mathsf{I}, \mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \left(\mathbb{T}^{\mathsf{I}}\right)$$

Proof. By the Décollage Theorem, the functor

$$\Gamma_{\mathsf{Exit}(\underline{\mathbb{S}^\infty})} := \Gamma\left(\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right) : \mathsf{Hom}_{\mathbb{Z}_{\geq 0}}(-, \mathsf{Exit}(\underline{\mathbb{S}^\infty})) : \mathsf{sd}(\mathsf{P})^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$$

is a spacial décollage. In particular, since the spacial décollage satisfies the Segal condition, the space $\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\mathsf{I}, \mathsf{Exit}(\underline{\mathbb{S}^\infty}))$ can be computed as the limit of the diagram

$$\begin{tikzcd}
 & & \dots \arrow[r] \arrow[d] & \Gamma_{\mathsf{Exit}(\underline{\mathbb{S}^\infty})}(\{i_{n-1} < i_n\}) \\
 & \Gamma_{\mathsf{Exit}(\underline{\mathbb{S}^\infty})}(i_1 < i_2) \arrow[r] \arrow[d] & \Gamma_{\mathsf{Exit}(\underline{\mathbb{S}^\infty})}(i_2) & \\
\Gamma_{\mathsf{Exit}(\underline{\mathbb{S}^\infty})}(\{i_0 < i_1\}) \arrow[r] & \Gamma_{\mathsf{Exit}(\underline{\mathbb{S}^\infty})}(\{i_1\}) & &
\end{tikzcd}$$

which can be computed as a series of pullbacks. The arrows in the diagram are the canonical morphisms indicated by the indexing. By Lemma 159 this is equivalent to computing the the limit of the diagram

$$\begin{tikzcd}
 & & & \mathbb{T} \times \mathbb{T} \arrow[d, "\pi_1"'] \\
 & & \dots \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbb{T} \\
 & \mathbb{T} \times \mathbb{T} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & \mathbb{T} & \\
\mathbb{T} \times \mathbb{T} \arrow[r, "\pi_2"'] & \mathbb{T} & &
\end{tikzcd} .$$

This limit upon inspection is the space $\mathbb{T}^{\mathsf{I}}$. □

Observation 161. The spacial décollage $\Gamma\left(\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right)$ is the presheaf

$$\begin{aligned}
\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))^{\mathsf{op}} : \mathsf{sd}(\mathbb{Z}_{\geq 0}) &\longrightarrow \mathcal{S}\mathsf{paces} \\
\mathsf{I} &\longmapsto \mathbb{T}^{\mathsf{I}}
\end{aligned} .$$

Proposition 162. *There is an equivalence of ∞-categories*

$$\mathsf{Exit}(\underline{\mathbb{S}^\infty}) \simeq \mathop{\bigstar}_{k \in \mathbb{Z}_{\geq 0}} \mathbb{T} .$$

Proof. Consider the composite functor

$$\begin{array}{rcl} \mathsf{E}_\bullet\mathbb{T} : \Delta^{\mathsf{op}} & \xrightarrow{\mathsf{fgt}} \mathsf{Fin}^{\mathsf{op}} & \xrightarrow{\mathbb{T}^{-}} \mathcal{S}\mathsf{paces} \\ \mathsf{I} = \{i_0 \leq \cdots \leq i_n\} & \longmapsto \{i_0, \ldots, i_n\} & \longmapsto \mathbb{T}^{\{i_0,\ldots,i_n\}} \end{array}$$

The canonical inclusion $\mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \hookrightarrow \Delta^{\mathsf{op}}$ induces a spacial décollege

$$\begin{array}{rcccl} \mathbb{T}_{\mathsf{Déc}} := \mathsf{sd}\left(\mathbb{Z}_{\geq 0}\right)^{\mathsf{op}} & \longrightarrow \Delta^{\mathsf{op}} & \longrightarrow \mathsf{Fin}^{\mathsf{op}} & \longrightarrow \mathcal{S}\mathsf{paces} \\ \mathbb{Z}_{\geq 0} \supset \mathsf{I} = \{i_0 \leq \cdots \leq i_n\} & \longmapsto \mathsf{I} & \longmapsto \{i_0, \ldots, i_n\} & \longmapsto \mathbb{T}^{\{i_0,\ldots,i_n\}} \end{array} .$$

Note that this functor $\mathbb{T}_{\mathsf{Déc}}$ is identified with the functor R in Definition 121 where each ∞-category is the space $\mathbb{T}$. In particular, the restriction of $\mathbb{T}_{\mathsf{Déc}}$ to the non-negative integers $\mathbb{Z}_{\geq 0}$ is the constant functor at $\mathbb{T} \in \mathcal{S}\mathsf{paces}$, and the functor $\mathbb{T}_{\mathsf{Déc}}$ is the right Kan extension therefrom:

$$\begin{tikzcd} \mathsf{Obj}\left(\mathbb{Z}_{\geq 0}\right) \arrow[r, "\mathsf{const}_{\mathbb{T}}"] \arrow[d] & \mathcal{S}\mathsf{paces} \\ \mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \arrow[ur, dashed, "\mathbb{T}_{\mathsf{Déc}}"'] & \end{tikzcd} .$$

Therefore, using the equivalence of ∞-categories in Theorem 155, the problem reduces to showing that there is an equivalence of spacial décollages

$$\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \mathbb{T}_{\mathsf{Déc}} .$$

Towards this goal, by Lemma 157, the functor $\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))$ restricts along $\mathsf{Obj}(\mathbb{Z}_{\geq 0})$ as the constant functor at $\mathbb{T}$

$$\begin{tikzcd} \mathsf{Obj}\left(\mathbb{Z}_{\geq 0}\right) \arrow[r, "\mathsf{const}_{\mathbb{T}}"] \arrow[d] & \mathcal{S}\mathsf{paces} \\ \mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \arrow[ur, dashed, "\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))"'] & \end{tikzcd} .$$

The spacial décollage $\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))$ also restricts on singleton subsets of $\mathbb{Z}_{\geq 0}$ as the constant functor at $\mathbb{T}$. Therefore there is a canonical choice for a natural transformation

$$\kappa : \Gamma\left(\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right) \to \mathbb{T}_{\mathsf{D\acute{e}c}}$$

implemented by the universal property of the right Kan extension

$$\begin{tikzcd}
\mathsf{Obj}\left(\mathbb{Z}_{\geq 0}\right) \arrow[r, "\mathsf{const}_{\mathbb{T}}"] \arrow[d] & \mathcal{S}\mathsf{paces} \\
\mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \arrow[ur, "\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))"] \arrow[ur, bend right, "\mathbb{T}_{\mathsf{D\acute{e}c}}"'] &
\end{tikzcd}$$

This natural transformation κ is clearly an equivalence on $\mathsf{Obj}(\mathbb{Z}_{\geq 0})$, the singleton subsets of $\mathbb{Z}_{\geq 0}$. The natural transformation $\kappa : \Gamma\left(\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right) \to \mathbb{T}_{\mathsf{D\acute{e}c}}$ is also an equivalence on cardinality 2 subsets of $\mathbb{Z}_{\geq 0}$ by Lemma 158. Finally, because both $\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))$ and $\mathbb{T}_{\mathsf{D\acute{e}c}}$ send pushouts in $\mathsf{sd}(\mathbb{Z}_{\geq 0})$ to pullbacks in $\mathcal{S}\mathsf{paces}$, the natural transformation κ is an equivalence

$$\kappa : \Gamma\left(\mathsf{Exit}(\underline{\mathbb{S}^\infty})\right) \xrightarrow{\simeq} \mathbb{T}_{\mathsf{D\acute{e}c}} \ .$$

□

Exit Path ∞-Category of $\underline{\mathbb{CP}^\infty}$

Notation 163. Define the ∞-category

$$\mathsf{E}_\infty := \mathsf{Exit}(\underline{\mathbb{CP}^\infty})$$

to be the exit path ∞-category of the stratification of $\mathbb{CP}^\infty$ induced by Lemma 54 and the closed $\mathbb{Z}_{\geq 0}$-filtration

$$\left(\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \dots\right)$$

Theorem 164. *The exit path ∞-category E_∞ is identified as follows.*

(1) E_∞ *is equivalent with the quotient of* $\mathsf{Exit}(\underline{\mathbb{S}^\infty})$ *by the action of* $\mathbb{T}$

$$\mathsf{E}_\infty \simeq \mathsf{Exit}(\underline{\mathbb{S}^\infty})_{/\mathbb{T}}$$

(2) The space of objects of E_∞ *is canonically identified with the non negative integers*

$$\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i\}, \mathsf{E}_\infty) \simeq \{i\} \ .$$

(3) Through the canonical identification of the objects of E_∞ *with* $\mathbb{Z}_{\geq 0}$*, the space of morphisms from* $i \leq j \in \mathbb{Z}_{\geq 0}$ *is given by* $\mathbb{T}$

$$\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i \leq j\}, \mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \simeq \mathbb{T} \ .$$

(4) The composition rule is the given by the group operation on $\mathbb{T}$

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \times \mathsf{Hom}_{\mathsf{E}_\infty}(k,l) \arrow[r, "\circ"] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathsf{E}_\infty}(j,l) \arrow[d, "\simeq"] \\
\mathbb{T} \times \mathbb{T} \arrow[r, "\mu"'] & \mathbb{T}
\end{tikzcd} \ .$$

Proof. The first statement is immediate from Proposition 145.

The second statement follows through the equivalences

$$\mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i\}, \mathsf{E}_\infty) \underset{\text{Proposition 145}}{\simeq} \mathsf{Hom}_{/\mathbb{Z}_{\geq 0}}(\{i\}, \mathsf{Exit}(\underline{\mathbb{S}^\infty}))_{/\mathbb{T}} \underset{\text{Lemma 157}}{\simeq} (\mathbb{T})_{/\mathbb{T}} \simeq * \ .$$

The last equivalence is since the action of $\mathbb{T}$ on $\mathbb{T}$ given by the group operation is free. The last statement reduces to showing there exists two homotopy equivalences

$$\begin{tikzcd} (\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} \arrow[r, "\pi_1\times\pi_3"] \arrow[d, dashed, "\simeq"'] & (\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} \arrow[d, dashed, "\simeq"] \\ \mathbb{T}\times\mathbb{T} \arrow[r, "\mu"] & \mathbb{T} \end{tikzcd} . \tag{2.1}$$

We begin by noting that the action of $\mathbb{T}$ on $\mathbb{T}\times\mathbb{T}\times\mathbb{T}$ is free, since it is free in each coordinate. For a free action of a group on a topological space, the homotopy quotient agrees with the quotient of topological spaces. Therefore we seek to identify the quotients of $\mathbb{T}^3$ and $\mathbb{T}^2$ respectively. First, consider the map

$$\begin{aligned} \phi : \mathbb{T}\times\mathbb{T}\times\mathbb{T} &\longrightarrow \mathbb{T}\times\mathbb{T} \\ (x,y,z) &\longmapsto (x^{-1}y, y^{-1}z) \end{aligned}$$

This map induces a map from the quotient space $(\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}}$ since for all $\alpha\in\mathbb{T}$

$$\phi_1(\alpha x,\alpha y,\alpha z) = ((\alpha x)^{-1}(\alpha y),(\alpha y)^{-1}(\alpha z) = (x^{-1}y, y^{-1}z)$$

where the last equality comes from the fact that $(\alpha x)^{-1} = x^{-1}\alpha^{-1}$. The map on the quotient will also be defined as $[\phi]$

$$\begin{aligned} [\phi] : (\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} &\longrightarrow \mathbb{T}\times\mathbb{T} \\ [x,y,z] &\longmapsto (x^{-1}y, y^{-1}z) \end{aligned}$$

. Similarly, there is a map

$$\begin{aligned} \psi : \mathbb{T}\times\mathbb{T} &\longrightarrow \mathbb{T} \\ (x,z) &\longmapsto (x^{-1}z) \end{aligned} ,$$

and the map descends to a map on the quotient $[\psi] : (\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} \to \mathbb{T}$.

The square (2.1) canonically

commute, since the top composite is given by

$$\begin{array}{ccccc} (\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} & \longrightarrow & (\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} & \longrightarrow & \mathbb{T} \\ [x,y,z] & \longmapsto & [x,z] & \longrightarrow & (x^{-1}z) \end{array}$$

and the bottom composite is given by

$$\begin{array}{ccccc} (\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} & \longrightarrow & (\mathbb{T}\times\mathbb{T}) & \longrightarrow & \mathbb{T} \\ [x,y,z] & \longmapsto & (x^{-1}y, y^{-1}z) & \longrightarrow & (x^{-1}yy^{-1}z) = (x^{-1}z) \end{array} .$$

Finally, we seek to show ϕ is a homotopy equivalence

$$\phi : (\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} \xrightarrow{\simeq} \mathbb{T}\times\mathbb{T} .$$

First, we show $[\phi]$ is surjective. Let $(x,z)\in\mathbb{T}\times\mathbb{T}$. The element $(x^{-1},1,z)\in\mathbb{T}\times\mathbb{T}\times\mathbb{T}$ gets sent through ϕ to $((x^{-1})^{-1},z)=(x,z)\in\mathbb{T}\times\mathbb{T}$. Therefore since the map $\phi:\mathbb{T}\times\mathbb{T}\times\mathbb{T}\to\mathbb{T}\times\mathbb{T}$ is surjective, the map $\phi:(\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}}$ is surjective.

Next, we show $[\phi]$ is injective. Let $(g,h)\in\mathbb{T}\times\mathbb{T}$. Consider the diagram

$$\begin{tikzcd} \phi^{-1}(g,h) \arrow[r] \arrow[d] & {[\phi^{-1}](g,h)} \arrow[r] \arrow[d] & * \arrow[d, "\langle(a,b)\rangle"] \\ \mathbb{T}\times\mathbb{T}\times\mathbb{T} \arrow[r] & (\mathbb{T}\times\mathbb{T}\times\mathbb{T})_{/\mathbb{T}} \arrow[r] & \mathbb{T}\times\mathbb{T} \end{tikzcd} .$$

Note that the outer square, and the right square are pullbacks by definition, which implies that the left square is a pullback. Note that $\mathbb{T}$ acts on $\pi^{-1}(g,h)$, and that

$$\phi(g,h)_{/\mathbb{T}} \simeq [\phi]^{-1}(g,h) .$$

Let $(x,y,z)\in\mathbb{T}\times\mathbb{T}\times\mathbb{T}$ be such that

$$\phi(x,y,z) = (g,h) .$$

By the definition of ϕ, the following two equations are true.

$$x^{-1}y = g \qquad y^{-1}z = h\ .$$

Note that if we fix y, then $x = yg^{-1}$ and $z = yh$ are uniquely determined. Fixing x uniquely determines y, which in turn uniquely determines z, and similarly the other direction. Therefore

$$\begin{tikzcd} \alpha \arrow[d, mapsto] \\ (g^{-1}\alpha, \alpha, \alpha h) \end{tikzcd} \qquad \begin{tikzcd} \phi^{-1}(g,h) \simeq \mathbb{T} \arrow[r, "!"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\langle g,h \rangle"] \\ \mathbb{T} \times \mathbb{T} \times \mathbb{T} \arrow[r] & \mathbb{T} \times \mathbb{T} \end{tikzcd}$$

is a pullback. The action of $\mathbb{T}$ on $\phi^{-1}(g,h) \simeq \mathbb{T}$ is given by the group multiplication, so

$$[\phi]^{-1}(g,h) = \phi^{-1}(g,h)_{/\mathbb{T}} \simeq \mathbb{T}_{/\mathbb{T}} = *.$$

Therefore $\phi^{-1}(g,h)$ is injective. $\square$

Theorem 165. *The ∞-category conservatively over the poset $\mathbb{Z}_{\geq 0}$, $\mathsf{E}_\infty \to \mathbb{Z}_{\geq 0}$, is the collage of the composite functor*

$$\mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \xrightarrow{(\mathsf{I} \subset \mathbb{Z}_{\geq 0}) \mapsto \mathsf{I}} \boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{Bar}_\bullet \mathbb{T}} \mathcal{S}\mathsf{paces}\ .$$

Proof. Recall again the functor

$$\begin{aligned} \mathsf{E}_\bullet \mathbb{T} : \Delta^{\mathsf{op}} &\xrightarrow{\quad \mathsf{fgt} \quad} \mathsf{Fin}^{\mathsf{op}} \xrightarrow{\quad \mathbb{T}^- \quad} \mathcal{S}\mathsf{paces} \\ \mathsf{I} = \{i_0 \leq \cdots \leq i_n\} &\longmapsto \{i_0, \ldots, i_n\} \longmapsto \mathbb{T}^{\{i_0,\ldots,i_n\}} \end{aligned}$$

The canonical inclusion $\mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \hookrightarrow \Delta^{\mathsf{op}}$ induces a spacial decollege

$$\begin{array}{ccccccc} \mathbb{T}_{\mathsf{D\acute{e}c}} := \mathsf{sd}\left(\mathbb{Z}_{\geq 0}\right)^{\mathsf{op}} & \longrightarrow & \Delta^{\mathsf{op}} & \longrightarrow & \mathsf{Fin}^{\mathsf{op}} & \longrightarrow & \mathcal{S}\mathsf{paces} \\ \mathbb{Z}_{\geq 0} \supset \mathsf{I} = \{i_0 \leq \cdots \leq i_n\} & \longmapsto & \mathsf{I} & \longmapsto & \{i_0, \ldots, i_n\} & \longmapsto & \mathbb{T}^{\{i_0,\ldots,i_n\}} \end{array} .$$

Proposition 162 witnesses the equivalence

$$\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty})) \simeq \mathbb{T}_{\mathsf{D\acute{e}c}} \ .$$

It is classically known that

$$(\mathsf{E}_\bullet \mathbb{T})_{/\mathbb{T}} \simeq \mathsf{Bar}_\bullet \mathbb{T} \ .$$

Therefore there is the sequence of equivalences

$$\mathsf{Exit}\left(\underline{\mathbb{CP}^\infty}\right) \underset{\text{Theorem 164}}{\simeq} \left(\Gamma(\mathsf{Exit}(\underline{\mathbb{S}^\infty}))\right)_{/\mathbb{T}} \underset{\text{Proposition 162}}{\simeq} \left(\mathbb{T}_{\mathsf{D\acute{e}c}}\right)_{/\mathbb{T}} \ .$$

where $\left(\mathbb{T}_{\mathsf{D\acute{e}c}}\right)_{/\mathbb{T}}$ is the spacial décollage

$$\mathsf{sd}(\mathbb{Z}_{\geq 0})^{\mathsf{op}} \xrightarrow{(\mathsf{I}\subset\mathbb{Z}_{\geq 0})\mapsto \mathsf{I}} \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{Bar}_\bullet \mathbb{T}} \mathcal{S}\mathsf{paces} \ .$$

$\square$

BACKGROUND ON STABLE ∞-CATEGORIES

Introduction to Stable ∞-Categories

Suspension and Loop Functors

The goal of this chapter is to supply facts about stable ∞-categories that will be used in chapter 4. First, before moving on to stable ∞-categories, we introduce the suspension and loop functors which are defined on a pointed ∞-category with finite limits and colimits. This serves two purposes. The suspension functor arises when talking about $\mathbb{T}$-modules in chapter 4, as well as the loop functor is used to generate many examples of stable ∞-categories, including the ∞-category of $\mathcal{S}\mathsf{p}$.

Definition 166. An ∞-category $\mathcal{K}$ is ***zero-pointed*** if there exists an object 0 that is both initial and terminal. The object 0 is called a ***zero object***.

Definition 167. Let $\mathcal{K}$ be a pointed ∞-category with pushouts. Let X be an object of $\mathcal{K}$. The ***suspension*** of X is the object ΣX defined to be the pushout in K

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, "!"'] & 0 \arrow[d, "!"] \\ 0 \arrow[r, "!"'] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd} .$$

Definition 168. Let $\mathcal{K}$ be a pointed ∞-category with pullbacks. Let X be an object of $\mathcal{K}$. The ***loops*** of X is the object ΩX of $\mathcal{K}$ that is defined by the pullback in K

$$\begin{tikzcd} \Omega X \arrow[r, "!"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d, "!"] \\ 0 \arrow[r, "!"'] & X \end{tikzcd} .$$

Convention 169. Throughout the remainded of this section, the ∞-category $\mathcal{K}$ will denote a pointed ∞-category that admits pushouts and pullbacks.

Observation 170. The suspension defines a functor between ∞-categories

$$\Sigma : \mathcal{K} \to \mathcal{K} \ .$$

Similarly, the loop space defines a functor between ∞-categories

$$\Omega : \mathcal{K} \to \mathcal{K} \ .$$

Observation 171. The suspension and loop space functors can be iterated by composition. Denote Σ^n as the composition of the functor Σ n times. Similarly, denote Ω^n as the composition of the functor Ω n times.

Remark 172. See [11] for more details on the suspension functor.

Observation 173. The suspension and loop functors define an adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left=2, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{K} \arrow[l, shift left=2, "\Omega"] \end{tikzcd} \ .$$

The unit morphism of this adjunction on an object X of $\mathcal{K}$ is given by the canonical diagram

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \Omega\Sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ & 0 \arrow[r] & \Sigma X \end{tikzcd} \ .$$

Similarly, the counit morphism on an object X of $\mathcal{K}$ is given by

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddr, bend left] & \\ 0 \arrow[r] \arrow[drr, bend right] & \Sigma\Omega X \arrow[dr, dashed] \arrow[ul, phantom, "\ulcorner"] & \\ & & X \end{tikzcd} .$$

Note that in particular, for objects X and Y of $\mathcal{K}$, the adjunction $\Sigma \dashv \Omega$ gives a canonical equivalence of spaces

$$\mathsf{Hom}_{\mathcal{K}}\left(\Sigma X, Y\right) \simeq \mathsf{Hom}_{\mathcal{K}}\left(X, \Omega Y\right) \ . \tag{3.1}$$

A particular instance of a pointed ∞-category is the pointed ∞-category of spaces

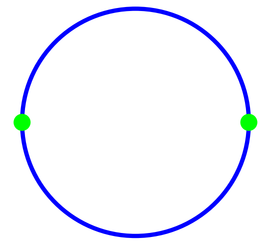
$$\mathcal{S}\mathsf{paces}_* := \mathcal{S}\mathsf{paces}^{*/} \ .$$

Recall that the category of spaces is the localization (Definition 337) of topological spaces on the weak homotopy equivalences[1] Weak homotopy equivalences participate in a Quillen model structure on the category Top. Therefore, by Corollary 4.2.4.8 of [10] , (co)limits in the ∞-category of $\mathcal{S}\mathsf{paces}$ can be computed as homotopy (co)limits in Top. A homotopy colimit in Top can be computed as the ordinary colimit after cofibrant replacement. For instance:

$$\mathsf{hocolim}\left(\begin{tikzcd} X \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}\right) \simeq \mathsf{colim}\left(\begin{tikzcd} X \arrow[r] \arrow[d] & \overline{\mathsf{C}}(X) \\ \overline{\mathsf{C}}(X) & \end{tikzcd}\right) .$$

In particular, for $X = \mathbb{S}^0$, $\overline{\mathsf{C}}(\mathbb{S}^0) \simeq [-1, 1]$ and $\Sigma\mathbb{S}^0 \simeq \mathbb{S}^1$, as indicated by the picture

[1] See [13] and [9], which show Kan complexes and the category of topological spaces restricted to CW complexes are quillen equivallent.



More generally, there is an equivalence

$$\Sigma\left(\mathbb{S}^{n-1}\right) \simeq \mathbb{S}^n \tag{3.2}$$

by noting that $\overline{\mathsf{C}}(\mathbb{S}^{n-1})$ is homeomorphic to a n-disk. Together, this gives

$$\Sigma^n \mathbb{S}^0 \simeq \mathbb{S}^n \ .$$

For X and Y based topological spaces, define the space $\mathsf{Map}_*(X, Y)$ to be the hom space

$$\mathsf{Map}_*(X, Y) := \mathsf{Hom}_{\mathcal{S}\mathsf{paces}_*}(X, Y)$$

Recall that the k-th homotopy group of a based space X is defined to be

$$\pi_k(X) := \mathsf{Map}_*(\mathbb{S}^k, X) \ .$$

Therefore using the adjunction (3.1), there is an adjunction

$$\pi_k(\Omega^p X) = \mathsf{Map}_*(\mathbb{S}^k, \Omega^p X) \underset{3.1}{\simeq} \mathsf{Map}_*(\Sigma^p \mathbb{S}^k, X) \underset{3.2}{\simeq} \mathsf{Map}_*(\mathbb{S}^{k+p}, X) = \pi_{k+p}(X) \ .$$

Stable ∞-categories

The material of this section is found in [11]. We recall some facts of stable ∞-categories that will be used in what follows.

Definition 174. An ∞-category $\mathcal{V}$ is a ***stable ∞-category*** if it satisfies the following conditions:

-) The ∞-category $\mathcal{V}$ is zero-pointed (Definition 166). In a stable ∞-category, denote 0 as the zero object.

-) For all morphisms $f : A \to B$, the following pushout and pullback of f exist

$$\begin{tikzcd} \mathsf{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & \mathsf{cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

The pullback, as the notation suggest, is the ***fiber*** of $f : A \to B$. Similarly, the pushout is the ***cofiber*** of $f : A \to B$.

-) Consider a diagram in $\mathcal{V}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

is a pullback diagram if and only it is a pushout diagram

Remark 175. A stable ∞-category plays the role in ∞-category theory of an ***abelian category*** in ordinary category theory. One important distinction between these notions, however, is that, while an abelian category is an ordinary category with additional *structure*, a stable ∞-category is an ∞-category satisfying *conditions*. This makes working with stable

∞-categories a more manageable task.

Remark 176. The final category $\{0\}$ is the only stable ∞-category that is an ordinary category. However, there are many ∞-categories that are stable. We will see that any pointed ∞-category with finite limits and colimits determines a stable ∞-category.

Lemma 177. *Let $\mathcal{V}$ be a pointed ∞-category. Then $\mathcal{V}$ is stable if and only if:*

-) *The ∞-category $\mathcal{V}$ admits finite limits and colimits.*

-) *Consider a diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{V}$. This diagram is a pushout if and only if it is a pullback diagram.

Proof. This is Proposition 1.1.3.4 of [11]. $\square$

This lemma gives important properties of stable ∞-categories. Note that clearly the set of conditions of Lemma 177 clearly imply the conditions of Definitoin 174, since fibers and cofibers are examples of pushouts and pullbacks. The conditions of Lemma 177 will be used frequently in what follows. The next lemma gives another important fact about stable ∞-categories, as well as provides an easily checkable condition if a category $\mathcal{V}$ is stable.

Lemma 178. *An ∞-category $\mathcal{V}$ is stable if and only if the following conditions are satisfied.*

-) *The ∞-category $\mathcal{V}$ is pointed.*

-) *The ∞-category $\mathcal{V}$ admits finite limits and colimits.*

-) *The endofunctor $\Sigma : \mathcal{V} \to \mathcal{V}$, or equivalently the endofunctor $\Omega : \mathcal{V} \to \mathcal{V}$, is an equivalence.*

Proof. This is Proposition 1.4.2.11 of [10]. □

Remark 179. Because the functors Σ and Ω are adjoint to one another, $\Sigma \dashv \Omega$, if one is an equivalence then so is the other.

Notation 180. In a stable ∞-category $\mathcal{V}$, we denote the loop functor as Σ^{-1}

$$\Sigma^{-1} := \Omega : \mathcal{V} \to \mathcal{V} \ .$$

Lemma 181. *Let $\mathcal{V}$ be a stable ∞-category, and let $\mathcal{K}$ be an arbitrary ∞-category. Then the ∞-category $\mathsf{Fun}(\mathcal{K}, \mathcal{V})$ is stable.*

Proof. This is Proposition 1.1.3.1 of [11]. □

<u>The Stable ∞-Category of Chain Complexes</u>

We introduce one of the main examples of a stable ∞-category, which is the ∞-category of chain complexes.

Definition 182. Let $\Bbbk$ be a commutative ring. A chain complex over $\Bbbk$ is

+) A set of $\Bbbk$-modules $\{M_i\}_{i \in \mathbb{Z}}$ indexed by the set $\mathbb{Z}$.

+) For each $\{i-1 < i\} \subset \mathbb{Z}$, a $\Bbbk$ linear map

$$d_i : M_i \to M_{i-1} \ .$$

such that:

-) The image of d_i is contained in the kernel of d_{i-1}.

$$\mathsf{im}(d_i) \subset \mathsf{ker}(d_{i-1}) \ .$$

Equivalently, the composition of d_i with d_{i-1} is 0

$$d_{i-1}d_i = 0$$

Definition 183. Let $M_\bullet$ and $N_\bullet$ be chain complexes. A map of chain complexes $\alpha : M_\bullet \to N_\bullet$ is

+) For each $n \in \mathbb{Z}$, a $\Bbbk$ linear map

$$\alpha_i : M_n \to N_n \ .$$

such that:

-) The maps α_n fits into a commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & M_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & M_n \arrow[r] \arrow[d, "\alpha_n"] & M_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \dots \\ \dots \arrow[r] & N_{n+1} \arrow[r] & N_n \arrow[r] & N_{n-1} \arrow[r] & \dots \end{tikzcd} .$$

Definition 184. Chain complexes and maps of chain complexes organize into an ordinary category $\mathsf{Chain}_\Bbbk$.

Definition 185. For each $n \in \mathbb{Z}$, the ***homology*** of a chain complex M in degree n is defined to be

$$H_n(M_\bullet) := \mathsf{ker}(f_n)/\mathsf{im}(f_{i+1}) \ .$$

Observation 186. A chain map $\alpha : M_\bullet \to N_\bullet$ induces a map on homology

$$H_n(M_\bullet) \to H_n(M_\bullet) \ .$$

Definition 187. A map of chain complexes $\alpha : M_\bullet \to N_\bullet$ is a ***quasi-isomorphism*** if for each n in $\mathbb{Z}$, the induced map on homology

$$H_n(M_\bullet) \to H_n(N_\bullet)$$

is an isomorphism.

Definition 188. Define the category $\mathcal{Q}$ to be the subcategory of $\mathsf{Chain}_\Bbbk$ consisting of all objects of $\mathsf{Chain}_\Bbbk$ with quasi-isomorphisms as morphisms.

Definition 189. The ∞-category of chain complexes $\mathcal{C}\mathsf{hain}_\Bbbk$ is defined to be the category of chain complexes localized (Definition 337) on quasi-isomorphisms

$$\mathcal{C}\mathsf{hain}_\Bbbk := \mathsf{Chain}_\Bbbk[\mathcal{Q}^{-1}]$$

Example 190. The suspension of a chain complex $M_\bullet$ is the chain complex

$$(\Sigma M_\bullet)_n := M_{n-1}$$

with differentials given by

$$(\Sigma M_\bullet)_{n+1} = M_n \xrightarrow{-d_n} M_{n-1} = (\Sigma M_\bullet)_n \ .$$

Note that clearly this suspension is invertible, and is given by shifting in the opposite direction and taking the negative of the differential map. Note that the fact that this satisfies the condition of Definition 167 is not obvious, and can be found in [11].

Corollary 191. The ∞-category $\mathcal{C}\mathsf{hain}_\Bbbk$ is a stable ∞-category.

Stabilization and Spectra

Lemma 178 states $\mathcal{V}$ is stable if and only if the loop functor $\Omega : \mathcal{V} \to \mathcal{V}$ is an equivalence. Therefore, a technique to produce new stable ∞-categories from previous ∞-categories. Namely, if we can formally invert the functor Ω, then we can produce stable ∞-categories from existing ∞-categories. This is the idea of stabilization that is introduced here. One key example of the stabilization of an ∞-category is the ∞-category of spectra, which is the stabilization of the ∞-category of $\mathcal{S}\mathsf{paces}$.

Definition 192. Let $\mathcal{K}$ be a be a pointed ∞-catergory that admits finite limits. The ***stabilization*** is defined to be the limit in $\mathsf{Cat}_{(\infty,1)}$

$$\mathsf{Stab}\,(\mathcal{K}) := \lim \left(\ldots \xrightarrow{\Omega} \mathcal{K} \xrightarrow{\Omega} \mathcal{K} \right) \ .$$

Definition 193. The ∞-category of ***spectra*** is defined to be the stabilization of the ∞-category of $\mathcal{S}\mathsf{paces}_*$

$$\mathcal{S}\mathsf{p} := \mathsf{Stab}\,(\mathcal{S}\mathsf{paces}_*) \ .$$

Remark 194. Note that if a pointed ∞-category $\mathcal{K}$ is already stable, the functor $\Omega : \mathcal{K} \to \mathcal{K}$ is an equivalence by Lemma 178. Therefore the canonical map

$$\mathcal{K} \to \mathsf{Stab}\,(\mathcal{K})$$

is an equivalence of ∞-categories.

An object in the ∞-category $\mathsf{Stab}\,(\mathcal{K})$ is presented by a sequence of spaces indexed by the non negative integers $\left(\{E_i\}_{i \in \mathbb{Z}_{\geq 0}}\right)$, together with an equivalence in $\mathcal{K}$ for each $i \in \mathbb{Z}_{>0}$

$$E_i \xrightarrow{\simeq} \Omega E_{i-1} \ .$$

Definition 195. There is a canonical forgetful functor from the stabilization of $\mathsf{Stab}(\mathcal{K})$ to the ∞-category $\mathcal{K}$ presented by

$$\Omega^\infty : \mathsf{Stab}(\mathcal{K}) \longrightarrow \mathcal{K}$$

$$E \longmapsto \mathsf{colim}\,(\Omega^1 E_1 \to \Omega^2 E_2 \to \dots)$$

Lemma 196. *Let $\mathcal{K}$ be a presentable pointed ∞-category. Then the functor $\Omega^\infty : \mathsf{Stab}(\mathcal{K}) \to \mathcal{K}$ admits a left adjoint*

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & \mathsf{Stab}(\mathcal{K}) \arrow[l, shift left, "\Omega^\infty"] \end{tikzcd} .$$

*A spectrum in the image of the functor Σ^∞ is a **suspension spectrum**.*

Proof. This is the content of section 1.4 of [11]. □

Example 197. Let $K \in \mathcal{K}$. The value of $\Sigma^\infty K$ is the sequence of objects of $\mathcal{K}$

$$\{\Sigma^i K\}_{i\in\mathbb{Z}_{\geq 0}}$$

together with the identify map

$$\Sigma(\Sigma^{i-1}K) \xrightarrow{\mathsf{id}} \Sigma^i K \ .$$

Observation 198. The forgetful functor

$$\mathcal{S}\mathsf{paces}_* \to \mathcal{S}\mathsf{paces}$$

admits a left adjoint given by adjoining a disjoint basepoint

$$(-)_+ : \mathcal{S}\mathsf{paces} \longrightarrow \mathcal{S}\mathsf{paces}_*$$
$$X \longmapsto X_+ := X \amalg *$$

Definition 199. The ***sphere spectrum*** $\mathbb{S}$ is the suspension spectrum of $\mathbb{S}^0$

$$\mathbb{S} := \Sigma^\infty \mathbb{S}^0 \ .$$

Remark 200. The ∞-category of $\mathcal{S}\mathsf{paces}$ is the free ∞-category on a point. Similarly, the ∞-category of $\mathcal{S}\mathsf{p}$ is the free stable ∞-category on a point.

Products and Coproducts of Stable ∞-categories

Lemma 201. *Let $\mathcal{K}$ be a pointed ∞-category with finite limits and colimits. Then the diagram*

$$\begin{tikzcd} U \amalg V \arrow[r, "\mathrm{id} \amalg 0"] \arrow[d, "0 \amalg \mathrm{id}"'] & U \arrow[d] \\ V \arrow[r] & 0 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout.

Proof. Note that there is a canonical diagram where each square is a pushout

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \arrow[d] \\ U \arrow[r] \arrow[d] & U \amalg V \arrow[r] \arrow[d] & U \\ 0 \arrow[r] & V & \end{tikzcd} \ .$$

Therefore the the colimit

$$\mathsf{colim} \left(\begin{tikzcd} U \amalg V \arrow[r] \arrow[d] & U \\ V & \end{tikzcd} \right) = 0 \ .$$

$\square$

Corollary 202. Let $\mathcal{V}$ be a stable ∞-category. Then finite coproducts agree with finite products.

Proof. Let U and V be objects in $\mathcal{V}$. Since the stable ∞-category V is zero-pointed, the diagram

$$\begin{tikzcd} U \amalg V \arrow[r, "\mathrm{id} \amalg 0"] \arrow[d, "0 \amalg \mathrm{id}"'] & U \arrow[d] \\ V \arrow[r] & 0 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram. In a stable ∞-category, pushout diagrams are pullback diagrams by Lemma 177. Applied to coproduct diagram above, this gives the diagram

$$\begin{tikzcd} U \amalg V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\ V \arrow[r] & 0 \end{tikzcd}$$

is a pullback diagram. This is true in a zero-pointed ∞-category if and only if $U \coprod V$ witnesses the product of U and V. Therefore, finite coproducts agree with finite products in the stable ∞-category $\mathcal{V}$. $\square$

Notation 203. We will denote the coproduct

$$U \oplus V := U \amalg V$$

in a stable ∞-category. This is to remind us that this coproduct is also the product.

Morphisms in Stable ∞-categories

We now discuss how morphisms in a stable ∞-category. The fact that finite coproducts agree with products endows the spaces of morphisms between any two objects therein with a structure aking to that of an abelian group. Furthermore, we show how morphisms between objects in a stable ∞-category can formally be represented as a matrix, with composition given by matrix multiplication.

Definition 204. Let U and V be objects of $\mathcal{V}$, and f, g morphisms in $\mathsf{Hom}_{\mathcal{V}}(U, V)$. We define the ***sum*** $f+g$ to be the following morphism in $\mathsf{Hom}_{\mathcal{V}}(U, V)$

$$f+g : U \xrightarrow{(\mathsf{id},\mathsf{id})} U \oplus U \xrightarrow{f \oplus g} V \oplus V \xrightarrow{(\mathsf{id},\mathsf{id})} V \ .$$

Lemma 205. *Let I and J be finite sets. Let $\bigoplus_{i\in I} V_i \ \bigoplus_{j\in J} W_j$ be objects of $\mathcal{V}$. Then*

$$\mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, \bigoplus_{j\in J} W_j) \simeq \bigoplus_{(i,j)\in I\times J} \mathsf{Hom}_{\mathcal{V}}\,(V_i, W_j)$$

Proof. This follows from the fact that a finitely indexed direct sum is both the product and the coproduct in $\mathcal{V}$. First, since the direct sum is the product, a morphism with target $\bigoplus\limits_{j\in J} W_j$ is given by a map for each j

$$\bigoplus_{i\in I} V_i \to W_j \ .$$

Therefore we have

$$\mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, \bigoplus_{j\in J} W_j) \simeq \bigoplus_{j\in J} \mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, W_j) \ .$$

The direct sum is also the coproduct, so for a fixed $j \in J$, a morphism $\bigoplus_{i\in I} V_i \to W_j$ is determined by a morphism for each $i \in I$

$$V_i \to W_j \ .$$

For each $i \in I$ and $j \in J$, we refer to the function $V_i \to W_j$ as a ***component function***. Thus the universal property gives that $\mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, W_j)$ splits

$$\mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, W_j) \simeq \bigoplus_{i\in I} \mathsf{Hom}_{\mathcal{V}}(V_i, W_j)$$

Altogether we have

$$\mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, \bigoplus_{j\in J} W_j) \simeq \bigoplus_{j\in J} \mathsf{Hom}_{\mathcal{V}}(\bigoplus_{i\in I} V_i, W_j) \simeq \bigoplus_{(i,j)\in I\times J} \mathsf{Hom}_{\mathcal{V}}(V_i, W_j) \ .$$

□

Notation 206. Consider an arbitrary map

$$F : \bigoplus_{i\in I} V_i \to \bigoplus_{j\in J} W_j$$

with each component function (Lemma 205) denoted

$$F^i_j : V_i \to W_j \ .$$

Then we define the following $J \times I$ matrix to be F.

$$[F^i_j]^{i\in I}_{j\in J} = F \ .$$

Notation 207. For a matrix $[F^i_j]^{i\in I}_{j\in J}$, if the I and J are clear from context, we denote

$$[F^i_j] := [F^i_j]^{i\in I}_{j\in J} \ .$$

Lemma 208. *Let I, J, and K be finite sets, and let $\bigoplus_{i\in I} U_i$, $\bigoplus_{j\in J} V_j$, and $\bigoplus_{k\in K} W_k$ be objects of $\mathcal{V}$. For two maps,*

$$F = [F^i_j]^{i\in I}_{j\in J} : \bigoplus_{i\in I} U_i \to \bigoplus_{j\in J} V_j$$

$$G = [G^j_k]^{j\in J}_{k\in K} : \bigoplus_{j\in J} V_j \to \bigoplus_{k\in K} W_k \ ,$$

their composition $G \circ F$ is given by

$$[(G \circ F)^i_k]^{i\in I}_{k\in K} = [\sum_{j\in J} G^j_k F^i_j]^{i\in I}_{j\in J}$$

This means that the composition map is just given by multiplication of matrices

$$[(G \circ F)^i_k]^{i\in I}_{k\in K} = [G^j_k]^{j\in J}_{k\in K} \circ [F^i_j]^{i\in I}_{j\in J}$$

Proof. Consider a sequence of composable morphisms

$$\bigoplus_{i\in I} U_i \xrightarrow{[F^i_j]} \bigoplus_{j\in J} V_j \xrightarrow{[G^j_k]} \bigoplus_{k\in K} W_k$$

Note then that a given $(G \circ F)^i_k$ is given by precomposing with the inclusion map with $U_i \to \bigoplus_{i\in I} U_i$ and postcomposing with the projection map $\bigoplus_{k\in K} W_k \to W_k$.

$$(G \circ F)^i_k : U_i \hookrightarrow \bigoplus_{i\in I} U_i \xrightarrow{F=[F^i_j]} \bigoplus_{j\in J} V_j \xrightarrow{G=[G^j_k]} \bigoplus_{k\in K} W_k \to W_k$$

Recall, that the morphism $\sum_{j\in J}(G^j_k F^i_j)$ is defined to be the following sequence of morphisms.

$$\sum_{j\in J} G^j_k F^i_j : U_i \xrightarrow{(\mathsf{id})_{j\in J}} \bigoplus_{j\in J} U_i \xrightarrow{(G^j_k F^i_j)_{j\in J}} \bigoplus_{j\in J} W_k \xrightarrow{(\mathsf{id})_{j\in J}} W_k$$

Consider the diagram, which we will need to check commutes

$$\begin{tikzcd}
 & \bigoplus_{i\in I} U_i \arrow[r, "F"] & \bigoplus_{j\in J} V_j \arrow[r, "G"] \arrow[dr, dashed, "(G^j_k)_{j\in J}"] & \bigoplus_{k\in K} W_k \arrow[dr] & \\
U_i \arrow[ur, hook] \arrow[r, "(\mathsf{id})_{j\in J}"'] & \bigoplus_{j\in J} U_i \arrow[ur, dashed, "(F^i_j)_{j\in J}"] \arrow[rr, "(G^j_k F^i_j)_{j\in J}"'] & & \bigoplus_{j\in J} W_k \arrow[r, "(\mathsf{id})_{j\in J}"'] & W_k
\end{tikzcd}$$

Note that the top composite is $(G \circ F)^i_k$, and the bottom composite is $\sum_{j\in J}(G^j_k F^i_k)$.

Therefore if the diagram commutes the two maps are equivalent. We begin by checking commutativity of the square

$$\begin{tikzcd}
\bigoplus_{i\in I} U_i \arrow[r, "F"] & \bigoplus_{j\in J} V_j \\
U_i \arrow[u, hook] \arrow[r, "(\mathsf{id})_{j\in J}"'] & \bigoplus_{j\in J} U_i \arrow[u, "(F_j^i)_{j\in J}"']
\end{tikzcd}$$

Since we are mapping from U_i into a product $\bigoplus_{j\in J} V_j$, we need to only check if each coordinate function matches.

Let us first inspect the top composite. The inclusion of $U_i \to \bigoplus_{i\in I} U_i$ is given by the identity map on the ith coordinate, and the zero map on every other coordinate. First by going around the top, we get each coordinate function for each j given by

$$U_i \hookrightarrow \bigoplus_{i\in I} U_i \xrightarrow{F} \bigoplus_{j\in J} V_j \to V_j$$

This is just the definition of F_j^i. Now consider the coordinate function given by the bottom composite:

$$U_i \xrightarrow{(\mathsf{id})_{j\in J}} \bigoplus_{j\in J} U_i \xrightarrow{(F_j^i)_{j\in J}} \bigoplus_{j\in J} V_j \to V_j \ .$$

The maps $\{F_j^i \circ \mathsf{id}_{U_i} : U_i \to V_j\}_{j\in J}$ assemble into a morphisms $U_i \to \bigoplus_{j\in J} V_j$ that factors through $\bigoplus\limits_{j\in J} U_i$:

$$\begin{tikzcd}
U_i \arrow[r, "F_j^i\,\mathsf{id}_{U_i}"] \arrow[d, "(\mathsf{id})'_{j\in J}"] & V_j \\
\bigoplus\limits_{j\in J} U_i \arrow[r] & \bigoplus\limits_{j\in J} V_j \arrow[u]
\end{tikzcd} .$$

Since the map $F_j^i \circ \mathsf{id}_{U_i} : U_i \to V_j$ is defined as a composition, it factors through U_i:

$$\begin{tikzcd}
 & U_i \arrow[dr, "F_j^i"] & \\
U_i \arrow[ur, "\mathsf{id}"] \arrow[rr, "F_j^i\,\mathsf{id}"] \arrow[d, "(\mathsf{id})_{j\in J}"'] & & V_j \\
\bigoplus_{i\in I} U_i \arrow[rr, "(F_j^i)_{j\in J}"'] & & \bigoplus_{j\in J} V_j \arrow[u]
\end{tikzcd} .$$

Going around the outside and collapsing the identity then gives that the coordinate function

$$U_i \xrightarrow{(\mathsf{id})_{j\in J}} \bigoplus_{j\in J} U_i \xrightarrow{(F_j^i)_{j\in J}} \bigoplus_{j\in J} V_j \to V_j$$

is F_j^i. Therefore we have the commutativity of

$$\begin{tikzcd}
\bigoplus_{i\in I} U_i \arrow[r, "F"] & \bigoplus_{j\in J} V_j \\
U_i \arrow[u, hook] \arrow[r, "(\mathsf{id})_{j\in J}"'] & \bigoplus_{j\in J} U_i \arrow[u, "(F_j^i)_{j\in J}"']
\end{tikzcd}$$

since the coordinate functions of the top and bottom composite are both given by F_j^i.

We now check the commutativity of the triangle

$$\begin{tikzcd}
 & \bigoplus_{j\in J} V_j \arrow[dr, "(G_k^j)_{j\in J}"] & \\
\bigoplus_{j\in J} U_i \arrow[ur, "(F_j^i)_{j\in J}"] \arrow[rr, "(G_k^j F_j^i)_{j\in J}"'] & & \bigoplus_{j\in J} W_k
\end{tikzcd} .$$

The coordinate functions of the top composite is just given by the $G_k^j F_j^i$ and so are the coordinate functions of the bottom composite, therefore this triangle commutes.

Finally we check the commutativity of the right square.

$$\begin{tikzcd}
\bigoplus_{j\in J} V_j \arrow[r, "G"] \arrow[d, "(G_k^j)_{j\in J}"'] & \bigoplus_{k\in K} W_k \arrow[d] \\
\bigoplus_{j\in J} W_k \arrow[r, "(\mathsf{id})_{j\in J}"'] & W_k
\end{tikzcd} .$$

This follows by using the same methods as checking the left square. Precompose by U_i to find the ith coordinate function, and we see that going along the top composite and the bottom composite both give G_k^j as the coordinate function. Therefore we have that

$$\begin{tikzcd}
 & \bigoplus_{i\in I} U_i \arrow[r, "F"] & \bigoplus_{j\in J} V_j \arrow[r, "G"] \arrow[dr, "(G_k^j)_{j\in J}"] & \bigoplus_{k\in K} W_k \arrow[dr] & \\
U_i \arrow[ur, hook] \arrow[r, "(\mathsf{id})_{j\in J}"'] & \bigoplus_{j\in J} U_i \arrow[ur, "(F_j^i)_{j\in J}"] \arrow[rr, "(G_k^j F_j^i)_{j\in J}"'] & & \bigoplus_{j\in J} W_k \arrow[r, "(\mathsf{id})_{j\in J}"'] & W_k
\end{tikzcd} .$$

commutes, and therefore $(GF)_j^i = \sum_{j\in J} G_k^j F_j^i$.

□

Notation 209. Define the map $\sigma : \mathbb{T} \to \mathbb{T}$ to be the inverse map for the natural group structure on $\mathbb{T}$

$$\begin{aligned} \sigma : \mathbb{T} &\longrightarrow \mathbb{T} \\ x &\longmapsto x^{-1} \end{aligned} .$$

Definition 210. Let $f : U \to V$ be an arrow in $\mathcal{V}$. Define $-f$ to be the morphism

$$-f = \Sigma^{-1}\left(\Sigma U \xrightarrow{\overline{\sigma}} \Sigma U \xrightarrow{\Sigma f} \Sigma V\right) .$$

Observation 211. Let $f : U \to V$ be a morphism in $\mathcal{V}$. Then $f + -f = 0$. To see this, note that

$$\Sigma\mathbb{S} \xrightarrow{\mathsf{id}+-\mathsf{id}} \Sigma\mathbb{S} = 0 ,$$

which follows from the fact that the composite

$$\mathbb{S}^1 \xrightarrow{\text{pinch}} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\mathsf{id} \vee \sigma} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\text{fold}} \mathbb{S}^1$$

is nullhomotopic, and that $\Sigma^\infty \mathbb{S}^1 = \Sigma\mathbb{S}$ after choosing a basepoint of $\mathbb{S}^1$. Since Σ is invertible in a stable ∞-category, this implies that

$$\left(\mathbb{S} \xrightarrow{\mathsf{id}_{\mathbb{S}} + -\mathsf{id}_{\mathbb{S}}} \mathbb{S}\right) = \left(\mathbb{S} \xrightarrow{0} \mathbb{S}\right)$$

Using the monoidal structure on $\mathcal{S}\mathsf{p}$, with $\mathbb{S}$ the unit, this implies that for an arbitary $W \in \mathcal{S}\mathsf{p}$, that

$$\left(W \simeq \mathbb{S} \otimes W \xrightarrow{(\mathsf{id}_{\mathbb{S}} + -\mathsf{id}_{\mathbb{S}}) \otimes \mathsf{id}_W} \mathbb{S} \otimes W \simeq W\right) \simeq \left(\mathbb{S} \xrightarrow{0} \mathbb{S}\right) .$$

Furthermore, using that an arbitrary stable ∞-category $\mathcal{V}$ is tensored over $\mathcal{S}\mathsf{p}$, this yields that for an arbitrary $U \in \mathcal{V}$

$$\left(U \xrightarrow{\mathsf{id}_U + -\mathsf{id}_U} U\right) \simeq \left(U \xrightarrow{0} U\right) .$$

Lastly, by inspecting the definition of the addition of morphisms (Definition 204), it follows that

$$f + -f = f \circ (\mathsf{id}_U + -\mathsf{id}_U) = f \circ 0 = 0$$

Lemma 212. *Consider the following pushout diagram in $\mathcal{V}$.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Then we have the following equivalence

$$\mathsf{coker}(A \to B) \simeq \mathsf{coker}(C \to D)$$

Proof. Since the cokernal is a pushout as well, we have the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[rr] \arrow[d] & & 0 \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & D \arrow[rr] \arrow[urr, phantom, "\ulcorner"] & & \mathsf{coker}(C \to D) \end{tikzcd} .$$

Since each square in the diagram is a pushout, that implies that the outer square is a pushout.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & \mathsf{coker}(C \to D) \end{tikzcd}$$

Finally, since colimits are unique up to unique isomorphism then we have

$$\mathsf{coker}(A \to B) \simeq \mathsf{coker}(C \to D)$$

$\square$

Co/Tensors

Definition 213. Let X be a space, and $V \in \mathcal{K}$ and object in a ∞-categorythat admits colimits. The ***tensor*** (sometimes called ***copowering***) of X with V is the following object of $\mathcal{V}$:

$$X \odot V := \mathsf{colim}\left(X \xrightarrow{!} * \xrightarrow{\langle V \rangle} \mathcal{K}\right) .$$

Definition 214. Let X be a space, and $V \in \mathcal{K}$ and object in an ∞-category that admits limits. The ***cotensor*** (Sometimes called ***powering***) of X with V is the following object of

$\mathcal{V}$:

$$V^{\pitchfork X} := \mathsf{lim}\left(X \xrightarrow{!} * \xrightarrow{\langle V\rangle} \mathcal{K}\right) \ .$$

Lemma 215. *Let $V =$ be an object in a presentable stable ∞-category $\mathcal{V}$.*

(1) The functor

$$- \odot V : \mathsf{\mathcal{S}paces} \to \mathcal{V}$$

is the unique colimit preserving functor from $\mathsf{\mathcal{S}paces}$ *to* $\mathcal{V}$ *such that the value on* $*$ *is* V*.*

(2) The functor

$$V^{\pitchfork -} : \mathsf{\mathcal{S}paces}^{\mathsf{op}} \to \mathcal{V}$$

is the unique limit colimit reflecting functor from $\mathsf{\mathcal{S}paces}$ *to* $\mathcal{V}$ *whose value on* $*$ *is* V*.*

Proof. The first statement follows from the fact that $\mathsf{\mathcal{S}paces}$ is freely generated by $*$ under colimits. The second follows as the dual of the first statement by taking opposites, and by noting that if $\mathcal{V}$ is stable, then $\mathcal{V}^{\mathsf{op}}$ is stable. □

Example 216. We give without proof the following examples of tensors and cotensors in specific stable infinity categories $\mathcal{V}$

$\mathcal{V}$	$X \odot V$	$V^{\pitchfork X}$
Sets	$V^{\coprod \pi_0(X)}$	$\mathsf{Hom}_{\mathsf{Sets}}(X, V)$
$\mathsf{\mathcal{S}paces}$	$X \times V$	$\mathsf{Map}(X, V)$
Chain Complexes	$C_*(X, V)$	$C^*(X, V)$
Spectra	$(\Sigma^\infty_+ X) \wedge V$	$\underline{\mathsf{Map}}_{\mathsf{Spectra}}(\Sigma^\infty_+ X, V)$

Hopf Map

Definition 217. The ***Hopf fibration*** is the quotient map of the action of the diagonal action of $\mathbb{T} \subset \mathbb{C}$ on $\mathbb{S}^3 \subset \mathbb{C} \times \mathbb{C}$

$$\begin{aligned} \eta : \mathbb{S}^3 &\longrightarrow \mathbb{CP}^1 := \mathbb{S}^3_{/\mathbb{T}} \\ \mathbb{C} \times \mathbb{C} \ni (x_1, x_2) &\longmapsto \mathsf{Span}\,(x_1, x_2) \end{aligned} .$$

The space $\mathbb{CP}^1$ is homeomorphic to $\mathbb{S}^2$. Indeed, $\mathbb{CP}^1$ can be identified as the one point compactification of the complex plane, and $\mathbb{S}^2$ can be identified as the one point compactification of $\mathbb{R}^2$. Therefore the Hopf fibration can be viewed as a map

$$\eta : \mathbb{S}^3 \to \mathbb{S}^2 \ .$$

The map η can be considered as a based map, and therefore as an element

$$\eta \in \pi_3(\mathbb{S}^2) \underset{3.1}{\simeq} \pi_1\left(\Omega^2\mathbb{S}^2\right) \ .$$

Notation 218. Let S be a set. Define $\mathbb{Z}\langle S\rangle$ to be the free abelian group on the set S.

Lemma 219. *The Hopf map η generates $\pi_1(\Omega^2\mathbb{S}^2)$.*

$$\pi_3\left(\mathbb{S}^2\right) \simeq \mathbb{Z} \simeq \mathbb{Z}\langle\eta\rangle \ .$$

Observation 220. Note that the hopf map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ induces a map on spectra

$$\Sigma^\infty\eta : \Sigma^3\mathbb{S} \to \Sigma^2\mathbb{S}.$$

Desuspending twice then gives a map of spectra

$$\Sigma^{-2}\Sigma^{\infty}\eta : \Sigma\mathbb{S} \to \mathbb{S}.$$

We will refer to this map of spectra as η as well. Note that tensoring by an object $V \in \mathcal{V}$ also yields a morphism in $\mathcal{V}$

$$\eta : \Sigma V \simeq V \otimes \Sigma\mathbb{S} \to V \otimes \mathbb{S} \simeq V.$$

Similarly, using the suspension functor, we can compose the morphism η by

$$\eta^2 := \Sigma^2 V \xrightarrow{\Sigma\eta} \Sigma V \xrightarrow{\eta} V \ .$$

In a similar way, for $n \in \mathbb{Z}_{\geq 2}$ define η^n to be the composite

$$\eta^n := \Sigma^n V \xrightarrow{\Sigma^{n-1}\eta} \Sigma^{n-1} V \xrightarrow{\Sigma^{n-2}\eta} \dots \xrightarrow{\Sigma\eta} \Sigma \xrightarrow{\eta} V \ .$$

Lemma 221. *The composition of η^4 is equivalent to the zero map*

$$\eta^4 \simeq 0 \ .$$

Proof. See [12]. □

Recollection of Stratified Noncommutative Geometry

Recalled in this section are the key results from [6] that will be used in identifying the category $\mathsf{Fun}(\mathsf{Exit}(\underline{\mathbb{CP}^{\infty}}, \mathcal{V})$. In the paper [6], there is a reconstruction theorem (Theorem A) and a "reflected" reconstruction theorem (Theorem F). The proof of Theorem 310 only relies on the reflected side, so we will only recall the reflected version, however see [6] for the full theory of stratifications of presentable stable ∞-categories.

In [6], Ayala, Mazel-Gee, and Rozenblyum develop a theory of stratifications for presentable stable ∞-categories. A main result from that work is that a stratification of a presentable stable ∞-category $\mathcal{X}$ can be used to reconstruct the ∞-category $\mathcal{X}$ in two ways. A key example is worked out in the Ret-modules section 3. Another useful instance of the reconstruction theorem gives an equivalence between filtered objects in $\mathcal{V}$ and downward finite chain complexes over $\mathcal{V}$

$$\mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \simeq \mathsf{Chain}_{\geq 0}(\mathcal{V}) \ .$$

This is an ∞-categorical version of the Dold-Kan theorem (Theorem 1.2.3.7 of [11]) See Example 1.10.6 of [6] and Theorem 1.2.4.1 of [11]

Definition 222. A ***closed*** ∞***-subcategory*** of $\mathcal{X}$ is

+) a full presentable stable ∞-subcategory $\mathcal{Z} \subseteq \mathcal{X}$

such that:

-) The inclusion i_L admits a right adjoint y, which admits a further right adjoint i_R:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, bend left, "i_L"] \arrow[r, phantom, shift left, "\perp"] \arrow[r, phantom, shift right, "\perp"] \arrow[r, hook, dashed, bend right, "i_R"'] & \mathcal{X} \arrow[l, dashed, "y"'] \end{tikzcd} \ .$$

Definition 223. Define $\mathbf{Cls}(\mathcal{X})$ to be the poset of closed subcategories of $\mathcal{X}$ ordered by inclusion.

Note that if $\mathcal{Z}_1 \leq \mathcal{Z}_2$ in $\mathbf{Cls}(\mathcal{X})$, then the inclusion $\mathcal{Z}_1 \hookrightarrow \mathcal{Z}_2$ factors as

$$\mathcal{Z}_1 \xrightarrow{i_L} \mathcal{X} \xrightarrow{y} \mathcal{Z}_2$$

and this inclusion is fully faithful.

Definition 224. An arbitrary map of posets

$$\mathcal{Z}_\bullet : \mathsf{P} \to \mathbf{Cls}(\mathcal{X})$$

is a ***prestratification***.

Notation 225. Let $\mathcal{Z}_\bullet : \mathsf{P} \to \mathbf{Cls}(\mathcal{X})$ be a prestratification, and let D be a subposet of P. Define $\mathcal{Z}_\mathsf{D}$ to be the colimit

$$\mathcal{Z}_\mathsf{D} := \mathsf{colim}\left(\mathsf{Obj}\,(\mathsf{D}) \hookrightarrow \mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}(\mathcal{X})\right) \ ,$$

where we regard $\mathsf{Obj}\,(\mathsf{D})$ as the discrete category on the elements of the poset D. This colimit is the least upper bound (Lemma 24) of the elements of D. Furthermore, define

$$\mathcal{Z}_{\leq q} := \mathcal{Z}_{\mathsf{P}_{\leq q}}$$

$$\mathcal{Z}_{< q} := \mathcal{Z}_{\mathsf{P}_{< q}}$$

$$\mathcal{Z}_{\leq p \cap \leq q} := \mathcal{Z}_{\mathsf{P}_{\leq q \cap \leq p}}$$

.

Definition 226. A stratification of $\mathcal{X}$ over P is:

+) A functor

$$\begin{aligned} \mathcal{Z}_\bullet : \mathsf{P} &\longrightarrow \mathbf{Cls}(\mathcal{X}) \\ p &\longmapsto \mathcal{Z}_p \end{aligned} .$$

such that:

-) The colimit $\mathcal{Z}_{\mathsf{P}}$ is the ∞-∞-category $\mathcal{X}$.

$$\mathcal{Z}_{\mathsf{P}} = \mathcal{X} \ .$$

-) For any $p, q \in \mathsf{P}$, there exists a factorization

$$\begin{tikzcd} \mathcal{Z}_{\leq p \cap \leq q} \arrow[r, "i_L"] & \mathcal{Z}_p \\ \mathcal{Z}_q \arrow[u, dashed] \arrow[r, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} .$$

This condition is the ***stratification condition***.

Observation 227. Let $\mathsf{P} \to \mathbf{Cls}(\mathcal{X})$ be a prestratification, and let $p \leq q$ in P. The colimit $\mathcal{Z}_{\leq p \cap \leq q}$ is a closed ∞-subcategory $\mathcal{Z}_{\leq q}$. Therefore there exists factorizations

$$\begin{tikzcd} \mathcal{Z}_q \arrow[r, "y \circ i_L"] & \mathcal{Z}_p \\ \mathcal{Z}_q \arrow[u, dashed, "\mathsf{id}"] \arrow[r, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Z}_q \arrow[r, "\mathsf{id}"] & \mathcal{Z}_q \\ \mathcal{Z}_p \arrow[u, dashed, "y \circ i_L"] \arrow[r, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} .$$

Therefore a prestratification by a linearly ordered poset P always satisfies the stratification condition. If a linearly ordered poset P is finite, then it suffices to check that the maximal element of the poset P is sent to $\mathcal{X} \in \mathbf{Cls}_{\mathcal{X}}$ in order to verify the prestratification is a stratification.

Notation 228. Throughout the remainder of this section, fix a P-stratification of the presentable stable ∞-category $\mathcal{X}$:

$$\mathcal{Z}_{\bullet} : \mathsf{P} \to \mathbf{Cls}(\mathcal{X}) \ .$$

Definition 229. A ***recollement*** is a diagram of presentable stable ∞-categories

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, bend left=40, "i_L"] \arrow[r, bend right=40, "i_R"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{X} \arrow[l, "y" description] \arrow[r, bend left=40, "p_L"] \arrow[r, bend right=40, "p_R"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{U} \arrow[l, "\nu" description]
\end{tikzcd} \quad ,$$

such that $\mathsf{ker}\,(p_L) = \mathsf{im}(i_L)$, $\mathsf{ker}\,(y) = \mathsf{im}(\nu)$, and $\mathsf{ker}\,(p_R) = \mathsf{im}\,(i_L)$

Definition 230. Define the p-***stratum*** $\mathcal{Z}_\bullet : \mathsf{P} \to \mathbf{Cls}(X)$ to be a stratification

$$\mathcal{X}_p := (\mathcal{Z}_p / \mathcal{Z}_{<p})$$

Observation 231. Given an element $p \in \mathsf{P}$, there is a canonical recollement

$$\begin{tikzcd}
\mathcal{Z}_{<p} \arrow[r, bend left=40, "i_L"] \arrow[r, bend right=40, "i_R"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{Z}_p \arrow[l, "y" description] \arrow[r, bend left=40, "p_L"] \arrow[r, bend right=40, "p_R"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{X}_p \arrow[l, "\nu" description]
\end{tikzcd} \quad .$$

Definition 232. For $p \in \mathsf{P}$, define λ_p and Ψ_p as the composites

$$\begin{tikzcd}
\lambda_p : \mathcal{X}_p \arrow[r, shift left=2, "\nu"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_p \arrow[l, shift left=2, "p_R"] \arrow[r, shift left=2, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{X} : \Psi_p \arrow[l, shift left=2, "y"]
\end{tikzcd} \quad .$$

The functor Ψ_p is called the ***geometric colocalization*** functor.

Definition 233. For all $p, q \in \mathsf{P}$, the corresponding ***glueing functor*** is the composite

$$\check{\Gamma}^p_q := \mathcal{X}_p \xrightarrow{\lambda_p} \mathcal{X} \xrightarrow{\Psi_q} \mathcal{X}_q \ .$$

Definition 234. The ***reflected gluing diagram*** is full ∞-subcategory of $\mathcal{X} \times \mathsf{P}^{\mathsf{op}}$

$$\check{\mathscr{G}}(\mathcal{X}) := \{(X, p^\circ) \mid X \in \lambda_p(\mathcal{X}_p)\} \subseteq \mathcal{X} \times \mathsf{P}^{\mathsf{op}} \ .$$

The reflected gluing diagram has a canonical functor to P^{op} given by projection.

Notation 235. We will refer to an object

$$(x, p^{\circ}) \in \check{\mathscr{G}}(\mathfrak{X})$$

as $\lambda_p(x_p)$, since the object x lies in the essential image of $\mathfrak{X}_p$.

In what follows, we make use of (locally) cartesian fibrations. For more details on cartesian fibrations see [2]. For how caretesian fibration fit into the theory of presentable stable ∞-categories, see [6].

Example 236. The map of posets

$$\mathsf{sd}(\mathsf{P})^{\mathsf{op}} \xrightarrow{\mathsf{Max}^{\mathsf{op}}} \mathsf{P}^{\mathsf{op}}$$

is a locally cartesian fibration. In the example of $\mathsf{P} = 2$, the non-identity cartesian morphisms of $\mathsf{sd}([2])^{\mathsf{op}}$ are highlighted in red

$$\begin{tikzcd}
 & & 1 & & \\
 & 01 \arrow[ur] \arrow[dl, red] & 012 \arrow[l, red] \arrow[r] \arrow[d] & 12 \arrow[ul, red] \arrow[dr] & \\
0 & & 02 \arrow[ll, red] \arrow[rr] & & 2
\end{tikzcd} .$$

See [6] for details.

Lemma 237. *The canonical projection functor*

$$\check{\mathscr{G}}(\mathfrak{X}) \to \mathsf{P}^{\mathsf{op}}$$

is a locally cartesian fibration.

Proof. The functor $\check{\mathscr{G}}(\mathfrak{X}) \to \mathsf{P}^{\mathsf{op}}$ is a locally cartesian fibration if for each morphism $[1] \xrightarrow{\langle i^\circ \to j^\circ \rangle} \mathsf{P}^{\mathsf{op}}$ and object $\lambda_j(x_j) \in \check{\mathscr{G}}(\mathfrak{X})$ in the fiber over $j^\circ \in \mathsf{P}^{\mathsf{op}}$, the restriction

$$\check{\mathscr{G}}(\mathfrak{X})_{|i^\circ \to j^\circ} \to \{i^\circ \to j^\circ\}$$

is a cartesian fibration. By Lemma 2.16 of [2], the restriction over the morphism $i^\circ \to j^\circ$ is a cartesian fibration if there exists a final lift

$$\begin{tikzcd} \{1\} \arrow[r, "\langle \lambda_j(x_j) \rangle"] \arrow[d] & \check{\mathscr{G}}(\mathfrak{X})_{|i^\circ \to j^\circ} \arrow[d] \\ {[1]} \arrow[r, "\langle i^\circ \to j^\circ \rangle"'] \arrow[ur, dashed] & \{i^\circ \to j^\circ\} \end{tikzcd} . \tag{3.3}$$

There is a canonical choice of lift given by the counit of the adjunction $\lambda_i \dashv \Psi_i$

$$\lambda_i \Psi_i \lambda_j(x_j) \xrightarrow{\kappa_i := \mathsf{counit}} \lambda_j(x_j) \; .$$

Therefore, it suffices to show that for any other filler $\lambda_i(x_i) \xrightarrow{f} \lambda_j(x_j)$ in the diagram 3.3, there is a unique filler

$$\begin{tikzcd} \lambda_i(x_i) \arrow[d, dashed, "\exists !"'] \arrow[dr, "f"] & \\ \lambda_i \Psi_i \lambda_j(x_j) \arrow[r, "\kappa_i"'] & \lambda_j(x_j) \end{tikzcd} . \tag{3.4}$$

Applying the composite functor $\lambda_i \Psi_i$ to the morphism $\lambda_i(x_i) \to \lambda_j(x_j)$ gives a morphism

$$\lambda_i \Psi_i \lambda_i(x_i) \to \lambda_i \Psi_i \lambda_j(x_j) \; .$$

Furthermore, since λ_i is fully faithful, the composite $\Psi_i \lambda_i$ is equivalent with the identity functor on $\mathcal{X}_i$. Therefore we get a morphism

$$\lambda_i(x_i) \to \lambda_i \Psi_i \lambda_j(x_j) \ ,$$

which is clearly a filler in the diagram (3.4) . Therefore what remains to show is that this filler is unique. Assume there is a filler $g : \lambda_i\left(x_i\right) \to \lambda_i \Psi_i \lambda_j\left(x_j\right)$

$$\begin{tikzcd}
\lambda_i(x_i) \arrow[d, "g"'] \arrow[dr] & \\
\lambda_i \Psi_i \lambda_j(x_j) \arrow[r, "\kappa_i"'] & \lambda_j(x_j)
\end{tikzcd} .$$

We seek to factor g by morphisms which do not depened on g. Therefore any other filler g' is identified in this same way, and so the filler is essential unique.

Applying the functor $\lambda_i \Psi_i$ to triangle above, and applying the counit κ_i gives

$$\begin{tikzcd}
\lambda_i \Psi_i \lambda_i(x_i) \arrow[rrrr, "\lambda_i \Psi_i(f)"] \arrow[ddd, "\kappa_i"] \arrow[drr, "\lambda_i \Psi_i(g)"] & & & & \lambda_i \Psi_i \lambda_j(x_j) \arrow[ddd, "\kappa_i"] \\
& & \lambda_i \Psi_i \lambda_i \Psi_i \lambda_j(x_j) \arrow[urr, "\lambda_i \Psi_i(\kappa_i)"] \arrow[ddd, "\kappa_i"] & & \\
& & & & \\
\lambda_i(x_i) \arrow[rrrr, "f"] \arrow[drr, "g"'] & & & & \lambda_j(x_j) \\
& & \lambda_i \Psi_i \lambda_j(x_j) \arrow[urr, "\kappa_i"'] & &
\end{tikzcd}$$

Here any application of κ_i on the ∞-subcategory $\lambda_i(\mathcal{X}_i)$ must be equivalences, since the functor $\lambda_i\Psi_i$ is idempotent. Therefore the morphisms indicated are equivalences

$$\begin{tikzcd}
\lambda_i\Psi_i\lambda_i(x_i) \arrow[rr, "\lambda_i\Psi_i(f)"] \arrow[dr, "\lambda_i\Psi_i(g)"] \arrow[dd, "\simeq"'] & & \lambda_i\Psi_i\lambda_j(x_j) \arrow[dd, "\kappa_i"] \\
& \lambda_i\Psi_i\lambda_i\Psi_i\lambda_j(x_j) \arrow[ur, "\simeq"] \arrow[dd, "\simeq"'] & \\
\lambda_i(x_i) \arrow[rr, "f"] \arrow[dr, "g"'] & & \lambda_j(x_j) \\
& \lambda_i\Psi_i\lambda_j(x_j) \arrow[ur, "\kappa_i"'] &
\end{tikzcd}$$

Note then this witnesses g as a composition of morphisms, all which only depend on f or κ_i. □

Before the statement of the next theorem, recall Example 236, which states $\mathsf{sd}(\mathsf{P})^{\mathsf{op}} \xrightarrow{\mathsf{Max}} \mathsf{P}^{\mathsf{op}}$ is a locally cartesian fibration.

Theorem 238 (Reflected Reconstruction Theorem)**.** *Let* P *be a down-finite poset.*

(1) ***(macrocosm)*** *For each* P*-stratified presentable stable* ∞*-category* $\mathcal{X} \in \mathsf{Strat}_{\mathsf{P}}^{\mathsf{strict}}$*, the equivalence in* (1) *determines an equivalence*

$$\mathsf{Fun}^{\mathsf{cart}}_{/\mathsf{P}^{\mathsf{op}}}\left(\mathsf{sd}(\mathsf{P})^{\mathsf{op}}, \check{\mathscr{G}}(\mathcal{X})\right) \underset{\check{g}}{\overset{\mathsf{colim}(-)}{\underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}}}} \mathcal{X} \quad .$$

where $\check{g}$ *is given by a unit map of a particular adjunction in Theorem F of [6].*

(2) ***(microcosm)*** *For each object* $X \in \mathcal{X}$*, there is an equivalence in* $\mathcal{X}$

$$\mathsf{colim}(\hat{g}(X)) \xrightarrow{\simeq} X \ .$$

determined by the equivalence of ∞*-categories in* (1)*.*

This is a part of Theorem F of [6].

Ret-Modules: The Key Example

This section seeks to apply the reflected reconstruction theorem to look at retractions in a presentable stable ∞-category $\mathcal{V}$. We begin by recalling the definition of retractions.

Definition 239. The category Ret is the ordinary category

$$\begin{tikzcd} E \arrow[loop left, "sr"] \arrow[r, shift left, "r"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

where the composition rule is uniquely determined by $s \circ r = sr$, and $r \circ s = \mathsf{id}_B$.

Definition 240. A ***retraction*** in an ∞-category $\mathcal{V}$ is a functor

$$\mathsf{Ret} \to \mathcal{V} \ .$$

The ∞-category of retractions in $\mathcal{V}$ is the ∞-category

$$\mathcal{V}^{\mathsf{Ret}} := \mathsf{Fun}(\mathsf{Ret}, \mathcal{V}) \ .$$

We seek then to stratify the ∞- category $\mathsf{Fun}\,(\mathsf{Ret}, \mathcal{V})$ and apply the reflected reconstruction theorem. The end result is the following theorem (Theorem 249), which says that retractions in a stable ∞-category functorially split.

Theorem. *The functor*

$$\begin{aligned} \mathcal{V}^{\mathsf{Ret}} &\longrightarrow \mathcal{V} \times \mathcal{V} \\ \mathsf{F} &\longmapsto (\mathsf{F}(B), \ker(\mathsf{F}(r))) \end{aligned}$$

is an equivalence of ∞-categories.

Lemma 241. *Recall ∞-category* Ret *(Definition 239). The categories* $\{B\}_{/E}$ *and* $\{B\}^{E/}$ *are equivalent with the terminal category:*

$$\{B\}_{/E} \simeq * \simeq \{B\}^{E/} \ .$$

Proof. We show $\{B\}_{/E} \simeq *$. Recall, that the the categories $\{B\}_{/E}$ and $\mathsf{Ret}_{/E}$ are defined (Definition 346) to be the following pullbacks

$$\begin{tikzcd}
\{B\}_{/E} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & * \arrow[d, "\langle B\rangle"] \\
\mathsf{Ret}_{/E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}([1],\mathsf{Ret}) \arrow[r, "\mathsf{ev}_0"'] \arrow[d, "\mathsf{ev}_1"] & \mathsf{Ret} \\
* \arrow[r, "\langle E\rangle"'] & \mathsf{Ret} &
\end{tikzcd} .$$

The limit of the outer ladder diagram is $\mathsf{Hom}_{\mathsf{Ret}}(B,E)$, and therefore

$$\{B\}_{/E} \simeq \mathsf{Hom}_{\mathsf{Ret}}(B,E) \simeq \{s\}$$

due to the fact

$$(sr)\circ s = s\circ(rs) = s\circ \mathsf{id}_B = s.$$

A similar argument shows $\{B\}^{E/} \simeq \mathsf{Hom}_{\mathsf{Ret}}(E,B) \simeq \{r\}$. $\square$

Lemma 242. *The inclusion functor* $\{B\} \to \mathsf{Ret}$ *induces a pair of adjunctions*

$$\begin{tikzcd}
\mathcal{V}^{\{B\}} \arrow[rr, bend left, "i_L"] \arrow[rr, bend right, "i_R"'] & & \mathcal{V}^{\mathsf{Ret}} \arrow[ll, "y" description] \arrow[ll, phantom, shift left=3, "\perp"] \arrow[ll, phantom, shift right=3, "\perp"]
\end{tikzcd}$$

where y is the restriction functor along $\{B\} \to \mathsf{Ret}$ *and i_L and i_R are left and right Kan* *extensions along* $\{B\} \to \mathsf{Ret}$ *respectively.*

Proof. This is a particular instance of Lemma 321. □

Lemma 243. *The left Kan extension functor*

$$i_L : \mathcal{V}^{\{B\}} \to \mathcal{V}^{\mathsf{Ret}}$$

extends a functor $\mathsf{F} \in \mathcal{V}^{\{B\}}$ *to the constant functor at* $\mathsf{F}(B)$*:*

$$i_L(\mathsf{F})(E) \simeq \mathsf{F}(B) \ .$$

Proof. The left Kan extension evaluated on E is given by

$$i_L(\mathsf{F})(E) = \mathsf{colim}\left(\{B\}_{/E} \to \{B\} \xrightarrow{\mathsf{F}} \mathcal{V}\right) \ .$$

By Lemma 242, the category $\{B\}_{/E}$ is contractible, so the colimit is equivalent to the value $\mathsf{F}(B)$.

$$\mathsf{colim}\left(\{B\}_{/E} \to \{B\} \xrightarrow{\mathsf{F}} \mathcal{V}\right) \simeq \mathsf{colim}(* \xrightarrow{\langle \mathsf{F}(B) \rangle} \mathcal{V}) \simeq \mathsf{F}(B) \ .$$

□

Lemma 244. *The right Kan extension functor*

$$i_R : \mathcal{V}^{\{B\}} \to \mathcal{V}^{\mathsf{Ret}}$$

extends a functor $\mathsf{F} \in \mathsf{Fun}(\{B\}, \mathcal{V})$ *to the constant functor at* $\mathsf{F}(B)$

$$i_R(\mathsf{F})(E) = \mathsf{F}(B) \ .$$

Proof. The proof follows by the same logic as Lemma 243, after noting the right Kan extension is computed as

$$\mathsf{lim}\left(\{B\}^{E/} \to \{B\} \xrightarrow{\mathsf{F}} \mathcal{V}\right) \ ,$$

and that by Lemma 242, the category $\{B\}^{E/}$ is contractible. □

Lemma 245. *The closed ∞-subcategory $\mathcal{V}^{\{B\}}$ of $\mathcal{V}^{\mathsf{Ret}}$ determines a recollement with $\mathcal{V}^{\{E\}} :=$ $\mathsf{Fun}\left(\{E\}, \mathcal{V}\right)$*

$$\begin{tikzcd} \mathcal{V}^{\{B\}} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{V}^{\mathsf{Ret}} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{V}^{\{E\}} \arrow[l, "\nu"'] \end{tikzcd} \ ,$$

where p_L is determined by the formula

$$\nu p_L = \mathsf{cofib}\left(i_L y \xrightarrow{i_L \dashv y} \mathsf{id}\right)$$

and p_R is determined by the formula

$$\nu p_R = \mathsf{fib}\left(\mathsf{id} \xrightarrow{y \dashv i_R} i_R y\right) \ .$$

Here the morphisms in the cofiber and fiber are the counit and unit morphisms respectively. The functor ν is given by extension by zero.

Proof. A closed ∞-subcategory always determines a recollement

$$\begin{tikzcd} \mathcal{V}^{\{B\}} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{V}^{\mathsf{Ret}} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{ker}(y) \arrow[l, "\nu"'] \end{tikzcd} \ ,$$

where p_L is determined by the formula

$$\nu p_L = \mathsf{cofib}\left(i_L y \xrightarrow{i_L \dashv y} \mathsf{id}\right)$$

and p_R is determined by the formula

$$\nu p_R = \mathsf{fib}\left(\mathsf{id} \xrightarrow{y \dashv i_R} i_R y\right) \ .$$

Here the morphisms in the cofiber and fiber are the counit and unit morphisms respectively.

Therefore what remains is to show that the equivalence

$$\mathcal{V}^{\{E\}} \simeq \mathsf{ker}(y) \ .$$

The equivalence follows after showing that $\{B\} \subset \mathsf{Ret}$ is Reedy closed (Definition 260).

We first check that $\{B\} \subset \mathsf{Ret}$ satisfies Condition A of Definition 260. Note that $\{E\} = \mathsf{Obj}(\mathsf{Ret}) \backslash \{B\}$, and that $\mathsf{Fact}_{\{B\}}(E, E) \simeq * \simeq \{sr\}$, since the morphism sr factors as

$$\begin{tikzcd} E \arrow[rr, "sr"] \arrow[dr, "s"'] & & E \\ & B \arrow[ur, "r"'] & \end{tikzcd} \ .$$

Therefore, $\{B\} \subset \mathsf{Ret}$ satsifies Condition A.

The subcategory $\{B\} \subset \mathsf{Ret}$ satisfies Condition B, since

$$|\{B\}^{E/}_{/E}| \to \mathsf{Fact}_{\{B\}}(E, E) \simeq \{sr\}$$

is an equivalence. Therefore by Theorem 276, there is a recollement

$$\begin{tikzcd} \mathcal{V}^{\{B\}} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{V}^{\mathsf{Ret}} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{V}^{\{E\}} \arrow[l, "\nu"'] \end{tikzcd} ,$$

and ν is given by extension by zero. $\square$

Observation 246. The inclusion $\mathcal{A}_1 \hookrightarrow \mathcal{A}$ determines a restriction functor

$$\mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \hookrightarrow \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})$$

which admits a left adjoint given by left Kan extension by Lemma 321. Denote the left Kan extension functor

$$\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \hookrightarrow \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$$

Lemma 247. *The functors p_L and p_R are identified as follows.*

(1) The functor p_L sends a functor $\mathsf{Ret} \xrightarrow{\mathsf{F}} \mathcal{V}$ to the functor $\{E\} \to \mathcal{V}$ that selects the cofiber of the morphism $\mathsf{F}(s)$

$$\begin{aligned} p_L : \mathcal{V}^{\mathsf{Ret}} &\longrightarrow \mathcal{V}^{\{E\}} \\ \mathsf{F} &\longmapsto \left\langle \mathsf{cofib}\left(\mathsf{F}(B) \xrightarrow{\mathsf{F}(s)} \mathsf{F}(E)\right)\right\rangle \end{aligned} .$$

(2) The functor p_R sends a functor $\mathsf{F} : \mathsf{Ret} \to \mathcal{V}$ to the functor $\{E\} \to \mathcal{V}$ that selects the fiber of the morphism $\mathsf{F}(r)$

$$\begin{aligned} p_R : \mathcal{V}^{\mathsf{Ret}} &\longrightarrow \mathcal{V}^{\{E\}} \\ \mathsf{F} &\longmapsto \left\langle \mathsf{fib}\left(\mathsf{F}(E) \xrightarrow{\mathsf{F}(r)} \mathsf{F}(B)\right)\right\rangle \end{aligned}$$

Proof. Let $\mathsf{F} : \mathsf{Ret} \to \mathcal{V}$. We seek to compute $p_L(\mathsf{F})$. Recall from Lemma 245 that functor p_L is computed by the formula

$$\nu p_L = \mathsf{cofib}(i_L y \to \mathsf{id}) \ .$$

The functor ν is given by extension by zero by Lemma 245 which gives $\nu p_L(\mathsf{F}(B)) = 0$.

Therefore, it suffices to compute by Lemma 245

$$\mathsf{cofib}(i_L y \mathsf{F} \to \mathsf{F})(E)\ .$$

The composite $i_L y$ sends a functor F to the constant functor at $\mathsf{F}(B)$.

$$\mathsf{cofib}(i_L y \mathsf{F} \to \mathsf{F})(E) = \mathsf{cofib}(\mathsf{const}_{\mathsf{F}(B)} \to \mathsf{F})(E)$$

The evaluation at E functor is a left adjoint, so it preserves colimits. Therefore, the cofiber in $\mathcal{V}^{\mathsf{Ret}}$ evaluated on the object E is the cofiber

$$\mathsf{cofib}(\mathsf{const}_{\mathsf{F}(B)}(E) \to \mathsf{F}(E)) = \mathsf{cofib}\left(\mathsf{F}(B) \xrightarrow{\mathsf{F}(s)} \mathsf{F}(E)\right)\ .$$

Therefore, p_L is the functor that selects the cofiber of $\mathsf{F}(s)$.

The computation of p_R follows a similar argument after noting by Lemma 245 that $\nu p_R \simeq \mathsf{fib}(\mathsf{id} \xrightarrow{i_L \dashv y} i_R y)$ □

Lemma 248. *The glueing functor and reflected glueing functor for the recollement*

$$\begin{tikzcd} \mathcal{V}^{\{B\}} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{V}^{\mathsf{Ret}} \arrow[l, "y" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{ker}(y) \arrow[l, "\nu" description] \end{tikzcd}\ ,$$

are identified as follows:

(1) The gluing functor $\Gamma = p_L i_R$ is equivalent to the zero functor

$$\begin{tikzcd} \mathcal{V}^{\{B\}} \arrow[r, "\Gamma"] \arrow[d, "\simeq"'] & \mathcal{V}^{\{E\}} \arrow[d, "\simeq"] \\ \mathcal{V} \arrow[r, "0"'] & \mathcal{V} \end{tikzcd}\ .$$

(2) The reflected gluing functor $\check{\Gamma} := p_R i_L$ *is equivalent to the zero functor*

$$\begin{tikzcd} \mathcal{V}^{\{B\}} \arrow[r, "\check{\Gamma}"] \arrow[d, "\simeq"'] & \mathcal{V}^{\{E\}} \arrow[d, "\simeq"] \\ \mathcal{V} \arrow[r, "0"'] & \mathcal{V} \end{tikzcd} .$$

Proof. We begin by showing $\Gamma = p_L i_R = 0$. Let $\langle V \rangle : \{B\} \to V$ select an object V. By Lemma 247, the functor Γ applied to $\langle V \rangle$ is

$$\Gamma(\langle V \rangle) = \langle \mathsf{cofib}\left(i_R(\langle V \rangle)(B) \xrightarrow{i_R(\langle V \rangle)(s)} i_R(\langle V \rangle)(E) \right) \rangle \ .$$

The functor i_R is right Kan extension along the inclusion $\{B\} \hookrightarrow \mathsf{Ret}$, so $\Gamma(\langle V \rangle)$ is the functor that selects the cofiber of the morphism

$$\mathsf{lim}\left(\{B\}^{B/} \to \{B\} \xrightarrow{\langle V \rangle} \mathcal{V} \right) \to \mathsf{lim}\left(\{B\}^{E/} \to \{B\} \xrightarrow{\langle V \rangle} \mathcal{V} \right),$$

that is induced by the map $\{B\}^{B/} \xrightarrow{-\circ r} \{B\}^{E/}$. This map is an equivalence, since each category is contractible by Lemma 241 . Therefore, the map between the limits is an equivalence, so its cofiber is 0:

$$\left\langle \mathsf{cofib}\left(\mathsf{lim}\left(\{B\}^{B/} \to \{B\} \xrightarrow{\langle V \rangle} \mathcal{V} \right) \xrightarrow{\simeq} \mathsf{lim}\left(\{B\}^{E/} \to \{B\} \xrightarrow{\langle V \rangle} \mathcal{V} \right) \right) \right\rangle = \langle 0 \rangle \ .$$

A dual argument follows for the computation of p_R. $\square$

Theorem 249. *The functor*

$$\begin{aligned} \mathcal{V}^{\mathsf{Ret}} &\longrightarrow \mathcal{V} \times \mathcal{V} \\ \mathsf{F} &\longmapsto (\mathsf{F}(B), \mathsf{ker}(\mathsf{F}(r))) \end{aligned}$$

is an equivalence of ∞-categories.

Proof. By Theorem 238, the adjunction

$$\mathsf{Fun}\left(\mathsf{Ret},\mathcal{V}\right) \underset{\mathsf{colim}_{\mathsf{sd}([1])^{\mathsf{op}}}}{\overset{\breve{g}}{\rightleftarrows}} \mathsf{lim}^{\mathsf{l.lax}}\left([1] \xrightarrow{\langle \mathsf{Fun}(\{B\},\mathcal{V}) \xrightarrow{\check{\Gamma}=0} \mathsf{Fun}(\{E\},\mathcal{V})\rangle} \mathsf{Pr}_{\mathsf{st}}\right)$$

is an equivalence of ∞-categories. The functor $\mathsf{sd}([1])^{\mathsf{op}} \xrightarrow{\mathsf{max}^{\mathsf{op}}} [1]^{\mathsf{op}}$ is a cartesian fibration (See [6]), and that the left lax limit is defined to be all functors $\mathsf{sd}\left([1]\right)^{\mathsf{op}} \to \breve{\mathscr{G}}\left(\mathsf{Fun}\left(\mathsf{Ret},\mathcal{V}\right)\right)$ over $[1]^{\mathsf{op}}$ that preserve cartesian morphism:

$$\mathsf{lim}^{\mathsf{l.lax}}\left([1] \xrightarrow{\langle \mathsf{Fun}(\{B\},\mathcal{V}) \xrightarrow{\check{\Gamma}=0} \mathsf{Fun}(\{E\},\mathcal{V})\rangle} \mathsf{Pr}_{\mathsf{st}}\right) := \mathsf{Fun}^{\mathsf{cart}}_{/[1]^\circ}\left(\mathsf{sd}([1])^{\mathsf{op}}, \breve{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})\right) \ .$$

Note that the only non-identity cartesian morphism of $\mathsf{sd}([1])^{\mathsf{op}}$ is the morphism

$$\{0,1\}^\circ \to \{0\}^\circ \ .$$

Therefore, for each $\mathsf{F}_0 \in \mathcal{V}^{\{B\}}$, this morphism in $\mathsf{sd}([1])^{\mathsf{op}}$ must be sent to a morphism in $\mathcal{V}^{\mathsf{Ret}}$ of the form

$$\nu p_R i_L(x_0) \xrightarrow{\mathsf{counit}(\nu \dashv \mathsf{p_R})} i_L(\mathsf{F}_0) \ .$$

By Lemma 248, the domain $\nu p_R i_L = \nu\Gamma$ is zero, and therefore, this is the zero morphism. Therefore, the composite functor between ∞-categories

$$\mathsf{Fun}^{\mathsf{cart}}_{/[1]^\circ}\left(\mathsf{sd}([1])^{\mathsf{op}}, \breve{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})\right) \xrightarrow{(ev_{0^\circ}, ev_{1^\circ})} \breve{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})_{|\{0\}^\circ} \times \breve{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})_{|\{1\}^\circ} \xrightarrow[\simeq]{\text{Lemma 245}} \mathcal{V}^{\{B\}} \times \mathcal{V}^{\{E\}}$$

is an equivalence. Indeed, there is a canonical inverse given by

$$\mathcal{V}^{\{B\}} \times \mathcal{V}^{\{E\}} \xrightarrow{i_L \times \nu} \mathcal{V}^{\mathsf{Ret}} \times \mathcal{V}^{\mathsf{Ret}} \xrightarrow{\mathsf{ext}_0} \mathsf{Fun}^{\mathsf{cart}}_{/[1]^\circ}\left(\mathsf{sd}([1])^{\mathsf{op}}, \breve{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})\right) \ ,$$

where the map ext_0 is the functor

$$\begin{tikzcd} \mathsf{ext}_0 : \mathcal{V}^{\mathsf{Ret}} \times \mathcal{V}^{\mathsf{Ret}} \arrow[r] & \mathsf{Fun}\left(\mathsf{sd}([1])^{\mathsf{op}}, \mathcal{V}\right) \\ (\mathsf{F}_0, \mathsf{F}_1) \arrow[r, mapsto] & \begin{tikzcd} & \mathsf{F}_1 \\ \mathsf{F}_0 & 0 \arrow[l] \arrow[u] \end{tikzcd} \end{tikzcd} .$$

Evaluated on objects in $\mathsf{Fun}(\{B\}, \mathcal{V}) \times \mathsf{Fun}\left(\{E\}, \mathcal{V}\right) \subset \mathcal{V}^{\mathsf{Ret}} \times \mathcal{V}^{\mathsf{Ret}}$, the functor ext_0 indeed takes values in $\mathsf{Fun}^{\mathsf{cart}}_{/[1]^\circ}\left(\mathsf{sd}([1])^{\mathsf{op}}, \check{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})\right)$.

In summary, we have a composite equivalence between ∞-categories:

$$\mathcal{V}^{\mathsf{Ret}} \underset{\text{Thm } 238}{\xrightarrow{\simeq}} \mathsf{Fun}^{\mathsf{cart}}_{/[1]^\circ}\left(\mathsf{sd}([1])^{\mathsf{op}}, \check{\mathscr{G}}(\mathcal{V}^{\mathsf{Ret}})\right) \underset{(245)}{\xrightarrow{\simeq}} \mathcal{V}^{\{B\}} \times \mathcal{V}^{\{E\}} \simeq \mathcal{V} \times \mathcal{V} ,$$

given by

$$\mathsf{F} \mapsto \left(\begin{tikzcd} & \nu p_R(\mathsf{F}) \\ i_L y(\mathsf{F}) & \nu p_R i_L y(\mathsf{F}) \arrow[l] \arrow[u] \end{tikzcd} \right) \mapsto \left(y i_L y(\mathsf{F}), p_R \nu p_R(\mathsf{F})\right) \underset{\text{Lemma } 247}{\simeq} \left(\mathsf{F}(B), \mathsf{ker}(\mathsf{F}(r))\right) .$$

□

Remark 250. Using the Reconstruction Theorem of [6], there is also an equivalence

$$\begin{aligned} \mathcal{V}^{\mathsf{Ret}} &\longrightarrow \mathcal{V} \times \mathcal{V} \\ \mathsf{F} &\longmapsto \left(\mathsf{F}(B), \mathsf{coker}(\mathsf{F}(s))\right) \end{aligned} .$$

Corollary 251. Let B be a retract of E in a stable ∞-category $\mathcal{V}$

$$\begin{tikzcd} E \arrow[r, shift right, "r"'] & B \arrow[l, shift right, "s"'] \end{tikzcd} .$$

Then the following diagram commutes

$$
\begin{tikzcd}
 & B \arrow[dl, "\begin{bmatrix}1\\0\end{bmatrix}"'] \arrow[d, "s"] \arrow[dr, "\begin{bmatrix}1\\0\end{bmatrix}"] & \\
B \oplus \mathsf{ker}(r) \arrow[dr, "\begin{bmatrix}1 & 0\end{bmatrix}"'] & E \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "r"] & B \oplus \mathsf{coker}(s) \arrow[dl, "\begin{bmatrix}1 & 0\end{bmatrix}"] \\
 & B &
\end{tikzcd}
$$

REEDY CLOSED

Introduction

Notation 252. The following notation is fixed throughout this chapter.

- $\mathcal{A}$ is an arbitrary ∞-category.
- $\mathcal{A}_0 \subset \mathcal{A}$ is a full ∞-subcategory.
- $i : \mathcal{A}_0 \to \mathcal{A}$ is the inclusion functor from $\mathcal{A}_0$ into $\mathcal{A}$.
- Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be an arbitrary functor between ∞-categories. $\mathsf{F}_!$, F^*, and F_* refer to the left Kan extension functor, restriction functor, and right Kan extension functors respectively:

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C}, \mathcal{S}\mathsf{paces}) \arrow[r, bend left, "\mathsf{F}_!"] \arrow[r, bend right, "\mathsf{F}_*"'] & \mathsf{Fun}(\mathcal{D}, \mathcal{S}\mathsf{paces}) \arrow[l, "\mathsf{F}^*" description] \end{tikzcd} .$$

- The classifying space of an ∞-category $\mathcal{C}$ is denoted $|\mathcal{C}|$.
- $\mathcal{S}\mathsf{paces}$ is the ∞-category of spaces. $\mathcal{S}\mathsf{paces}$ can be presented as the localization of the ordinary category of topological spaces that are homotopy equivalent to CW complexes, localized at homotopy equivalences.
- The ∞-category $\mathcal{S}\mathsf{paces}^{*/}$ is the ∞-category of pointed $\mathcal{S}\mathsf{paces}$.
- For an object c in an arbitrary ∞-category $\mathcal{C}$, we will denote the object c in the opposite ∞-category $\mathcal{C}^{\mathsf{op}}$ as c°.
- Let $\mathcal{V}$ be an arbitrary presentable stable ∞-category.

The theory of stratifications has been used in many different contexts. In the paper [6], a theory of stratifications for presentable stable ∞-categories is developed. A host of important examples are provided by considering the ∞-category $\mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$ of $\mathcal{A}$-modules in $\mathcal{S}\mathsf{p}$. Given a full ∞-subcategory $\mathcal{A}_0$, there is a pair of adjunctions

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p}) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[l, "i^*" description] \end{tikzcd} . \tag{4.1}$$

This situation is a particular instance of a ***closed ∞-subcategory***, in the sense of Definition 1.3.1 of [6], where $\mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p})$ is a closed ∞-subcategory of $\mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$. For $\mathcal{U} := \mathsf{ker}(i^*)$, the inclusion functor $\nu : \mathcal{U} \to \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$ admits two adjoints p_L and p_R, such that $\mathsf{ker}(p_L) = \mathsf{im}(i_!)$ and $\mathsf{ker}(p_R) = \mathsf{im}(i_*)$. Altogether the data fits into a ***recollement***

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p}) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[l, "i^*" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{U} \arrow[l, "\nu" description] \end{tikzcd} . \quad .$$

In favorable situations, which we will refer to as $\mathcal{A}_0 \subset \mathcal{A}$ being ***Reedy Closed***, the ∞-category $\mathcal{U} = \mathsf{ker}(i^*)$ has a description as $\mathcal{A}_1$-modules, where $\mathcal{A}_1$ is an ∞-subcategory of $\mathcal{A} \backslash \mathcal{A}_0$ that is entirely determined by $\mathcal{A}_0 \subset \mathcal{A}$. In what follows in this chapter, we will articulate checkable conditions on $\mathcal{A}_0 \subset \mathcal{A}$ that ensure this favorable situation.

Reedy Closed Condition

The goal is to identify conditions on a closed ∞-subcategory $\mathcal{A}_0 \subset \mathcal{A}$ (Definition 222) that ensures $\mathsf{ker}(i^*) = \mathsf{Fun}\left(\mathcal{A}_1, \mathcal{V}\right)$ for a suitable compliment $\mathcal{A}_1$ of $\mathcal{A}_0 \subset \mathcal{A}$. A naive thought is to take $\mathcal{A}_1 \subset \mathcal{A}$ to be the full ∞-subcategory on those objects in $\mathcal{A}$ that are not in $\mathcal{A}_0$. However, consider the following example.

Example 253. Consider $i\colon \Delta^{\mathsf{op}}_{\leq 0} \hookrightarrow \Delta^{\mathsf{op}}_{\leq 1}$. Consider the restriction functor

$$i^* : \mathsf{Fun}(\Delta^{\mathsf{op}}_{\leq 1}, \mathcal{V}) \to \mathsf{Fun}(\Delta^{\mathsf{op}}_{\leq 0}, \mathcal{V}) \ .$$

By definition, the kernel of this restriction functor consists of those functors $\mathbf{\Delta}^{\mathsf{op}}_{\leq 1} \to \mathcal{V}$ that evaluate on $[0]$ as $0 \in \mathcal{V}$. Using that $0 \in \mathcal{V}$ is a zero-object, evaluation at $[1] \in \mathbf{\Delta}^{\mathsf{op}}_{\leq 1}$ defines an equivalence:

$$\mathsf{ev}_{[1]}\colon \mathsf{ker}(i^*) \xrightarrow{\simeq} \mathcal{V} \ \simeq \ \mathsf{Fun}(\{[1]^\circ\}, \mathcal{V}) \ .$$

However, the full ∞-subcategory $\mathcal{A}' \subset \mathbf{\Delta}^{\mathsf{op}}_{\leq 1}$ consisting of those objects not in $\mathbf{\Delta}^{\mathsf{op}}_{\leq 0}$ has the single object $[1]^\circ$ and two distincts non-identity idempotents. Therefore, $\mathsf{ker}(i^*) \simeq \mathsf{Fun}(\{[1]^\circ\}, \mathcal{V}) \not\simeq \mathsf{Fun}(\mathcal{A}', \mathcal{V})$. Observe that the morphisms in $\mathcal{A}'$ that are not in $\{[1]^\circ\}$ are those that factor through $\Delta^{\mathsf{op}}_{\leq 0}$.

Definition 254. Let $f : q \to p$ be a morphism in $\mathcal{A}$. The morphism f ***factors through*** $\mathcal{A}_0$ if there exists an $a_0 \in \mathcal{A}_0$, and a triangle

$$\begin{tikzcd} q \arrow[rr, "f"] \arrow[dr] & & p \\ & a_0 \arrow[ur] & \end{tikzcd}$$

in $\mathcal{A}$. Define the subspace

$$\mathsf{Fact}_{\mathcal{A}_0} \subset \mathsf{Mor}(\mathcal{A})$$

to be the subspace consisting of those morphisms in $\mathcal{A}$ that factor through $\mathcal{A}_0$.

For a fixed p, q in $\mathcal{A}$, define $\mathsf{Fact}_{\mathcal{A}_0}(q,p)$ to be the subspace of $\mathsf{Hom}_{\mathcal{A}}(q,p)$ on those morphisms $q \to p$ that factor through $\mathcal{A}_0$. Similarly, we say that a morphism $q \to p$ does not factor through $\mathcal{A}_0$ if the morphism $q \to p$ is in the compliment of $\mathsf{Fact}_{\mathcal{A}_0}(q,p)$.

This leads us to the first condition needed on $\mathcal{A}_0 \subset \mathcal{A}$, which is a condition on the morphisms that do not factor through $\mathcal{A}_0$.

Condition A. *Let $q \xrightarrow{f} r \xrightarrow{g} p$ be a pair of composable morphisms in $\mathcal{A}$. If neither f nor g factors through $\mathcal{A}_0$, then the composite morphism $gf : q \to p$ does not factor through $\mathcal{A}_0$.*

Observation 255. Consider a pair of composible morphisms $q \xrightarrow{f} r \xrightarrow{g} p$ in $\mathcal{A}$. If f factors through $\mathcal{A}_0$, then the composite gf must also factor through $\mathcal{A}_0$. Indeed, we can choose a factorization of f through $\mathcal{A}_0$

$$\begin{tikzcd} a_0 \arrow[r, "j"] & b \arrow[dr, "g"] & \\ a \arrow[u, "k"] \arrow[ur, "f"] \arrow[rr, "gf"'] & & c \end{tikzcd} ,$$

which gives a triangle

$$\begin{tikzcd} & a_0 \arrow[dr, "gj"] & \\ a \arrow[ur, "k"] \arrow[rr, "gf"'] & & c \end{tikzcd}$$

in $\mathcal{A}$ such that a_0 is in $\mathcal{A}_0$. A similar argument follows using a factorization of g. Therefore gf factors through $\mathcal{A}_0$.

This leads to a construction of the ∞-subcategory $\mathcal{A}_1$. Condition A, ensures that $\mathcal{A}_1$ is well defined.

Definition 256. Suppose $\mathcal{A}_0 \subset \mathcal{A}$ satisfies Condition A. The ∞-subcategory $\mathcal{A}_1 \subset \mathcal{A}$ is characterized by the following identities:

$$\mathsf{Obj}(\mathcal{A}_1) := \mathcal{A} \backslash \mathcal{A}_0 \ ;$$

$$\mathsf{Mor}(\mathcal{A}_1) := \mathsf{Mor}(\mathcal{A}) \backslash \mathsf{Fact}_{\mathcal{A}_0} \ .$$

Therefore for $p, q \in \mathcal{A} \backslash \mathcal{A}_0$

$$\mathsf{Hom}_{\mathcal{A}_1}(q,p) := \mathsf{Hom}_{\mathcal{A}}(q,p) \backslash \mathsf{Fact}_{\mathcal{A}_0}(q,p).$$

Denote the inclusion of $\mathcal{A}_1$ into $\mathcal{A}$ as

$$j : \mathcal{A}_1 \to \mathcal{A} \ .$$

Lemma 257. *Suppose $\mathcal{A}_0 \subset \mathcal{A}$ satisfies Condition A. The ∞-subcategory $\mathcal{A}_1$ exists.*

Proof. As $\mathcal{A}$ is a complete Segal space, the Segal condition states that the $[p]$ points are determined by the [1] and [0] points. Therefore, by naming subspaces of the [1] and [0] spaces such that the composition rule is still satisfied determines all of the $[p]$ points. Therefore, all that needs to be checked is that the composition rule for $\mathcal{A}_1$ is well defined. This is precisely Condition A on $\mathcal{A}_0 \subset \mathcal{A}$. $\square$

Observation 258. There is a canonical colimit preserving functor from $\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})$ to $\mathcal{U}$ given by

$$\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \xrightarrow{\hat{j}} \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \xrightarrow{p_L} \mathcal{U} \ ,$$

where $\hat{j}$ is given by left Kan extension.

The goal is to show that under a second condition on $\mathcal{A}_0 \subset \mathcal{A}$ that functor

$$\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \xrightarrow{p_L \hat{j}} \mathcal{U}$$

is an equivalence. The following definitions are to set up the statement of this condition.

Definition 259. Let p, q be objects of $\mathcal{A}$. Then the ∞-category $\mathcal{A}^{q/}_{/p}$ is defined to be the pullback

$$\begin{tikzcd}
\mathcal{A}^{q/}_{/p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_{/p} \arrow[d, "\mathsf{fgt}"] \\
\mathcal{A}^{q/} \arrow[r, "\mathsf{fgt}"'] & \mathcal{A}
\end{tikzcd} .$$

The category ${\mathcal{A}_0}^{q/}_{/p}$ is defined the be the pullback.

$$\begin{tikzcd} {\mathcal{A}_0}^{q/}_{/p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{q/}_{/p} \arrow[d, "\mathsf{fgt}"] \\ \mathcal{A}_0 \arrow[r, "i"'] & \mathcal{A} \end{tikzcd}.$$

Consider the functor

$$\circ : {\mathcal{A}_0}^{q/}_{/p} \to \mathsf{Fun}([2], \mathcal{A}) \to \mathsf{Fun}(\{0 < 2\}, \mathcal{A})$$

that sends a pair of composable morphisms to their composite in $\mathcal{A}$. The image of $\circ$ consists of arrows in $\mathcal{A}$ such that evaluation at 0 is identical with q, and evaluation at 2 is identical with p. Therefore the functor $\circ$ factors through $\mathsf{Hom}_{\mathcal{A}}(p, q)$. By the definition of $\mathsf{Fact}_{\mathcal{A}_0}(q, p)$, the functor $\circ$ also factors through $\mathsf{Fact}_{\mathcal{A}_0}(q, p)$

$$\begin{tikzcd} & \mathsf{Fact}_{\mathcal{A}_0}(q, p) \arrow[d] \\ & \mathsf{Hom}_{\mathcal{A}}(q, p) \arrow[d] \\ {\mathcal{A}_0}^{q/}_{/p} \arrow[r, "\circ"'] \arrow[ur, dashed] \arrow[uur, dashed] & \mathsf{Mor}(\mathcal{A}) \end{tikzcd}.$$

Since each codomain is a ∞-groupoid, the functor $\circ$ factors through the classifying space of the ∞-category ${\mathcal{A}_0}^{q/}_{/p}$:

$$\begin{tikzcd} {\mathcal{A}_0}^{q/}_{/p} \arrow[r, "\circ"] \arrow[d] & \mathsf{Fact}_{\mathcal{A}_0}(q, p) \\ |{\mathcal{A}_0}^{q/}_{/p}| \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}.$$

Denote the unique morphism given by the universal property of the classifying space

$$|\circ| : |{\mathcal{A}_0}^{q/}_{/p}| \to \mathsf{Fact}_{\mathcal{A}_0}(q,p) \ .$$

By inspection, for an arbitrary $\mathcal{A}_0 \subset \mathcal{A}$, the functor $|\circ|$ is always surjective. Indeed, given a morphism $f : q \to p$ in $\mathsf{Fact}_{\mathcal{A}_0}(q,p)$, there is a factorization of f given by $q \to a_0 \to p$, which is an object of ${\mathcal{A}_0}^{q/}_{/p}$. However, the functor $|\circ|$ being surjective will not be sufficient to ensure $p_L\hat{\jmath}$ is an equivalence, and we will need a second condition on $\mathcal{A}_0 \subset \mathcal{A}$ to ensure there is an identification of $\mathcal{U}$ as an ∞-category of $\mathcal{A}_1$-modules.

Condition B. *For each p and q in $\mathcal{A}_1$, the functor*

$$|\circ| : |{\mathcal{A}_0}^{q/}_{/p}| \xrightarrow{\simeq} \mathsf{Fact}_{\mathcal{A}_0}(q,p)$$

is an equivalence.

Definition 260. An ∞-subcategory $\mathcal{A}_0$ of $\mathcal{A}$ is ***Reedy-Closed*** if it satisfies both Condition A and Condition B.

Convention 261. For the rest of this article, we will assume that $\mathcal{A}_0 \subset \mathcal{A}$ is Reedy-Closed.

There is an equivalent condition on the morphism $|\circ|$ being an equivalence that will at times be conducive to work with.

Lemma 262. *The following conditions are equivalent.*

(1) $|\circ| : |{\mathcal{A}_0}^{q/}_{/p}| \xrightarrow{\simeq} \mathsf{Fact}_{\mathcal{A}_0}(q,p)$ *is an equivalence (Condition B).*

(2) For all morphisms $f : p \to q \in \mathsf{Fact}_{\mathcal{A}_0}(q,p)$*, the ∞-category* $|{\mathcal{A}_0}^{q/}_{/f:q\to p}|$ *is contractible*

$$|{\mathcal{A}_0}^{q/}_{/f:q\to p}| \simeq *.$$

Proof. The proof is an application of Quillen's Theorem B (Proposition 349.)

We seek to show the diagram

$$\begin{tikzcd}
|{\mathcal{A}_0}^{q/}_{/f:q\to p}| \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & |{\mathcal{A}_0}^{q/}_{/p}| \arrow[d] \\
* \arrow[r] & \mathsf{Fact}_{\mathcal{A}_0}(q,p)
\end{tikzcd}$$

is a pullback diagram. Let $(f : q \to p) \to (g : q \to p)$ be a morphism in $\mathsf{Fact}_{\mathcal{A}_0}(q,p)$. Post-composition with this morphism determines an equivalence between ∞-overategories

$$\mathsf{Fact}_{\mathcal{A}_0}(q,p)_{/f} \xrightarrow{\simeq} \mathsf{Fact}_{\mathcal{A}_0}(q,p)_{/g}$$

by Lemma 345, and therefore

$$\left({\mathcal{A}_0}^{q/}_{/p}\right)_{/f} \to \left({\mathcal{A}_0}^{q/}_{/p}\right)_{/g}$$

is an equivalence. Therefore, by Quillen's theorem B and Lemma 347, the diagram

$$\begin{tikzcd}
|\left({\mathcal{A}_0}^{q/}_{/p}\right)_{|f}| \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & |{\mathcal{A}_0}^{q/}_{/p}| \arrow[d] \\
* \arrow[r] & \mathsf{Fact}_{\mathcal{A}_0}(q,p)
\end{tikzcd}$$

is a pullback diagram. Next, note the equivalence

$$\left({\mathcal{A}_0}^{q/}_{/p}\right)_{|f} \xrightarrow{\simeq} {\mathcal{A}_0}^{q/}_{/f:q\to p} .$$

On the level of objects, this is readily seen. An object of both the left side and the right side is equivalent with a commutative triangle in $\mathcal{A}$

$$\begin{tikzcd} & r \arrow[dr] & \\ q \arrow[ur] \arrow[rr, "f"'] & & p \end{tikzcd} .$$

Therefore the diagram

$$\begin{tikzcd} |{\mathcal{A}_0}^{q/}_{/f:q\to p}| \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & |{\mathcal{A}_0}^{q/}_{/p}| \arrow[d] \\ * \arrow[r] & \mathsf{Fact}_{\mathcal{A}_0}(q,p) \end{tikzcd}$$

is a pullback diagram. Condition B states that the right arrow in the pullback diagram is an equivalence. Since the diagram is a pullback, the arrow ${\mathcal{A}_0}^{q/}_{/f:q\to p} \to *$ is also an equivalence. Similarly, if the left arrow in the pullback diagram is an equivalence then this implies Condition B.

$\square$

In summary, this section gives two checkable conditions, Condition A and Condition B, on an ∞-subcategory $\mathcal{A}_0 \subset \mathcal{A}$. There is also a canonical colimit preserving functor

$$\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \xrightarrow{\hat{j}} \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \xrightarrow{P_l} \mathcal{U} \ .$$

The main goal is to show that this composite morphism $p_L\hat{j}$ is an equivalence of ∞-categories. A corollary of this fact is that we then have a recollement

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p}) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[l, "i^*" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{U} \arrow[l, "\nu" description] & \simeq & \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \end{tikzcd} .$$

The next two sections verify that the functor $p_L\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathcal{U}$ is fully faithful and surjective, which provides the equivaelnce

$$\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \xrightarrow{\simeq} \mathcal{U} \ .$$

The next section shows that the functor $p_L\hat{j}$ is surjective.

<u>Surjectivity of $p_L\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{V}) \to \mathcal{U}$</u>

The goal of this section is to verify that the functor $p_L\hat{j}$ is surjective. The most difficult part of this is in computing the cofiber of the inclusion of $\Sigma^\infty_+ |\mathcal{A}_0{}^{q/}_{/p}| \to \Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q,p)$. The following lemmas are used to build to computing this cofiber.

Lemma 263. *The diagram*

$$\begin{tikzcd}
\mathsf{colim}(\mathcal{A}_{0/p} \to \mathcal{A}_0 \hookrightarrow \mathcal{A} \xrightarrow{\mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{paces}) \arrow[r, "\circ"] \arrow[d] & \mathsf{Hom}_{\mathcal{A}}(p,q) \arrow[d] \\
* \arrow[r, "\langle + \rangle"'] & \mathsf{Hom}_{\mathcal{A}_1}(p,q)_+
\end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{paces}$.

Proof. First, we seek to show the colimit

$$\mathsf{colim}\left(\mathcal{A}_{0/p} \to \mathcal{A}_0 \to \mathcal{A} \xrightarrow{\mathsf{Hom}_{\mathcal{A}(q,-)}} \mathcal{S}\mathsf{paces}\right) \simeq |\mathcal{A}_0{}^{q/}_{/p}| \ .$$

is equivalent with the ∞-category $|\mathcal{A}_0{}^{q/}_{/p}|$. The colimit is valued in $\mathcal{S}\mathsf{paces}$, therefore by Lemma 352, the colimit is computed as the classifying space of the unstraightening of the composite functor $\mathcal{A}_{0/p} \to \mathcal{S}\mathsf{paces}$. By Lemma 351 the unstraightening of the functor can be computed as a series of pullbacks

$$\begin{tikzcd}
{\mathcal{A}_0}^{q/}_{/p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {\mathcal{A}_0}^{q/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{q/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Spaces}^{*/} \arrow[d] \\
\mathcal{A}_{0/p} \arrow[r] & \mathcal{A}_0 \arrow[r] & \mathcal{A} \arrow[r, "\mathsf{Hom}_{\mathcal{A}}(q,-)"'] & \mathsf{Spaces}
\end{tikzcd} .$$

The first pullback is because the over and under categories are the unstraightenings of the yoneda embeddings. Therefore the colimit is

$$\mathsf{colim}\left(\mathcal{A}_{0/p} \to \mathcal{A}_0 \to \mathcal{A} \xrightarrow{\mathsf{Hom}_{\mathcal{A}(q,-)}} \mathsf{Spaces}\right) \simeq |{\mathcal{A}_0}^{q/}_{/p}| \ .$$

Next, recall that the morphisms of $\mathsf{Hom}_{\mathcal{A}_1}(q,p)$ are defined to be the compliment of $\mathsf{Fact}_{\mathcal{A}_0}(q,p)$ in $\mathsf{Hom}_{\mathcal{A}}(q,p)$. There is then the following equivalences

$$\mathsf{Hom}_{\mathcal{A}}(p,q) \simeq \mathsf{Fact}_{\mathcal{A}_0}(q,p) \amalg \mathsf{Hom}_{\mathcal{A}_1}(q,p) \simeq |{\mathcal{A}_0}^{q/}_{/p}| \amalg \mathsf{Hom}_{\mathcal{A}_1}(q,p) \ .$$

where the last equivalence is given by Condition B. Therefore there is an equivalence between the two squares

$$\begin{tikzcd}
{|{\mathcal{A}_0}^{q/}_{/p}|} \arrow[rr] \arrow[dd] \arrow[dr, "\mathsf{id}"'] & & {|{\mathcal{A}_0}^{q/}_{/p}| \amalg \mathsf{Hom}_{\mathcal{A}_1}(q,p)} \arrow[dd] \arrow[dr, "\simeq"] & \\
& {|{\mathcal{A}_0}^{q/}_{/p}|} \arrow[rr] \arrow[dd] & & \mathsf{Hom}_{\mathcal{A}}(q,p) \arrow[dd] \\
* \arrow[rr] \arrow[dr, "\mathsf{id}"'] & & \mathsf{Hom}_{\mathcal{A}_1}(q,p)_+ \arrow[dr, "\mathsf{id}"] & \\
& * \arrow[rr] & & \mathsf{Hom}_{\mathcal{A}_1}(q,p)_+
\end{tikzcd} .$$

Finally, the back square is a pullback by Lemma 341, which implies the front square is also a pullback. $\square$

Lemma 264. *The diagram*

$$\begin{tikzcd}
\mathsf{colim}(\mathcal{A}_{0/p} \to \mathcal{A}_0 \hookrightarrow \mathcal{A} \xrightarrow{\mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{paces} \xrightarrow{(-)+} \mathcal{S}\mathsf{paces}^{*/}) \arrow[r, "\circ"] \arrow[d] & \mathsf{Hom}_{\mathcal{A}}(p,q)_+ \arrow[d] \\
* \arrow[r, "\langle + \rangle"'] & \mathsf{Hom}_{\mathcal{A}}(p,q)_+
\end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{paces}^{*/}$.

Proof. First note that the functor $\mathcal{S}\mathsf{paces} \xrightarrow{(-)_+} \mathcal{S}\mathsf{paces}^{*/}$ preserves colimits, as it is the left adjoint to the forgetful functor $\mathcal{S}\mathsf{paces}^{*/} \to \mathcal{S}\mathsf{paces}$. Therefore it suffices to show the diagram

$$\begin{tikzcd}
\mathsf{colim}(\mathcal{A}_{0/p} \to \mathcal{A}_0 \hookrightarrow \mathcal{A} \xrightarrow{\mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{paces})_+ \arrow[r, "\circ"] \arrow[d] & \mathsf{Hom}_{\mathcal{A}}(p,q)_+ \arrow[d] \\
* \arrow[r, "\langle + \rangle"'] & \mathsf{Hom}_{\mathcal{A}_1}(p,q)_+
\end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{paces}$, which follows from Lemma 342 and Lemma 263. $\square$

Lemma 265. *The diagram*

$$\begin{tikzcd}
\mathsf{colim}(\mathcal{A}_{0/p} \to \mathcal{A}_0 \hookrightarrow \mathcal{A} \xrightarrow{\mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{paces} \xrightarrow{+} \mathcal{S}\mathsf{paces}^{*/} \xrightarrow{\Sigma^\infty} \mathcal{S}\mathsf{p}) \arrow[r, "\circ"] \arrow[d] & \Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(p,q) \arrow[d] \\
0 \arrow[r, "\langle + \rangle"'] & \Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}_1}(p,q)
\end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{p}$.

Proof. This follows immediately from Lemma 264, and the fact that $\Sigma^\infty : \mathcal{S}\mathsf{paces}^{*/} \to \mathcal{S}\mathsf{p}$ preserves colimits since it is a left adjoint to Ω^∞. $\square$

This verifies that

$$\mathsf{cofib}\left(\Sigma^\infty_+ |{\mathcal{A}_0}^{q/}_{/p}| \to \Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(p,q)\right) \simeq \Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}_1}(q,p) \ .$$

Lemma 266. *The diagram*

$$\begin{tikzcd} \mathcal{A}_1^{\mathsf{op}} \arrow[r, "j^\circ"] \arrow[d, "\mathsf{yoneda}"'] & \mathcal{A}^{\mathsf{op}} \arrow[r, "\mathsf{yoneda}"] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{paces}) \\ \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{paces}) \arrow[urr, bend right, "j_!"'] & & \end{tikzcd}$$

witnesses $j_!$ as the left Kan extension.

Proof. We first show that the diagram

$$\begin{tikzcd} \mathcal{A}_1^{\mathsf{op}} \arrow[r, "j^{\mathsf{op}}"] \arrow[d, "\mathsf{yoneda}"'] & \mathcal{A}^{\mathsf{op}} \arrow[d, "\mathsf{yoneda}"] \\ \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{paces}) \arrow[r, "j_!"] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{paces}) \end{tikzcd}$$

commutes. First, let $q^\circ \in \mathcal{A}_1^{\mathsf{op}}$ and $p \in \mathcal{A}$. The left composite applied to q° gives a functor $j_!(\mathcal{A}_1 \xrightarrow{\mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{paces})$. Evaluating this functor on the point p gives

$$j_!\left(\mathcal{A}_1 \xrightarrow{\mathsf{Hom}_{\mathcal{A}_1}(q,-)} \mathcal{S}\mathsf{paces}\right)(p) := \mathsf{colim}\left(\mathcal{A}_{1/p} \to \mathcal{A}_1 \xrightarrow{\mathsf{Hom}_{\mathcal{A}_1}(q,-)} \mathcal{S}\mathsf{paces}\right)$$

since $j_!$ is the left Kan extension functor. The colimit is valued in $\mathcal{S}\mathsf{paces}$, so to evaluate the colimit one can take the classifying space of the unstraightening of the functor by Lemma 352. The unstraightening can be computed as a series of pullbacks by Lemma 351

$$\begin{tikzcd} \mathcal{A}_1{}^{q/}_{/p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A}_1^{q/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}\mathsf{paces}^{*/} \arrow[d] \\ \mathcal{A}_{1/p} \arrow[r] & \mathcal{A}_1 \arrow[r, "\mathsf{Hom}_{\mathcal{A}_1}(q,)"'] & \mathcal{S}\mathsf{paces} \end{tikzcd} .$$

Therefore there is the equivalence

$$\mathsf{colim}\left(\mathcal{A}_{1/p} \to \mathcal{A}_1 \xrightarrow{\mathsf{Hom}_{\mathcal{A}_1}(q,-)} \mathcal{S}\mathsf{paces}\right) \simeq |\mathcal{A}_1{}^{q/}_{/p}| .$$

The right composite is equivalent to $\mathsf{Hom}_{\mathcal{A}}(q,p)$, as it is the yoneda functor $\mathsf{Hom}_{\mathcal{A}}(q,-)$ applied to $p \in \mathcal{A}$. Therefore we seek to show that $\mathsf{Hom}_{\mathcal{A}}(q,p)$ is equivalent to $|{\mathcal{A}_1}^{q/}_{/p}|$. Note the equivalence

$$\mathsf{Hom}_{\mathcal{A}}(q,p) \simeq \left(\mathcal{A}_{1/p}\right)_{|q}$$

as both are the limit of the diagram

$$\begin{tikzcd} & & \ast \arrow[d, "\langle p \rangle"] \\ & \mathsf{Fun}([1], \mathcal{S}\mathsf{paces}) \arrow[r, "\mathsf{ev}_1"'] \arrow[d, "\mathsf{ev}_0"] & \mathcal{A} \quad . \\ \ast \arrow[r, "\langle q \rangle"'] & \mathcal{A} & \end{tikzcd}$$

There is also a canonical functor

$$\left(\mathcal{A}_{1/p}\right)_{|q} \to \left({\mathcal{A}_1}^{q/}_{/p}\right)$$

that extends a $f : q \to p$ to $f\,\mathsf{id}_q : q \to q \to p$. This functor is the left adjoint to the composition functor $\circ : {\mathcal{A}_1}^{q/}_{/p} \to \left(\mathcal{A}_{1/p}\right)_{|q}$. Taking classifying spaces of both sides then gives the equivalence

$$|{\mathcal{A}_1}^{q/}_{/p}| \simeq |\left(\mathcal{A}_{1/p}\right)_{|q}| \simeq \mathsf{Hom}_{\mathcal{A}}(q,p)$$

since the classifying space functor preserves the adjunctions, and adjunctions in $\mathcal{S}\mathsf{paces}$ are always equivalences.

The next step is to show the functor $j_! : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{paces}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{paces})$ also witnesses a left Kan extension. This is to show that for each $\mathsf{F} : \mathcal{A}_1 \to \mathcal{S}\mathsf{paces}$ and each $p \in \mathcal{A}$ the spaces

$$\mathsf{colim}\left({\mathcal{A}_1^{\mathsf{op}}}_{/\mathsf{F}} \to \mathcal{A}_1^{\mathsf{op}} \to \mathcal{A}^{\mathsf{op}} \xrightarrow{\mathsf{yoneda}} \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{paces}) \xrightarrow{\mathsf{ev}_p} \mathcal{S}\mathsf{paces}\right) \simeq j_!(\mathsf{F})(p) \ . \qquad (4.2)$$

are equivalent. First consider $\mathsf{colim}\left({\mathcal{A}_1}^{\mathsf{op}}_{/\mathsf{F}} \to \mathcal{A}_1^{\mathsf{op}} \to \mathcal{A}^{\mathsf{op}} \xrightarrow{\mathsf{yoneda}} \mathsf{Fun}(\mathcal{A}, \mathsf{\mathcal{S}paces}) \xrightarrow{\mathsf{ev}_p} \mathsf{\mathcal{S}paces}\right)$. By inspection, the composite $\mathcal{A}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{A}, \mathcal{V}) \to \mathsf{\mathcal{S}paces}$ is the functor $\mathsf{Hom}_{\mathcal{A}}(-, p)$. The unstraightening of $\mathsf{Hom}_{\mathcal{A}}(-, p)$ is $(\mathcal{A}^{\mathsf{op}})^{p^\circ/}$. Therefore the colimit is the classifying space of the limit of the diagram

$$\begin{tikzcd} & ({\mathcal{A}_1}^{\mathsf{op}})^{p^\circ/} \arrow[d] \\ {\mathcal{A}_1}^{\mathsf{op}}{}_{/\mathsf{F}} \arrow[r] & \mathcal{A}_1^{\mathsf{op}} \end{tikzcd} .$$

By Lemma 355, this is equivalent with the classifying space of the limit

$$\begin{tikzcd} & \left(\mathcal{A}_{1/p}\right)^{\mathsf{op}} \arrow[d] \\ \left({\mathcal{A}_1}^{\mathsf{F}^\circ/}\right)^{\mathsf{op}} \arrow[r] & {\mathcal{A}_1}^{\mathsf{op}} \end{tikzcd} .$$

Since the classifying space of an ∞-category $\mathcal{C}$ is equivalent with the classifying space of its opposite $\mathcal{C}^{\mathsf{op}}$, this is equivalent with taking the classifying space of the limit of the opposite diagram

$$\begin{tikzcd} & \mathcal{A}_{1/p} \arrow[d] \\ {\mathcal{A}_1}^{\mathsf{F}^\circ/} \arrow[r] & \mathcal{A}_1 \end{tikzcd} .$$

The value of $j_!(\mathsf{F})(p)$ is given by

$$j_!(\mathsf{F})(p) = \mathsf{colim}\left(\mathcal{A}_{1/p} \to \mathcal{A}_1 \to \mathcal{A} \xrightarrow{\mathsf{F}} \mathsf{\mathcal{S}paces}\right)$$

The unstraightening of $\mathcal{A}_1 \to \mathcal{A} \xrightarrow{\mathsf{F}} \mathsf{\mathcal{S}paces}$ is ${\mathcal{A}_1}^{\mathsf{F}^\circ/}$, so the colimit is the classifying space of the limit of the diagram

$$\begin{tikzcd} & \mathcal{A}_{1/p} \arrow[d] \\ {\mathcal{A}_1}^{\mathsf{F}^\circ/} \arrow[r] & \mathcal{A}_1 \end{tikzcd} .$$

Therefore the two colimits are equivalent. $\square$

Notation 267. Define the ***spectral yoneda*** functor to be the composite functor

$$\text{よ} := \mathcal{A}^{\mathsf{op}} \xrightarrow{\mathsf{yoneda}} \mathsf{Psh}(\mathcal{A}) \xrightarrow{\Sigma^\infty_+} \mathcal{S}\mathsf{p} \ .$$

Similarly, the spectral yoneda functor for $\mathcal{A}_1$ is denoted $\text{よ}_{\mathcal{A}_1}$.

Lemma 268. *The diagram*

$$\begin{tikzcd}
\mathcal{A}_1^{\mathsf{op}} \arrow[r, "\text{よ}\circ j"] \arrow[d, "\text{よ}_{\mathcal{A}_1}"'] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[r, "p_L"] & \mathcal{U} \arrow[r, "\nu"] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[r, "j^*"] & \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \\
\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \arrow[ur, "\hat{j}"] \arrow[urr, "\nu\hat{j}"] \arrow[urrr, "\nu p_L \hat{j}"] \arrow[urrrr, "\mathsf{id}"] & & & &
\end{tikzcd} .$$

commutes, and in particular

$$j^* \nu p_L \hat{j} = \mathsf{id} \ .$$

Proof. Let $q \in \mathcal{A}_1^{\mathsf{op}}$. The composite νp_L is equivalent to the functor

$$\nu p_L = \mathsf{cofib}\left(i_! i^* \to \mathsf{id}\right) \ .$$

We seek to evalute νp_L on $\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q, -)$. Let $p \in \mathcal{A}$. Evaluating the functor $i_! i^* \left(\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q, -)\right)$ on p gives

$$i_! i^* \left(\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q, -)\right)(p) = \mathsf{colim}({\mathcal{A}_0^{\mathsf{op}}}_{/p} \to {\mathcal{A}_0}^{\mathsf{op}} \to \mathcal{A}^{\mathsf{op}} \xrightarrow{\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{p})$$

where the colimit arises since $i_!$ is the left Kan extension. Therefore

$$\nu p_L(\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q, -) = \mathsf{cofib}(\mathsf{colim}({\mathcal{A}_0^{\mathsf{op}}}_{/p} \to \mathcal{A}_0^{\mathsf{op}} \to \mathcal{A}^{\mathsf{op}} \xrightarrow{\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}}(q,-)} \mathcal{S}\mathsf{p}) \to \mathsf{Hom}_{\mathcal{A}}(q, p)) \ .$$

By Lemma 265, this cofiber is $\Sigma^\infty_+ \mathsf{Hom}_{\mathcal{A}_1}(q,p)$. Therefore

$$j^* \nu p_L \hat{j} = \mathsf{id} \ .$$

∎

Proposition 269 (Surjective)**.** *The functor* $p_L \hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathcal{U}$ *is surjective.*

Proof. Let $\mathsf{F} \in \ker(y) \subset \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$. Consider the morphisms $j^*\nu\mathsf{F} \in \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})$. We seek to show the morphism

$$p_L \hat{j} j^* \nu \mathsf{F} \xrightarrow{\mathsf{counit}} p_L \nu \mathsf{F} \xrightarrow{\mathsf{counit}} \mathsf{F} \ .$$

is an equivalence. Since ν is conservative, it suffices to show the morphism

$$\nu p_L \hat{j} j^* \nu \mathsf{F} \xrightarrow{\mathsf{counit}} \nu p_L \nu \mathsf{F} \xrightarrow{\mathsf{counit}} \nu \mathsf{F}$$

is an equivalence in $\mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$. The functor

$$\mathcal{A}_0 \amalg \mathcal{A}_1 \xrightarrow{(i,j)} \mathcal{A}$$

is surjective, and therefore the functor

$$\mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \xrightarrow{(y, j^*)} \mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p}) \times \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})$$

is conservative. Therefore it suffices to show that the two morphisms

$$y \nu p_L \hat{j} j^* \nu \mathsf{F} \to y \nu p_L \nu \mathsf{F} \to y \nu \mathsf{F} \tag{4.3}$$

$$j^* \nu p_L \hat{j} j^* \nu \mathsf{F} \to j^* \nu p_L \nu \mathsf{F} \to j^* \nu \mathsf{F} \tag{4.4}$$

are equivalences. The morphism (4.3) is an equivalence, since $y\nu \simeq 0$. The morphism (4.4) is an equivalence, by Lemma 268 since $j^*\nu p_L\hat{j} \simeq \mathsf{id}$. $\square$

<u>Fully Faithfulness of the functor $p_L\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathcal{U}$</u>

The next section is devoted to showing the functor

$$p_l\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \to \mathcal{U}$$

is fully faithful. Recall that to show $\hat{j}p_L$ is fully faithful is to show that for two functors F and G in $\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})$, that the functor between spaces

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})}(\mathsf{F}, \mathsf{G}) \to \mathsf{Hom}_{\mathcal{U}}(\hat{j}p_L(\mathsf{F}), \hat{j}p_L(\mathsf{G}))$$

induced by $\hat{j}p_L$ is an equivalence of spaces. By Lemma 338, this is equivalent with showing that limits $\mathsf{lim}(\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{S}\mathsf{paces})$ and $\mathsf{lim}(\mathsf{TwAr}(\mathcal{A}) \to \mathcal{S}\mathsf{paces})$ are equivalent, where the map between the limits is given by the inclusion of $\mathsf{TwAr}(\mathcal{A}_1)$ into $\mathsf{TwAr}(\mathcal{A})$. In order to show the morphism in $\mathcal{S}\mathsf{paces}$ between the two limits is an equivalence, we factor inclusion of $\mathsf{TwAr}(\mathcal{A}_1) \to \mathsf{TwAr}(\mathcal{A})$ through an ∞-subcategory of $\mathsf{TwAr}(\mathcal{A})$, which we introduce now.

Definition 270. Define $\mathcal{T}$ to be the full ∞-subcategory of $\mathsf{TwAr}(\mathcal{A})$ on those arrows $(q \to p)$ such that $(q \to p)$ is in $\mathsf{TwAr}(\mathcal{A}_1)$, or p is in $\mathcal{A}_0$.

Lemma 271. *The inclusion functor $\mathcal{T} \to \mathsf{TwAr}(\mathcal{A})$ is initial.*

Proof. By Quillen's Theorem A, the functor $\mathcal{T} \to \mathsf{TwAr}(\mathcal{A})$ is initial if and only if for all $f : q \to p \in \mathsf{TwAr}(\mathcal{A})$, the classifying space

$$|\,\mathcal{T}_{/(q\to p)}\,| \simeq *$$

is contractible. There are 2 cases to consider.

(1) $f : q \to p$ is in $\mathcal{T}$.

(2) $f : q \to p$ is not in $\mathcal{T}$.

The first case follows since if $f : q \to p$ is in $\mathcal{T}$, then $\mathcal{T}_{/(q \to p)}$ has an initial object. For the second case, define $\mathcal{T}_0$ as the full ∞-subcategory of $\mathcal{T}_{/f:(q \to p)}$ on the objects

$$\begin{tikzcd} a \arrow[r] & b \arrow[d] \\ q \arrow[u] \arrow[r, "f"'] & p \end{tikzcd}$$

such that $q \to a$ is an equivalence.

Note that the the canonical map

$$\mathcal{T}_0 \longrightarrow \mathcal{A}_0{}^{q/}_{/f:q\to p}$$

$$\begin{tikzcd} a \arrow[r, "\beta"] & b \arrow[d, "g"] \\ q \arrow[u, "\alpha"] \arrow[r, "f"] & p \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & b \arrow[dr] & \\ q \arrow[ur] \arrow[rr, "f"] & & p \end{tikzcd}$$

admits a right adjoint given by

$$\mathcal{A}_0{}^{q/}_{/f:q\to p} \longrightarrow \mathcal{T}_0$$

$$\begin{tikzcd} & b \arrow[dr] & \\ q \arrow[ur] \arrow[rr] & & p \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} q \arrow[r] & b \arrow[d] \\ q \arrow[u, "\mathrm{id}"] \arrow[r] & p \end{tikzcd} .$$

Furthermore, the inclusion of $\mathcal{T}_0$ to $\mathcal{T}_{/f:q\to p}$ admits a left adjoint given by

$$\mathcal{T}_{/f:q\to p} \longrightarrow \mathcal{T}_0$$

$$\begin{tikzcd} a \arrow[r, "h"] & b \arrow[d] \\ q \arrow[u, "g"] \arrow[r, "f"] & p \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} q \arrow[r, "hg"] & b \arrow[d] \\ q \arrow[u, "\mathsf{id}"] \arrow[r, "f"] & p \end{tikzcd} .$$

Therefore after applying classifying spaces the spaces

$$* \underset{\text{Lemma 262}}{\simeq} |\mathcal{A}_0{}^{q/}_{/f:q\to p}| \simeq |\,\mathcal{T}_0\,| \simeq |\,\mathcal{T}_{/f:q\to p}\,|$$

are equivalent. $\square$

Lemma 272. *The functor* $\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{T}$ *is fully faithful.*

Proof. We show that $\mathsf{TwAr}(\mathcal{A}_1) \to \mathsf{TwAr}(\mathcal{A})$ is fully faithful. Since $\mathcal{T} \to \mathsf{TwAr}(\mathcal{A})$ is fully faithful by definition, then $\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{T}$ is fully faithful since fully faithful functors must satisfy a 2 out of 3 property, ie for any triangle in $\mathsf{Cat}_{(\infty,1)}$ such that 2 of the arrows are fully faithful, then the

Let $(f_s : q_s \to p_s)$ and $(f_t : p_t \to q_t)$ be objects of $\mathsf{TwAr}(\mathcal{A}_1)$. A morphism from $(f_s \to f_t)$ in $\mathsf{TwAr}(\mathcal{A})$ is a square

$$\begin{tikzcd} q_s \arrow[r, "f_s"] & p_s \arrow[d, "f_t"] \\ q_t \arrow[u] \arrow[r] & p_t \end{tikzcd}$$

in $\mathcal{A}$. Since $q_t \to p_t$ is an arrow in $\mathcal{A}_1$, then it must not factor through $\mathcal{A}_0$. By Observation

255, it must be that the arrows $q_t \to q_s$ and $p_s \to p_t$ also do not factor through $\mathcal{A}_0$, since otherwise $q_t \to p_t$ would factor through $\mathcal{A}_0$. Therefore this square is a square in $\mathsf{TwAr}(\mathcal{A}_1)$. Therefore the functor is fully faithful. $\square$

Lemma 273. *The functor* $\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{T}$ *is a monomorphism.*

Proof. First note that monomorphisms must satisfy a 2 out of 3 property. Therefore it suffices to show that $\mathsf{TwAr}(\mathcal{A}_1) \to \mathsf{TwAr}(\mathcal{A})$ is a monomorphism. This is immediate since $\mathcal{A}_1 \to \mathcal{A}$ is a monomorphism, so by Lemma 353 the functor $\mathsf{TwAr}(\mathcal{A}_1) \to \mathsf{TwAr}(\mathcal{A})$ is a monomorphism. $\square$

Lemma 274. *The functor* $\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{T}$ *is a fully faithful right fibration.*

Proof. The functor $\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{T}$ is a right fibration if there is a unique lift

$$\begin{tikzcd} * \arrow[r, "\langle f_t \rangle"] \arrow[d, "\langle 1 \rangle"'] & \mathsf{TwAr}(\mathcal{A}_1) \arrow[d] \\ {[1]} \arrow[r, "\langle (f_s \to f_t) \rangle"'] \arrow[ur, dashed, "\exists!"] & \mathcal{T} \end{tikzcd}$$

for every commutative square, where where $f_s : q_s \to p_s$ is in $\mathsf{TwAr}(\mathcal{A}_1)$ and $f_t : q_t \to p_t$ is in $\mathcal{T}$. An arrow $[1] \xrightarrow{\langle f_s \to f_t \rangle} \mathcal{T}$ consists of a diagram

$$\begin{tikzcd} q_s \arrow[r, "f_s"] & p_s \arrow[d] \\ q_t \arrow[u] \arrow[r, "f_t"'] & p_t \end{tikzcd} \tag{4.5}$$

in $\mathcal{A}$. There is a lift $[1] \to \mathsf{TwAr}(\mathcal{A}_1)$ if the two vertical morphisms in (4.5) do not factor through $\mathcal{A}_0$. Note that the morphism $q_t \to p_t$ lies in $\mathsf{TwAr}(\mathcal{A}_1)$, so it does not factor through $\mathcal{A}_0$. By the same reasoning as Observation 255, if either of the two vertical morphisms factored through $\mathcal{A}_0$, then the morphism f_t would factor through $\mathcal{A}_0$, which contradicts f_t

being in $\mathsf{TwAr}(\mathcal{A}_1)$. Therefore the two vertical morphism do not factor through $\mathcal{A}_0$, and so diagram (4.5) can be regarded as a morphism in $\mathsf{TwAr}(\mathcal{A}_1)$. Therefore there is a lift. □

Proposition 275. *The functor*

$$p_L\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathcal{U}$$

is fully faithful.

Proof. Since ν is fully faithful, it suffices to check that

$$\nu p_L\hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$$

is fully faithful. Let F and G be in $\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})$. Then by Lemma 338, each hom space is presented by a limit

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{A}_1,\mathcal{S}\mathsf{p})}(\mathsf{F}, \mathsf{G}) \simeq \lim(\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{A}_1^{\mathsf{op}} \times \mathcal{A}_1 \xrightarrow{\mathsf{F}^{\mathsf{op}} \times \mathsf{G}} \mathcal{V} \times \mathcal{V} \xrightarrow{\mathsf{Hom}_{\mathcal{V}}(-,-)} \mathcal{S}\mathsf{paces})\ .$$

Define H to be the composite $\mathsf{TwAr}(\mathcal{A}_1) \to \mathcal{S}\mathsf{paces}$. Next, note by Lemma 268 that diagram

$$\begin{tikzcd}
\mathsf{TwAr}(\mathcal{A}_1) \arrow[rr] \arrow[d] & & \mathsf{TwAr}(\mathcal{A}) \arrow[d] \\
\mathcal{A}_1^{\mathsf{op}} \times \mathcal{A}_1 \arrow[dr] & & \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \arrow[dl] \\
& \mathcal{S}\mathsf{paces} &
\end{tikzcd}$$

commutes. It also witnesses an extension by 0 since $i^*\nu = 0$. Factoring through $\mathcal{T}$ gives

$$\begin{tikzcd}
\mathsf{TwAr}(\mathcal{A}_1) \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[dd, "\mathsf{H}_*"] & \mathsf{TwAr}(\mathcal{A}) \arrow[d] \\
\mathcal{A}_1^{\mathsf{op}} \times \mathcal{A}_1 \arrow[dr] & & \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \arrow[dl] \\
& \mathcal{S}\mathsf{paces} &
\end{tikzcd}$$

where H_* is given by extension by 0. The functor H_* also witnesses a right Kan extension by Lemma 350, since $\mathsf{TwAr}\mathcal{A}_1 \to \mathcal{T}$ is a fully faithful right fibration (Lemma 274). Therefore by Lemma 354,

$$\lim(\mathsf{TwAr}(\mathcal{A}_1) \xrightarrow{\mathsf{H}} \mathcal{S}\mathsf{paces}) \xrightarrow{\simeq} \lim(\mathcal{T} \xrightarrow{\mathsf{H}_*} \mathcal{S}\mathsf{paces}) \ .$$

is an equivalence. Finally, by Lemma 271 the inclusion $\mathcal{T} \to \mathcal{A}$ is initial. Therefore,

$$\lim(\mathcal{T} \xrightarrow{\mathsf{H}_*} \mathcal{A}) \simeq \lim(\mathsf{TwAr}(\mathcal{A}) \to \mathcal{S}\mathsf{paces}) \ .$$

Again, by Lemma 338 the limit is equivalent to the $\mathsf{Hom}_{\mathsf{Fun}(\mathcal{A},\mathcal{S}\mathsf{p})}(p_L\hat{j}\mathsf{F}, p_L\hat{j}`)_{\mathsf{G}})$ space

$$\lim\left(\mathsf{TwAr}(\mathcal{A}) \to \mathcal{S}\mathsf{paces}\right) \simeq \mathsf{Hom}_{\mathsf{Fun}(\mathcal{A},\mathcal{S}\mathsf{p})}(\nu p_L\hat{j}\mathsf{F}, \nu p_L\hat{j}\mathsf{G}) \ .$$

$\square$

<u>Reedy Closed Recollement</u>

Theorem 276. *Let $\mathcal{A}_0 \overset{i}{\hookrightarrow} \mathcal{A}$ be Reedy-closed. Then the category $\mathcal{A}_1 \overset{j}{\hookrightarrow} \mathcal{A}$ given by Definition 256 gives a recollement*

$$\begin{tikzcd}
\mathsf{Fun}\left(\mathcal{A}_0, \mathcal{V}\right) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Fun}\left(\mathcal{A}, \mathcal{V}\right) \arrow[l, "i^*"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{Fun}\left(\mathcal{A}_1, \mathcal{V}\right) \arrow[l, "\nu_{\mathcal{A}_1}"']
\end{tikzcd} \ .$$

where $\nu_{\mathcal{A}_1} := \nu p_L \hat{j}$ is the extension by zero functor.

Proof. The closed ∞-subcategory $\mathcal{A}_0 \subset \mathcal{A}$ determines a recollement

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p}) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[l, "i^*"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{U} \arrow[l, "\nu"'] \end{tikzcd} .$$

Lemma 268 and 272 verifies that the functor

$$p_L \hat{j} : \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p})$$

is fully faithful and essential surjective, and so there is an equivalence of ∞-categories

$$\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \simeq \mathcal{U} .$$

Therefore the recollement is given by

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{A}_0, \mathcal{S}\mathsf{p}) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Fun}(\mathcal{A}, \mathcal{S}\mathsf{p}) \arrow[l, "i^*"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p}) \arrow[l, "\nu_{\mathcal{A}_1}"'] \end{tikzcd} .$$

where the functor $\nu_{\mathcal{A}_1}$ is the composite of $\nu p_L \hat{j}$. We next seek to verify that the functor $\nu p_l \hat{j}$ is the extension by zero functor. The restriction to the ∞-category $\mathcal{A}_1$ is given by composition with the functor j^* gives the identity on $\mathcal{A}_1$, which is the identity functor by Lemma 268

$$j^* \nu p_L \hat{j} = \mathsf{id}_{\mathsf{Fun}(\mathcal{A}_1, \mathcal{S}\mathsf{p})} .$$

The restriction to the ∞-category $\mathcal{A}_0$ is zero since $y\nu$ is zero since $\mathcal{U}$ is the kernel of y

$$y \nu p_L \hat{j} = 0 p_l \hat{j} = 0 .$$

Therefore $\nu_{\mathcal{A}_1}$ is given by extension by zero. Finally, by tensoring over $\mathcal{V}$, we get the recollement

$$\begin{tikzcd} \mathsf{Fun}\,(\mathcal{A}_0, \mathcal{V}) \arrow[rr, bend left, "i_!"] \arrow[rr, bend right, "i_*"'] & & \mathsf{Fun}\,(\mathcal{A}, \mathcal{V}) \arrow[ll, "i^*"'] \arrow[rr, bend left, "p_L"] \arrow[rr, bend right, "p_R"'] & & \mathsf{Fun}\,(\mathcal{A}_1, \mathcal{V}) \arrow[ll, "\nu_{\mathcal{A}_1}"'] \end{tikzcd} .$$

where $\nu_{\mathcal{A}_1}$ is extension by zero. $\square$

Stratification of $\mathsf{Exit}(\underline{\mathbb{CP}^\infty})$-modules

Let G be a group object in the ∞-category of $\mathcal{S}\mathsf{paces}$. A G-action on an object V in a presentable stable ∞-category $\mathcal{V}$ is defined as a functor from the classifying space $\mathsf{B}G$ of the group G to $\mathcal{V}$

$$\mathsf{B}G \to \mathcal{V} \ .$$

The ∞***-category of*** G***-modules*** is defined to be the ∞-category

$$\mathsf{Mod}_G(\mathcal{V}) := \mathsf{Fun}\left(\mathsf{B}G, \mathcal{V}\right) \ .$$

This section considers the case of $G = \mathbb{T}$, where $\mathbb{T}$ is the circle group. Recall that the classifying space of $\mathbb{T}$ is the complex projective space $\mathbb{CP}^\infty$

$$\mathsf{B}\mathbb{T} \simeq \mathbb{CP}^\infty \ .$$

We seek to provide an equivalent, and conceptually simpler description of the ∞-category

$$\mathsf{Mod}_{\mathbb{T}}(\mathcal{V}) := \mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{V}) \ .$$

The stratification of $\mathbb{CP}^\infty$ by the submanifolds $\mathbb{CP}^i$ simplifies the situation. For each $n \in \mathbb{Z}_{\geq 0}$, the restriction of the stratification $\mathbb{CP}^\infty \to \mathbb{Z}_{\geq 0}$ to $\mathbb{CP}^n$ is conically smoooth by Lemma139. Therefore the exit path ∞-category $\mathsf{E}_n := \mathsf{Exit}(\underline{\mathbb{CP}^n})$ exists for each n. This allows for the definition of $\mathsf{E}_\infty := \mathsf{Exit}(\mathbb{CP}^\infty)$, which is identified in Theorem 164

$$\mathsf{E}_\infty \underset{\text{Theorem 164}}{\simeq} \left(\underset{\mathbb{Z}_{\geq 0}}{\bigstar} \mathbb{T}\right)_{/\mathbb{T}} \ .$$

Because refinements between conically smooth stratified spaces are carried by $\mathsf{Exit}(-)$ to localizations, for each $n \geq 0$, the canonical functor

$$\mathsf{Exit}(\underline{\mathbb{CP}^n}) \longrightarrow \mathbb{CP}^n$$

witnesses an ∞-groupoid completion:

$$|\mathsf{Exit}(\underline{\mathbb{CP}^n})| \ \simeq \ \mathbb{CP}^n \ . \tag{5.1}$$

The functor $\mathsf{Cat}_\infty \to \mathcal{S}\mathsf{paces}$ given by ∞-groupoid completion is defined as left adjoint to the canonical inclusion $\mathcal{S}\mathsf{paces} \hookrightarrow \mathsf{Cat}_\infty$. Because left adjoints preserve colimits, we have an equivalence between spaces:

$$|\mathsf{Exit}\left(\underline{\mathbb{CP}^\infty}\right)| \simeq |\mathsf{colim}_{n\geq 0}\mathsf{Exit}\left(\underline{\mathbb{CP}^n}\right)| \xleftarrow{\simeq} \mathsf{colim}_{n\geq 0}\left|\mathsf{Exit}\left(\underline{\mathbb{CP}^n}\right)\right| \underset{(5.1)}{\simeq} \mathsf{colim}_{n\geq 0}\mathbb{CP}^n \ \simeq \ \mathbb{CP}^\infty \ .$$

In particular, the canonical functor

$$\mathsf{Exit}(\underline{\mathbb{CP}^\infty}) \longrightarrow \mathbb{CP}^\infty \ \simeq \ \mathsf{B}\mathbb{T} \tag{5.2}$$

is a localization. Appliying $\mathsf{Fun}(-,\mathcal{V})$ to this localization results in a fully faithful functor

$$\mathsf{Mod}_{\mathbb{T}}(\mathcal{V}) \ := \ \mathsf{Fun}(\mathsf{B}\mathbb{T},\mathcal{V}) \ \simeq \ \mathsf{Fun}(\mathbb{CP}^\infty,\mathcal{V}) \xrightarrow{(5.2)^*} \mathsf{Fun}(\mathsf{Exit}(\underline{\mathbb{CP}^\infty}),\mathcal{V}) \ =: \ \mathcal{V}^{\mathsf{E}_\infty} \ ,$$

whose image consists of those functors $\mathsf{E}_\infty \to \mathcal{V}$ that carry each exiting-path to an equivalence in $\mathcal{V}$.

Applying the functor $\mathsf{Exit}(-)$ to the filtration

$$\underline{\mathbb{CP}^0} \xrightarrow{\iota_1^0} \underline{\mathbb{CP}^1} \xrightarrow{\iota_2^1} \dots \mathsf{colim}(\underline{\mathbb{CP}^0} \xrightarrow{\iota_1^0} \underline{\mathbb{CP}^1} \xrightarrow{\iota_2^1} \dots) \simeq \underline{\mathbb{CP}^\infty}$$

of $\underline{\mathbb{CP}^\infty}$ gives rise to a filtration in $\mathsf{Cat}_{(\infty,1)}$.

$$\begin{tikzcd}
\mathsf{E}_0 \arrow[rr] \arrow[dr] & & \mathsf{E}_1 \arrow[rr] \arrow[d] & & \dots \arrow[dll] \\
& & (\mathsf{E}_0 \to \mathsf{E}_1 \to \dots) \simeq \mathsf{E}_\infty & &
\end{tikzcd}$$

The following lemma gives this induces a canonical stratification on $\mathcal{V}^{\mathsf{E}_\infty}$.

Lemma 277. *The inclusion*

$$\mathbb{CP}^i \xrightarrow{\iota^i} \mathbb{CP}^\infty$$

induces an inclusion given by left Kan extension

$$\mathcal{V}^{\mathsf{E}_i} \xrightarrow{\iota^i{}_!} \mathcal{V}^{\mathsf{E}_\infty} \ .$$

This extends to a diagram

$$\begin{tikzcd}
\mathcal{V}^{\mathsf{E}_i} \arrow[r, bend left, "\iota^i{}_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, dashed, bend right, "\iota^i{}_*"'] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{V}^{\mathsf{E}_\infty} \arrow[l, dashed, "\iota^{i*}"']
\end{tikzcd}$$

that realizes $\mathcal{V}^{\mathsf{E}_i}$ as a closed ∞-subcategory in the sense of Definition 222.

Proof. The left and right Kan extension functors, if they exist, are the left and right adjoints to the restriction functor ι^{i*} by Lemma 321.

The ∞-category $\mathcal{V}$ is a presentable stable ∞-category. In particular, the ∞-category $\mathcal{V}$ admits all limits and colimits, and for every functor $\mathsf{E}_i \xrightarrow{\mathsf{F}} \mathcal{V}$ and $k \in \mathsf{E}_\infty$, the colimit

$$\mathsf{colim}\left(\mathsf{E}_{i/j} \to \mathsf{E}_i \xrightarrow{\mathsf{F}} \mathcal{V}\right)$$

and the limit

$$\mathsf{lim}\left(\mathsf{E}_i{}^{j/} \to \mathsf{E}_i \xrightarrow{\mathsf{F}} \mathcal{V}\right)$$

exist. The functors from the slice categories to E_i are the canonical forgetful functor. By Proposition 321, the existence of these colimits and limits ensures that the left and right Kan extensions exist respectively. Furthermore, since the inclusion functor $\iota^i : \mathsf{E}_i \to \mathsf{E}_\infty$ is fully faithful, the left and right Kan extension functors are fully faithful by Lemma 336. $\square$

The left and right Kan extensions ensure the ∞-category $\mathcal{V}^{\mathsf{E}_i}$ is a closed ∞-subcategory of $\mathcal{V}^{\mathsf{E}_\infty}$ for each $i \in \mathbb{Z}_{\geq 0}$. Therefore the filtration of $\mathbb{CP}^\infty$ by the submanifolds $\mathbb{CP}^i$ gives a canonical stratification of the ∞-category $\mathcal{V}^{\mathsf{E}_\infty}$ by the poset $\mathbb{Z}_{\geq 0}$.

Lemma 278. *The map of posets*

$$\begin{aligned} \mathcal{Z}_\bullet : \mathbb{Z}_{\geq 0} &\longrightarrow \mathbf{Cls}(\mathcal{V}^{\mathsf{E}_\infty}) \\ i &\longmapsto \mathcal{V}^{\mathsf{E}_i} \end{aligned}$$

defines a stratification of the ∞-category $\mathcal{V}^{\mathsf{E}_\infty}$.

Proof. For each $i \in \mathbb{Z}_{\geq 0}$, the ∞-subcategory $\mathcal{V}^{\mathsf{E}_i}$ is a closed ∞-subcategory of $\mathcal{V}^{\mathsf{E}_\infty}$ by Lemma 277. Furthermore, for each $i < j$, the composition

$$\mathcal{V}^{\mathsf{E}_i} \xrightarrow{\iota^i{}_!} \mathcal{V}^{\mathsf{E}_\infty} \xrightarrow{\iota^{j*}} \mathcal{V}^{\mathsf{E}_j}$$

gives an inclusion of $\mathcal{V}^{\mathsf{E}_i}$ into $\mathcal{V}^{\mathsf{E}_j}$. This inclusion witnesses a left Kan extension as well. Therefore the map of posets

$$\begin{aligned} \mathcal{Z}_\bullet : \mathbb{Z}_{\geq 0} &\longrightarrow \mathbf{Cls}(\mathcal{V}^{\mathsf{E}_\infty}) \\ i &\longmapsto \mathcal{V}^{\mathsf{E}_i} \end{aligned}$$

is well defined.

Next, we seek to verify that

$$\mathsf{colim}\,(\mathcal{Z}_\bullet) \simeq \mathcal{V}^{\mathsf{E}_\infty}\ .$$

Consider the colimit diagram of categories

$$\mathsf{colim}(\mathsf{E}_0 \xrightarrow{\iota_1^0} \mathsf{E}_1 \to \dots) \simeq \mathsf{E}_\infty \ .$$

Applying the colimit preserving functor of Observation 323 implies that

$$\mathsf{colim}(\mathcal{V}^{\mathcal{E}_0} \xrightarrow{\iota_{1!}^0} \mathcal{V}^{\mathcal{E}_1} \to \dots) \simeq \mathcal{V}^{\mathsf{E}_\infty} \ .$$

Finally, the stratification condition is satisfied as $\mathbb{Z}_{\geq 0}$ is linearly ordered. Therefore the functor $\mathcal{Z}_\bullet$ is a stratification of the ∞-category $\mathcal{V}^{\mathsf{E}_\infty}$ by the poset $\mathbb{Z}_{\geq 0}$. $\square$

Lemma 278 verifies that the filtration of E_∞

$$\mathsf{colim}\,(\mathsf{E}_0 \hookrightarrow \mathsf{E}_1 \hookrightarrow \dots) \simeq \mathsf{E}_\infty$$

induces a canonical stratification of the ∞-category $\mathcal{V}^{\mathsf{E}_\infty}$. The stratification of the ∞-category $\mathcal{V}^{\mathsf{E}_\infty}$ allows one to reconstruct the ∞-category of E_∞-modules in two ways. The one used in this paper to reconstruct $\mathcal{V}^{\mathsf{E}_\infty}$ is the reflected reconstruction theorem (Theroem 238) – it is a description of $\mathcal{V}^{\mathsf{E}_\infty}$ in terms of the strata of this stratificaiton and its "(reflected) gluing functors".

Our next goal is to identify the strata of the stratificaiton $\mathcal{Z}_\bullet$. Recall that a stratification

$$\mathcal{Z}_\bullet : \mathbb{Z}_{\geq 0} \to \mathbf{Cls}(\mathcal{V}^{\mathsf{E}_\infty})$$

gives rise, for each $i \in \mathbb{Z}_{\geq 0}$, to a recollement

$$\begin{tikzcd} \mathcal{V}^{\mathsf{E}_{i-1}} \arrow[r, bend left, "{\iota_i^{i-1}}_!"] \arrow[r, bend right, "{\iota_i^{i-1}}_*"'] & \mathcal{V}^{\mathsf{E}_i} \arrow[l, "\iota_i^{i-1*}"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{ker}((\iota_{i-1}^i)^*) \arrow[l, "\nu"'] \end{tikzcd} \quad .$$

The i-stratum of the stratification defined in Lemma 278 is defined to be this ∞-subcategory $\mathsf{ker}((\iota_i^{i-1})^*) \subset \mathcal{V}^{\mathsf{E}_\infty}$. Our next goal is to identify ν, p_R and the kernel of the functor

$$(\iota_i^{i-1})^* : \mathcal{V}^{\mathsf{E}_i} \to \mathcal{V}^{\mathsf{E}_{i-1}} .$$

This situation is simplified by the fact that the ∞-subcategory $\mathsf{E}_{i-1} \subset \mathcal{V}^{\mathsf{E}_\infty}$ is Reedy closed in the sense of Definition 260, which provides a description of the i^{th} stratum as the ∞-category of $\{i\}$-modules.

Observation 279. For $i \in \mathbb{Z}_{\geq 0}$, there is a canonical inclusion

$$\mathbf{Cls}(\mathcal{V}^{\mathsf{E}_i}) \to \mathbf{Cls}(\mathcal{V}^{\mathsf{E}_\infty})$$

given by composition of the closed diagrams. That is, for a closed ∞-subcategory $\mathcal{Z} \subset \mathcal{V}^{\mathsf{E}_i}$, we can compose the maps

$$\begin{tikzcd} \mathcal{Z} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{V}^{\mathsf{E}_i} \arrow[l, "y"'] \arrow[r, bend left, "\iota^i{}_!"] \arrow[r, bend right, "\iota^i{}_*"'] & \mathcal{V}^{\mathsf{E}_\infty} \arrow[l, "\iota^{i*}"'] \end{tikzcd}$$

in a canonical way, which realizes $\mathcal{Z} \subset \mathcal{V}^{\mathsf{E}_\infty}$ as a closed ∞-subcategory.

Lemma 280. *The canonical functor of $[i] \to \mathbb{Z}_{\geq 0}$ gives a composite $[i] \to \mathbb{Z}_{\geq} \to \mathbf{Cls}\left(\mathcal{V}^{\mathsf{E}_\infty}\right)$. This map factors through $\mathbf{Cls}(\mathcal{V}^{\mathsf{E}_i})$, giving a stratification of $\mathcal{V}^{\mathsf{E}_i}$ by the poset $[i]$*

$$\begin{tikzcd} {[i]} \arrow[r] \arrow[d] & \mathbf{Cls}(\mathcal{V}^{\mathsf{E}_i}) \arrow[d] \\ \mathbb{Z}_{\geq 0} \arrow[r] & \mathbf{Cls}(\mathcal{V}^{\mathsf{E}_\infty}) \end{tikzcd} .$$

Lemma 281. *The closed ∞-subcategory $\mathsf{E}_{i-1} \subset \mathsf{E}_i$ is Reedy-closed.*

Proof. Recall from Definition 260 that the ∞-category $\mathsf{E}_i \subset \mathsf{E}_\infty$ is Reedy-closed if the following two conditions are met:

(1) The ∞-category $\mathsf{E}_i \backslash \mathsf{E}_{i-1} \simeq \{i\}$ satisfies Condition A.

(2) There is an equivalence

$$|{\mathsf{E}_{i-1}}^{i/}_{/i}| \simeq \mathsf{Fact}_{\mathsf{E}_{i-1}}(i,i) \simeq \emptyset \ ,$$

which implies that $\mathsf{E}_{i-1} \subset \mathsf{E}_i$ satisfies Condition B .

By Lemma 165, it follows the space of objects of the complement of E_{i-1} in E_i is contractible

$$\mathsf{Obj}(\mathsf{E}_i) \backslash \mathsf{Obj}(\mathsf{E}_{i-1}) \simeq \{i\} \ .$$

The space of endomorphisms of $\{i\}$ is also contractible by the calculation of E_i. Therefore a morphism in $\mathsf{Hom}_{\mathsf{E}_\infty}(i,i)$ does not factor through E_{i-1}. Thus, the category $\{i\}$ satisfies Condition A and is the subcategory of the compliment of E_{i-1} in E_i that contains all morphisms that do not factor through E_{i-1}.

Condition B is immediate since

$$\mathsf{E}_{i-1}^{i/} \simeq \emptyset$$

since for all $k < i \in \mathbb{Z}_{\geq 0}$

$$\mathsf{Hom}_{\mathsf{E}_i}(i,k) \simeq \emptyset \ .$$

by Theorem 165. □

Lemma 282. *For each $i \in \mathbb{Z}_{\geq 0}$, the stratification of $\mathcal{V}^{\mathsf{E}_\infty}$ (Lemma 278) determines a recollement*

$$\begin{tikzcd} \mathcal{V}^{\mathsf{E}_{i-1}} \arrow[r, bend left, "(\iota^i_{i-1})_!"] \arrow[r, bend right, "(\iota^i_{i-1})_*"'] & \mathcal{V}^{\mathsf{E}_i} \arrow[l, "(\iota^i_{i-1})^*" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{Fun}(\{i\}, \mathcal{V}) \arrow[l, "\nu" description] \end{tikzcd} \quad .$$

where ν is given by extension by zero.

Proof. This follows from Lemma 281 and Theorem 276. □

Lemma 283. *The inclusion $\{i\} \hookrightarrow \mathsf{E}_i$ is a fully faithful left fibration.*

Proof. The inclusion $\{i\} \to \mathsf{E}_i$ is fully faithful, since

$$\mathsf{Hom}_{\mathsf{E}_i}(i,i) = \mathsf{Hom}_{\mathsf{Exit}(\underline{\mathbb{S}^{2i+1}})}(i,i)_{/\mathbb{T}} \simeq \mathbb{T}_{/\mathbb{T}} \simeq * \ .$$

Therefore it suffices to check the inclusion $\{i\} \to \mathsf{E}_\infty$ is a left fibration. By Proposition 2.33 of [2] it suffices to show that

$$[1] \xrightarrow{!} \{i\}$$

is a cocartesian morphism with respect to the inclusion $\{i\} \to \mathsf{E}_i$. The morphisms $[1] \to \{i\}$ is cocartesian if the diagram

$$\begin{tikzcd} \{i\}^{i/} \arrow[r] \arrow[d] & \{i\}^{i/} \arrow[d] \\ (\mathsf{E}_i)^{i/} \arrow[r] & (\mathsf{E}_i)^{i/} \end{tikzcd} \tag{5.3}$$

is a pullback. Each of the categories is contractible, since

$$\mathsf{Hom}_{\mathsf{E}_i}(j,i) = \begin{cases} \emptyset & j < i \\ * & j = i \ . \end{cases}$$

Therefore every morphism in (5.3) is an equivalence, and hence the diagram is a pullback. □

Lemma 284. *Consider the recollement of Lemma 282 .*

(1) The functor ν witnesses a left Kan extension along the inclusion $\{i\} \to \mathsf{E}_i$.

(2) The functor p_R witnesses restriction along the inclusion $\{i\} \to \mathsf{E}_i$.

Proof. Recall that the functor ν is given by extension by zero, and that by Lemma 283, the inclusion $\{i\} \to \mathsf{E}_i$ is a fully faithful left fibration. Therefore by Lemma 322, the functor ν witnesses a left Kan extensnion.

The statemenmt that p_R is given by restriction immediately follows, as the right adjoint to a left Kan extension functor is the restriction functor. $\square$

Definition 285. The ***reflected localization functor*** Ψ_i and its left adjoint λ_i for the stratification of $\mathcal{V}^{\mathsf{E}_\infty}$ are defined to be the composite functors

$$\begin{tikzcd} \lambda_i : \mathsf{Fun}(\{i\}, \mathcal{V}) \arrow[r, shift left, "\nu"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathsf{E}_i} \arrow[l, shift left, "p_R"] \arrow[r, shift left, "{\iota^i}_!"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathsf{E}_\infty} : \Psi_i \arrow[l, shift left, "\iota^{i*}"] \end{tikzcd} .$$

Observation 286. For each $i \in \mathbb{Z}_{\geq 0}$, there is an adjunction

$$\begin{tikzcd} \lambda_i : \mathsf{Fun}(\{i\}, \mathcal{V}) \arrow[r, shift left, "\nu"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathsf{E}_i} \arrow[l, shift left, "p_R"] \arrow[r, shift left, "{\iota^i}_!"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathsf{E}_\infty} : \Psi_i \arrow[l, shift left, "\iota^{i*}"] \end{tikzcd} .$$

The reflected localization functor Ψ_i is given by restriction along $\{i\} \to \mathsf{E}_\infty$ since ι^{i*} is restriction along $\mathsf{E}_i \hookrightarrow \mathsf{E}_\infty$, and p_R is restriction along $\{i\} \to \mathsf{E}_i$. The inclusion functor λ_i is a composite of left Kan extension functors ν (Lemma 284) and ${\iota^i}_!$. Therefore the functor λ_i witnesses a left Kan extension functor along the inclusion $\{i\} \to \mathsf{E}_\infty$.

We seek to compute the left Kan extension functor

$$\lambda_i : \mathsf{Fun}(\{i\}, \mathcal{V}) \longrightarrow \mathcal{V}^{\mathsf{E}_\infty} .$$

The functor λ_i extends a functor $\{i\} \xrightarrow{\langle V_i \rangle} \mathcal{V}$ to a functor $\mathsf{E}_\infty \xrightarrow{\lambda_i(\langle V_i \rangle)} \mathcal{V}$ that evaluates on an object $j \in \mathsf{Obj}(\mathsf{E}_\infty)$ as the colimit

$$\lambda_i(\langle V_i \rangle)(j) = \mathsf{colim}\left(\{i\}_{/j} \to \{i\} \xrightarrow{\langle V_i \rangle} \mathcal{V}\right) .$$

Consider then the ∞- category $\{i\}_{/j}$. By definition of the ∞-overcategory, it fits in the limit diagram among ∞-categories:

$$\begin{tikzcd}
\{i\}_{/j} \arrow[rr] \arrow[dd] & & \{j\} \arrow[d, hook] \\
& \mathsf{Ar}(\mathsf{E}_\infty) \arrow[r, "t"] \arrow[d, "s"'] & \mathsf{E}_\infty \\
\{i\} \arrow[r, hook] & \mathsf{E}_\infty &
\end{tikzcd} .$$

Therefore there is an equivalence

$$\{i\}_{/j} \simeq \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \underset{\text{Theorem 164}}{=} \begin{cases} \emptyset & j < i \\ * & j = i \\ \mathbb{T} & j > i \end{cases} .$$

Since the category $\{i\}$ is contractible, then the colimit evaluates as

$$\mathsf{colim}\left(\{i\}_{/j} \to \{i\} \to \mathcal{V}\right) = \{i\}_{/j} \odot V_i \simeq \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \odot V_i \ .$$

Next, consider the space of morphisms between two objects j,k in E_∞. The functor $\lambda_i(\langle V_i \rangle)$ is a map of spaces

$$\mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \odot V_i \to \mathsf{Hom}_{\mathcal{V}}(\mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \odot V_i, \mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \odot V_i) \ .$$

This a map between colimits

$$\mathsf{colim}\left(\mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \to \{i\} \to \mathcal{V}\right) \xrightarrow{\mathsf{Hom}_{\mathsf{E}_\infty}(j,k)} \mathsf{colim}\left(\mathsf{Hom}_{\mathsf{E}_\infty}(i,k) \to \{i\} \to \mathcal{V}\right)$$

is the morphism in $\mathcal{V}$ induced by the map on diagrams

$$\mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \to \mathsf{Hom}_{\mathsf{Spaces}}(\mathsf{Hom}_{\mathsf{E}_\infty}(i,j), \mathsf{Hom}_{\mathsf{E}_\infty}(i,k))$$

which is adjoint to the composition map

$$\mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \times \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \xrightarrow{\circ} \mathsf{Hom}_{\mathsf{E}_\infty}(i,k) \ .$$

By Theorem 164, the composition is the map of spaces

$$\mathbb{T} \to \mathsf{Hom}_{\mathsf{Spaces}}(\mathbb{T}, \mathbb{T})$$

that is adjoint to the multiplication map on the circle group

$$\mathbb{T} \times \mathbb{T} \to \mathbb{T} \ .$$

Lemma 287. *Let i and j be objects of E_∞. The reflected glueing functor $\check{\Gamma}^i_j$ is the functor*

$$\begin{aligned} \check{\Gamma}^i_j : \mathsf{Fun}(\{i\}, \mathcal{V}) &\longrightarrow \mathsf{Fun}(\{j\}, \mathcal{V}) \\ \langle V_i \rangle &\longmapsto \langle \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \odot V_i \rangle \end{aligned}$$

This functor is canonically equivalent to the functor

$$\begin{aligned} \check{\Gamma}^i_j : \mathcal{V} &\longrightarrow \mathcal{V} \\ V_i &\longmapsto \mathbb{T} \odot V_i \end{aligned} \ .$$

Proof. Recall that the reflected gluing functor is defined to be the composite

$$\check{\Gamma}^i_j : \mathsf{Fun}(\{i\}, \mathcal{V}) \xrightarrow{\lambda_i} \mathcal{V}^{\mathsf{E}_\infty} \xrightarrow{\Psi_j} \mathsf{Fun}(\{j\}, \mathcal{V}) \ .$$

The functor λ_i is given by left Kan extension, and the functor Ψ_j is restriction onto the object j of E_∞. Therefore $\check{\Gamma}^i_j$ is the functor

$$\begin{aligned} \check{\Gamma}^i_j : \mathcal{V}^{\{i\}} &\longrightarrow \mathcal{V}^{\{j\}} \\ \langle V_i \rangle &\longmapsto \langle \mathsf{colim}\left(\mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \to \{i\} \xrightarrow{\langle V_i \rangle} \mathcal{V}\right)\rangle \end{aligned} .$$

By Theorem 164, there is an identification $\mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \simeq \mathbb{T}$, and by the canonical identification $\mathsf{Fun}(\{i\},\mathcal{V}) \simeq \mathsf{Fun}(\{j\},\mathcal{V}) \simeq \mathcal{V}$, the functor $\check{\Gamma}^i_j$ is the endofunctor of $\mathcal{V}$

$$\begin{aligned} \check{\Gamma}^i_j : \mathcal{V} &\longrightarrow \mathcal{V} \\ V_i &\longmapsto \mathbb{T} \odot V_i \end{aligned} .$$

□

Corollary 288. Let i and j be objects of E_∞. The reflected glueing functor $\check{\Lambda}^i_j$ is identified as the functor

$$\begin{aligned} \check{\Gamma}^i_j : \mathcal{V} &\longrightarrow \mathcal{V} \\ V_i &\longmapsto V_i \oplus \Sigma V_i \end{aligned} .$$

Proof. This follows from Lemma 289

$$\mathbb{T} \odot V \simeq V \oplus \Sigma V \ .$$

□

Lemma 289. *Let $\mathcal{V}$ be a stable ∞-category and let $V \in \mathcal{V}$. Then we have the following equivalence*

$$\mathbb{T} \odot V \simeq V \oplus \Sigma V \ .$$

Proof. First note the following retraction in $\mathcal{S}\mathsf{paces}$.

$$\begin{tikzcd} \{1\} \arrow[r, hook, shift left] & \mathbb{T} \arrow[l, shift left, "!"] \end{tikzcd} .$$

Since retractions are preserved by functors, then we have the following is a retraction in $\mathcal{V}$:

$$\begin{tikzcd} 1 \odot V \arrow[r, shift left] & \mathbb{T} \odot V \arrow[l, shift left] \end{tikzcd}$$

By Theorem 249, we have that

$$\mathbb{T} \odot V \simeq * \odot V \oplus \mathsf{coker}(* \odot V \to \mathbb{T} \odot V) .$$

Therefore what remains is to identify the cokernel:

$$\mathsf{coker}(* \odot V \to \mathbb{T} \odot V) \simeq \Sigma V .$$

Consider the following diagram, which witnesses $\mathbb{T}$ as the suspension of $\mathbb{S}^0$ in the ∞-category of $\mathcal{S}\mathsf{paces}$

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{T} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} .$$

By Lemma 215, the diagram

$$\begin{tikzcd} \mathbb{S}^0 \odot V \arrow[r] \arrow[d] & * \odot V \arrow[d] \\ * \odot V \arrow[r] & \mathbb{T} \odot V \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram. By Lemma 212, there is an equivalence

$$\mathsf{coker}(\mathbb{S}^0 \odot V \to * \odot V) \simeq \mathsf{coker}(* \odot V \to \mathbb{T} \odot V) .$$

Since $\mathbb{S}^0 \simeq * \coprod *$ and the functor $- \odot V$ preserves coproducts, then

$$\mathbb{S}^0 \odot V \simeq V \oplus V \ .$$

Note the following commutative diagram

$$\begin{tikzcd} V \oplus V \arrow[r, "{[1,1]}"] \arrow[d, "{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}"'] & V \arrow[d, "{[1]}"] \\ V \oplus V \arrow[r, "{[1,0]}"'] & V \end{tikzcd} \ .$$

Since both vertical arrows are equivalences, then we have the identification between the cokernels

$$\mathsf{coker}(V \oplus V \xrightarrow{[1,1]} V) \simeq \mathsf{coker}(V \oplus V \xrightarrow{[1,0]} V)$$

Finally, we can compute the cokernel of the projection map:

$$\mathsf{coker}(V \oplus V \xrightarrow{[1,0]} V) \simeq \Big(\mathsf{coker}(V \xrightarrow{\mathsf{id}} V) \oplus \mathsf{coker}(V \xrightarrow{0} V)\Big) \simeq 0 \oplus \Sigma V \simeq \Sigma V \ .$$

Therefore, we have that

$$\mathbb{T} \odot V \simeq V \oplus \Sigma V \ .$$

$\square$

Lemma 290. *Let $0 \leq i < j < k$. The counit of the adjunction $\lambda_j \dashv \Psi_j$ induces a natural transormation $\check{\Gamma}^j_k \check{\Gamma}^i_j \to \check{\Gamma}^i_k$ that is equivalent to the natural transformation $\mu_\odot$*

$$\begin{tikzcd} \check{\Gamma}^j_k \check{\Gamma}^i_j \arrow[r] \arrow[d, "\simeq"'] & \check{\Gamma}^i_k \arrow[d, "\simeq"] \\ \mathbb{T} \times \mathbb{T} \odot - \arrow[r, "\mu_\odot"'] & \mathbb{T} \odot - \end{tikzcd} \ .$$

Proof. Recall that the gluing functors $\check{\Gamma}^j_k\check{\Gamma}^i_j$ are defined to be $\Psi_k\lambda_j\Psi_j\lambda_i$, and is computed as

$$\check{\Gamma}^j_k\check{\Gamma}^i_j \simeq \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \times \mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \odot - \simeq \mathbb{T} \times \mathbb{T} \odot - \ .$$

Similarly, the endofunctor $\check{\Gamma}^i_k$ is defined to be $\Psi_k\lambda_i$ and by Lemma 287 the glueing functor is computed as

$$\check{\Gamma}^i_k \simeq \mathsf{Hom}_{\mathsf{E}_\infty}(i,k) \odot - \simeq \mathbb{T} \odot - \ .$$

The natural transformation from the endofunctor $\check{\Gamma}^j_k\check{\Gamma}^i_j$ to the endofunctor $\check{\Gamma}^i_k$ is induced by the adjunction $\lambda_j \dashv \Psi_j$, where λ_j is the left Kan extension and Ψ_j is restriction along the inclusion $\{j\} \hookrightarrow \mathsf{E}_j$. Therefore by unpacking an adjunction between the left Kan extension and the restriction functor, this natural transformation evaluated on an object $V_i \in \mathcal{V}$ is determined by a map

$$\mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \to \mathsf{Hom}_{\mathcal{V}}\left(\mathsf{Hom}_{\mathsf{E}_\infty}(i,k) \odot V_i, \mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \odot V_i\right)$$

induced by the map of spaces

$$\mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \to \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}\left(\mathsf{Hom}_{\mathsf{E}_\infty}(i,k), \mathsf{Hom}_{\mathsf{E}_\infty}(j,k)\right)$$

which through the tensor-hom adjunction is identified as the map of spaces

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{E}_\infty}(i,j) \times \mathsf{Hom}_{\mathsf{E}_\infty}(j,k) \arrow[r, "\circ"] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathsf{E}_\infty}(i,k) \arrow[d, "\simeq"] \\ \mathbb{T} \times \mathbb{T} \arrow[r, "\mu"'] & \mathbb{T} \end{tikzcd} .$$

□

The next goal is to identify the filler in the diagram

$$\begin{tikzcd}
\mathbb{T} \times \mathbb{T} \odot V \arrow[r, "\mu_\odot"'] \arrow[d, "\simeq"'] & \mathbb{T} \odot V \arrow[d, "\simeq"] \\
V \oplus \Sigma V \oplus \Sigma V \oplus \Sigma^2 V \arrow[r, dashed] & V \oplus \Sigma V
\end{tikzcd}$$

The morphism

$$V \oplus \Sigma V \oplus \Sigma V \oplus \Sigma^2 V \to V \oplus \Sigma V$$

is represented by a 4 by 2 matrix. The most technical computation involved is the computation of the map

$$\Sigma^2 V \to \Sigma V\ .$$

Working toward the goal of this computation we introduce the J-homomorphism.

<u>The J-Homomorphism</u>

Definition 291. The ***orthogonal group*** $O(n)$ is the group of isometries of $\mathbb{R}^n$ equipped with the standard inner product on $\mathbb{R}^n$. The ***stable orthogonal group*** O is defined to be the colimit

$$O := \mathsf{colim}(O(1) \to O(2) \to \dots)$$

where the inclusion $O(i) \to O(k)$ is given by extending by the identity on the orthogonal complement of the inclusion $\mathbb{R}^i \hookrightarrow \mathbb{R}^k$ of the first i coordinates.

Definition 292. The ***J-Homomorphism*** is the map of spaces

$$J : O := \mathsf{colim}(O(1) \hookrightarrow O(2) \hookrightarrow O(3)) \to \mathsf{colim}(\Omega^1\mathbb{S}^1 \hookrightarrow \Omega^2\mathbb{S}^2 \hookrightarrow \dots) =: \Omega^\infty\mathbb{S}$$

that is the colimit of the homomorphisms

$$\begin{aligned} O(n) &\longrightarrow \Omega^n\mathbb{S}^n \\ T &\longmapsto (T^+ : (\mathbb{R}^n)^+ \to (\mathbb{R}^n)^+) \end{aligned}$$

that extends the isometry T of $\mathbb{R}^n$ to a map T^+ on the one point compactification of $\mathbb{R}^n$, which is $\mathbb{S}^n$.

By definition of the J-homomorphism, there is a commutative diagram among spaces

$$\begin{tikzcd} O(2) \arrow[r] \arrow[d] & \Omega^2\mathbb{S}^2 \arrow[d] \\ O \arrow[r] & \Omega^\infty\mathbb{S}. \end{tikzcd}$$

In this diagram, the rightward maps carry the identity element $\mathbb{1}$ to the identity maps $\mathbb{S}^n \xrightarrow{\mathsf{id}} \mathbb{S}^n$. Regarding such elements as base points, the diagram (5.4) is one among based spaces. Applying π_1 results in a commutative diagram among groups

$$\begin{tikzcd} \pi_1(O(2);\mathbb{1}) \arrow[r] \arrow[d] & \pi_1(\Omega^2\mathbb{S}^2;\mathsf{id}) \arrow[d] \\ \pi_1(O;\mathbb{1}) \arrow[r] & \pi_1(\Omega^\infty\mathbb{S};\mathsf{id}). \end{tikzcd} \tag{5.4}$$

The following well-know result is a consequence of Whitehead's work [14] in which the J-homomorphism was introduced, as it generalizes the "Hopf Construction", which in turn generalizes the construction of the Hopf map $\mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2$.

Lemma 293. *The horizontal homomorphisms in (5.4) are isomorphisms. In particular, the homomorphism*

$$\pi_1(O(2)) \longrightarrow \pi_1(\Omega^2\mathbb{S}^2;\mathsf{id}) \;\cong\; \pi_1(\Omega^2\mathbb{S}^1;*) \;=\; \pi_3(\mathbb{S}^2)$$

carries a generator to the Hopf map $\mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2$.

Lemma 294. *Consider the multiplication map* $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ *on the group* $\mathbb{T}$. *The*

(1) $\Sigma(\mu)$ *is a morphism in* Spaces

$$\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^3 \to \mathbb{S}^2$$

(2) The restriction of $\Sigma(\mu)$ *to* $\mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2$ *is given by the identity map on each component.*

(3) The restriction of $\Sigma(\mu)$ *to* $\mathbb{S}^3 \to \mathbb{S}^2$ *is given by the Hopf fibration.*

Proof. The space $\mathbb{T} \times \mathbb{T}$ is presented as the following pushout in Spaces_*

$$\begin{tikzcd}
\mathbb{T} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T} \arrow[r] & \mathbb{T} \times \mathbb{T} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where the vertical map $\mathbb{T} \to \mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T}$ is the map that represents the commutator $aba^{-1}b^{-1}$ in $\pi_1(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T}) \simeq \mathbb{F}\langle a, b\rangle$, which is a free group on two generators. The reduced suspension Σ is a left adjoint, so it preserves colimits, and in particular pushouts. Therefore the diagram

$$\begin{tikzcd}
\Sigma\mathbb{T} \simeq \mathbb{S}^2 \arrow[r] \arrow[d] & \Sigma(0) \simeq 0 \arrow[d] \\
\Sigma(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T}) \arrow[r] & \Sigma(\mathbb{T} \times \mathbb{T}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout. Since the vertical map $\mathbb{S}^2 \to \Sigma(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T})$ is given by the suspension of the commutator map, and $\pi_2(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T})$ is abelian, the map $\mathbb{S}^2 \to \Sigma(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T})$ is nullhomotopic. Therefore we can compute $\Sigma(\mathbb{T} \times \mathbb{T})$ as the pushout

$$\begin{tikzcd}
\Sigma(\mathbb{T}) \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
0 \arrow[rr] \arrow[d] & & \Sigma(\Sigma\mathbb{T}) \simeq \mathbb{S}^3 \arrow[d] \\
\Sigma(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T}) \simeq \mathbb{S}^2 \vee \mathbb{S}^2 \arrow[rr] & & \Sigma(\mathbb{T} \times \mathbb{T}) \simeq \mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^2
\end{tikzcd} .$$

Note the equivalence $\Sigma(\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T}) \simeq \mathbb{S}^2 \vee \mathbb{S}^2$, which follows from the fact that $\vee$ is the coproduct in $\mathcal{S}\mathsf{paces}_*$, and since Σ is colimit preserving, it commutes with $\vee$. The top square is a pushout by definition of Σ. The bottom square then is a pushout since in any ∞-category with a zero object, a pushout with zero in the upper left agrees with the coproduct. Therefore, there is an equivalence in $\mathcal{S}\mathsf{paces}_*$:

$$\Sigma(\mathbb{T} \times \mathbb{T}) \simeq \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^3 .$$

Next, we seek to show that the restriction of $\Sigma(\mu)$ to $\mathbb{S}^2 \vee \mathbb{S}^2$ gives

$$\mathbb{S}^2 \vee \mathbb{S}^2 \xrightarrow{\mathsf{id} \vee \mathsf{id}} \mathbb{S}^2 .$$

This follows by observing the commutative triangle in $\mathcal{S}\mathsf{paces}$

$$\begin{tikzcd}
& \mathbb{T} \times \mathbb{T} \arrow[dr, "\mu"] & \\
\mathbb{T} \times \{1\} \vee \{1\} \times \mathbb{T} \arrow[ur, hook] \arrow[rr, "\mathsf{id} \vee \mathsf{id}"'] & & \mathbb{T}
\end{tikzcd}$$

using that 1 is the unit for $\mathbb{T}$. Therefore applying Σ gives the result.

Next we seek to show that the restriction of $\Sigma(\mu)$ to $\mathbb{S}^3$ is the hopf fibration η. Consider the adjoint map of μ under the tensor-hom adjunction

$$\mathbb{T} \xrightarrow{\mu} \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(\mathbb{T}, \mathbb{T}) .$$

Note that the suspension functor S on Top determines a functor

$$\mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(\mathbb{T},\mathbb{T}) \xrightarrow{S} \mathsf{Hom}_{\mathcal{S}\mathsf{paces}^*}(S\mathbb{T},S\mathbb{T})$$

where the base point is chosen to be one of the two cone points in the suspension. Furthermore, there is an inclusion

$$\begin{array}{ccccc} \mathbb{T} & \xrightarrow{\quad\simeq\quad} & SO(2) & \hookrightarrow & O(2) \\ p = e^{i\theta} & \longmapsto & \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} & \longmapsto & \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \end{array}.$$

Together this compiles to a commutative square in $\mathcal{S}\mathsf{paces}$

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\mu"] \arrow[d, "\mathsf{Rot}_{-}"'] & \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(\mathbb{T},\mathbb{T}) \arrow[d, "S"] \\ O(2) \arrow[r, "J"'] & \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(S\mathbb{T},S\mathbb{T}) \end{tikzcd}.$$

Applying the functor π_1 yields commutative square

$$\begin{tikzcd} \pi_1(\mathbb{T}) \arrow[r, "\mu"] \arrow[d, "\mathsf{Rot}_{-}"'] & \pi_1(\mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(\mathbb{T},\mathbb{T})) \arrow[d, "S"] \\ \pi_1(O(2)) \arrow[r, "J"'] & \pi_1(\mathsf{Hom}_{\mathcal{S}\mathsf{paces}_*}(S\mathbb{T},S\mathbb{T})) \end{tikzcd}.$$

The bottom composite is an equivalence by Lemma 293, which implies that the top composite is and equivalence. Therefore the composite

$$S\mu : \pi_1(\mathbb{T}) \xrightarrow{\simeq} \pi_1(\mathsf{Hom}_{\mathcal{S}\mathsf{paces}_*}(S\mathbb{T},S\mathbb{T}))$$

is an equivalence. Note $\pi_1(\mathsf{Hom}_{\mathcal{S}\mathsf{paces}_*}(S\mathbb{T},S\mathbb{T})) \simeq \pi_1(\Omega^2\mathbb{S}^2) \simeq \pi_3(\mathbb{S}^2)$, and the Hopf map η

generates $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$. Therefore

$$[\mathbb{T} \to \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(\mathbb{T}, \mathbb{T}) \to \Omega^2\mathbb{S}^2] \simeq [\eta] \in \pi_3(\mathbb{S}^2) \ .$$

$\square$

Lemma 295.

$$\begin{tikzcd}
\mathbb{T} \times \mathbb{T} \odot V \arrow[r, "\mu_\odot"] \arrow[d, "\simeq"'] & \mathbb{T} \odot V \arrow[d, "\simeq"] \\
V \oplus \Sigma V \oplus \Sigma V \oplus \Sigma^2 V \arrow[r, "\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix}"'] & V \oplus \Sigma V
\end{tikzcd}$$

Proof. This follows from Lemma 294, and from the fact that $\mathbb{S}$ generates $\mathcal{S}\mathsf{p}$ by colimits. $\square$

<u>Groupoid Completion Condition of $\mathbb{CP}^\infty$-modules</u>

The ∞-category $\mathbb{CP}^\infty$ witnesses the groupoid completion of the ∞-category of E_∞. A map

$$\mathsf{E}_\infty \to \mathcal{V}$$

factors through the groupoid completion, if every morphism in E_∞ is sent to an equivalence in $\mathcal{V}$. This states that for each $\{i \leq k\}$ in $\mathcal{V}$, that the map

$$(\mathbb{T} \odot V_i \to V_j) \ ,$$

which through the tensor-hom adjunction is equivalent to a map

$$\mathbb{T} \to \mathsf{Hom}_{\mathcal{V}(V_i, V_j)}$$

factors as

$$\begin{tikzcd} & \mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_j) \arrow[d] \\ \mathbb{T} \arrow[r, "F"] \arrow[ur, dashed, "\exists"] & \mathsf{Hom}_{\mathcal{V}}(V_i, V_j) \end{tikzcd}$$

where $\mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_j)$ is the subspace of the equivalences from V_i to V_j .

The composition rule in $\mathcal{E}_\infty$ gives that it is sufficient to check for $j = i + 1$. The next lemma simplifies the groupoid completion condition further.

Lemma 296. *Let V_i and V_{i+1} be objects of $\mathcal{V}$ and consider a map of spaces*

$$F : \mathbb{T} \to \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1})$$

There exists a lift

$$\begin{tikzcd} & \mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_{i+1}) \arrow[d] \\ \mathbb{T} \arrow[r, "F"] \arrow[ur, dashed, "\exists"] & \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1}) \end{tikzcd}$$

if and only if there exists a lift

$$\begin{tikzcd} & & \mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_{i+1}) \arrow[d] \\ * \arrow[r, "\langle 1 \rangle"'] \arrow[urr, dashed, "\exists"] & \mathbb{T} \arrow[r, "F"] & \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1}) \end{tikzcd} .$$

Proof. The forward direction is immediate by precomposing the lift by $* \xrightarrow{\langle 1 \rangle}$. Therefore assume that there exists a lift

$$\begin{tikzcd} & & \mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_{i+1}) \arrow[d] \\ * \arrow[r, "\langle 1 \rangle"'] \arrow[urr, dashed, "\exists"] & \mathbb{T} \arrow[r, "F"] & \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1}) \end{tikzcd} .$$

The result follows if for every $p \in \mathbb{T}$, there is a lift

$$\begin{tikzcd} & & \mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_{i+1}) \arrow[d] \\ * \arrow[r, "\langle p \rangle"'] \arrow[urr, dashed, "\exists"] & \mathbb{T} \arrow[r, "F"] & \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1}) \end{tikzcd} .$$

Choose a path γ in $\mathbb{T}$ from 1 to p. The path γ determines a diagram in $\mathcal{S}\mathsf{paces}$

$$\begin{tikzcd} \{1\} \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\gamma"] & \{p\} \arrow[dl] \\ & \mathbb{T} & \end{tikzcd} .$$

Applying the functor $- \odot V_i$ to the diagram gives a commutative diagram in $\mathcal{V}$, where the arrows mapping into V_{i+1} are determined by the data of the map $\mathbb{T} \to \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1})$. Furthermore the arrow $\{1\} \odot V_i \to V_{i+1}$ is an equivalence by the assumption of the lift of $\{p\} \to \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1})$. Therefore since the map $\{1\} \odot V_i \to \{p\} \odot V_i$ is an equivalence as well, then by the 2-of-3 property the arrow

$$\{p\} \odot V_i \to V_{i+1}$$

is an equivalence, which implies there is a lift

$$\begin{tikzcd} & & \mathsf{Hom}^{\simeq}_{\mathcal{V}}(V_i, V_{i+1}) \arrow[d] \\ * \arrow[r, "\langle p \rangle"'] \arrow[urr, dashed, "\exists"] & \mathbb{T} \arrow[r, "F"] & \mathsf{Hom}_{\mathcal{V}}(V_i, V_{i+1}) \end{tikzcd} .$$

□

Notation 297. Define the ∞-category $\mathsf{Fun}^{\simeq}(\mathbb{Z}_{\geq 0}, \mathcal{V})$ to be the full ∞-subcategory of

$\mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V})$ on those functors F such that $\mathsf{F}(i) \xrightarrow{\simeq} \mathsf{F}(i+1)$

$$\mathsf{Fun}^{\simeq}(\mathbb{Z}_{\geq 0}, \mathcal{V}) := \{\mathsf{F} \in \mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \mid \mathsf{F}(i) \to \mathsf{F}(i+1)\}$$

Lemma 298. *Consider the map of posets*

$$\langle 0 \rangle : * \to \mathbb{Z}_{\geq 0}$$

which induces an adjunction

$$\begin{tikzcd} \mathsf{Fun}\,(*, \mathcal{V}) \arrow[r, shift left, "\langle 0\rangle_!"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}\,(\mathbb{Z}_{\geq 0}, \mathcal{V}) \arrow[l, shift left, "\langle 0\rangle^*"] \end{tikzcd} \quad .$$

The functor $\langle 0 \rangle_!$ is fully faithful with image $\mathsf{Fun}^{\simeq}(\mathbb{Z}_{\geq 0}, \mathcal{V})$

Proof. The functor $\langle 0 \rangle$ is the inclusion of an initial object, since 0 is the minimal object of $\mathbb{Z}_{\geq 0}$. Therefore the functor $\langle 0 \rangle$ admits a left adjoint

$$\begin{tikzcd} * \arrow[r, shift left, "\langle 0\rangle"] \arrow[r, phantom, "\perp"] & \mathbb{Z}_{\geq 0} \arrow[l, shift left, "!"] \end{tikzcd} \quad .$$

Applying the functor $\mathsf{Fun}(-, \mathcal{V})$ gives

$$\langle 0 \rangle_! = (!)^* \; .$$

The image of $(!)^*$ consists of those functors $\mathbb{Z}_{\geq 0} \to \mathcal{V}$ that admit a factorization through $*$. Lastly, note that the consecutive morphisms $i < i+1$ in $\mathbb{Z}_{\geq 0}$ generate all morphisms in $\mathbb{Z}_{\geq 0}$. Therefore for any functor $\mathsf{F} : \mathbb{Z}_{\geq 0} \to \mathcal{V}$ that sends consecutive morphisms to equivalences, the functor F factors through $\mathbb{Z}_{\geq 0}[\mathbb{Z}_{\geq 0}{}^{-1}] \simeq *$

$$\begin{tikzcd} \mathbb{Z}_{\geq 0} \arrow[r, "\mathsf{F}"] \arrow[d] & \mathcal{V} \\ * \arrow[ur, dashed, bend right, "\exists!"'] & \end{tikzcd} .$$

Therefore the image of $(!)^* = \langle 0 \rangle$ is $\mathsf{Fun}^{\simeq}(\mathbb{Z}_{\geq 0}, \mathcal{V})$. ☐

Construction 299. We seek to construct a section of the conservative functor

$$\mathsf{E}_\infty \to \mathbb{Z}_{\geq 0} .$$

Consider for each $n \in \mathbb{Z}_{\geq 0}$ the map of spaces

$$\begin{array}{rcl} \mathsf{F}_n : \mathsf{Hom}_{\mathsf{Cat}}([n], \mathbb{Z}_{\geq 0}) & \longrightarrow & \mathsf{Hom}_{\mathsf{Cat}_{(\infty,1)}}([n], \mathsf{E}_\infty) \\ \left([n] \xrightarrow{\{i_0,\ldots,i_n\}} \mathbb{Z}_{\geq 0}\right) & \longmapsto & \left(\underline{\Delta^n} \longrightarrow \underline{\mathbb{CP}^\infty}\right) \\ & & \big((t_0, \ldots, t_n) \longmapsto \mathbb{C}\langle t_0 e_{i_0} + \ldots t_n e_{i_n} \rangle\big) \end{array} .$$

Note that the map $\underline{\Delta^n} \to \underline{\mathbb{CP}^\infty}$ map is indeed stratified, and that

$$\dim(\mathbb{C}\langle t_0 e_{i_0} + \ldots t_n e_{i_n} \rangle) = 1$$

since the e_{i_k} are linearly independent, and $(t_0 + \ldots t_n) = 1$. Moreover this maps F_n assemble into a morphism in $\mathsf{Psh}(\mathbf{\Delta})$. Therefore since

$$\mathsf{Hom}_{\mathsf{Cat}}([\bullet], -) : \mathsf{Cat} \xrightarrow{ff} \mathsf{Psh}(\Delta) ,$$

this corresponds to a functor

$$\sigma : \mathbb{Z}_{\geq 0} \to \mathsf{E}_\infty \ .$$

that is a section to the functor $\mathsf{E}_\infty \to \mathbb{Z}_{\geq 0}$.

Lemma 300. *The ∞-category $\mathsf{Mod}_{\mathbb{T}} \subset \mathcal{V}^{\mathsf{E}_\infty}$ fits into a pullback diagram*

$$\begin{tikzcd} \mathsf{Mod}_{\mathbb{T}} \arrow[r] \arrow[d, "\mathsf{ev}_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}^{\mathsf{E}_\infty} \arrow[d, "\sigma^*"] \\ \mathcal{V} \arrow[r, "\langle 0 \rangle_!"'] & \mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \end{tikzcd} \ .$$

Proof. Lemma 296 verifies that the ∞-category $\mathsf{Mod}_{\mathbb{T}}(\mathcal{V})$ is the pullback

$$\begin{tikzcd} \mathsf{Mod}_{\mathbb{T}} \arrow[r] \arrow[d, "\mathsf{ev}_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}^{\mathsf{E}_\infty} \arrow[d, "\sigma^*"] \\ \mathsf{Fun}^{\simeq}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \arrow[r, hook] & \mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \end{tikzcd} \ . \tag{5.5}$$

Lemma 298 gives an equivalence of ∞-categories

$$\mathcal{V} \simeq \mathsf{Fun}^{\simeq}(\mathbb{Z}_{\geq 0}, \mathcal{V})) \ .$$

Therefore, the pullback diagram (5.5) is equivalent to the diagram

$$\begin{tikzcd} \mathsf{Mod}_{\mathbb{T}} \arrow[r] \arrow[d, "\mathsf{ev}_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}^{\mathsf{E}_\infty} \arrow[d, "\sigma^*"] \\ \mathcal{V} \arrow[r, "\langle 0 \rangle_!"'] & \mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \end{tikzcd} \ .$$

$\square$

The Identification of $\mathsf{Exit}\left(\underline{\mathbb{CP}^\infty}\right)$-modules

The stratification of $\mathcal{V}^{\mathsf{E}_\infty}$ determines a glueing diagram

$$\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right) := \{(\mathsf{F}, i^\circ) \in \mathcal{V}^{\mathsf{E}_\infty} \times \mathbb{Z}_{\geq 0}^{\mathsf{op}} \mid \mathsf{F} \in \lambda_i(\mathcal{V}^{\{i\}})\} \ .$$

The canonical projection map to $\mathbb{Z}_{\geq 0}^{\mathsf{op}}$ is a locally cartesian fibration by Lemma 237

$$\left(\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right) \to \mathbb{Z}_{\geq 0}^{\mathsf{op}}\right) \in \mathsf{LMod}_{\mathsf{r.lax}.\mathbb{Z}_{\geq 0}} := \mathsf{loc.Cart}(\mathbb{Z}_{\geq 0}^{\mathsf{op}}) \ .$$

Theorem 2.2.1.2 of [10] states that the straightening and unstraightening construction gives an equivalence

$$\mathsf{Un} : \mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathsf{Cat}_{(\infty,1)}) \rightleftarrows \mathsf{LMod}_{\mathbb{Z}_{\geq 0}} := \mathsf{Cart}(\mathbb{Z}_{\geq 0}^{\mathsf{op}}) : \mathsf{St} \quad .$$

In the same way, one would want a straightening and unstraightening equivalence for the notion of a ***right-lax functor***

$$\mathsf{Un} : \mathsf{Fun}^{\mathsf{r.lax}}(\mathbb{Z}_{\geq 0}, \mathsf{Pr}_{\mathsf{st}}) \rightleftarrows \mathsf{LMod}_{\mathsf{r.lax}.\mathbb{Z}_{\geq 0}} := \mathsf{loc.Cart}(\mathbb{Z}_{\geq 0}^{\mathsf{op}}) : \mathsf{St} \quad .$$

However, the left hand term is not defined. Therefore, in [6], they define the left hand side to be the locally cartesian fibrations over $\mathbb{Z}_{\geq 0}^{\mathsf{op}}$

$$\mathsf{Fun}^{\mathsf{r.lax}}(\mathbb{Z}_{\geq 0}, \mathsf{Cat}) := \mathsf{LMod}_{\mathsf{r.lax}.\mathbb{Z}_{\geq 0}} \ .$$

This allows the glueing diagram $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right)$ to be thought of as a right-lax functor

$$\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right) : \mathbb{Z}_{\geq 0} \xrightarrow{\mathsf{r.lax}} \mathsf{Pr}_{\mathsf{st}} \ .$$

We seek to unpack the data of a right-lax functor

$$\mathbb{Z}_{\geq 0} \xrightarrow{\mathsf{r.lax}} \mathsf{Pr}_{\mathsf{st}} \ .$$

For ordinary categories, lax functors correspond to functors between 2 categories. Therefore, we begin by considering how to regard $\mathbb{Z}_{\geq 0}$ as a 2-category, in the "right" way. This is the ***right-laxification*** of the poset $\mathbb{Z}_{\geq 0}$ (See Observation A.7.2 of [6]).

(1) An object of the 2-category $(\mathbb{Z}_{\geq 0})^{\mathsf{r.lax}}$ is a non-negative integer $i \in \mathbb{Z}_{\geq 0}$.

(2) A morphism from i to j in $(\mathbb{Z}_{\geq 0})^{\mathsf{r.lax}}$ is a subset $\mathsf{I} \subset \mathbb{Z}_{\geq 0}$, such that:

$$\mathsf{Min}(\mathsf{I}) = i \qquad \mathsf{Max}(\mathsf{I}) = j \ .$$

(3) There is a unique 2 morphism from $\mathsf{I} \subset \mathbb{Z}_{\geq 0}$ to $\mathsf{J} \subset \mathbb{Z}_{\geq 0}$ if and only if there is an inclusion of $\mathsf{J} \subset \mathsf{I}$. The composition is given by the composition of inclusions.

The data then of the right-lax functor $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right)$ is a functor from the 2-category $(\mathbb{Z}_{\geq 0})^{\mathsf{r.lax}}$ to $\mathsf{Pr}_{\mathsf{st}}$

$$\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right) : (\mathbb{Z}_{\geq 0})^{\mathsf{r.lax}} \to \mathsf{Pr}_{\mathsf{st}}$$

which we unpack shortly. Before doing so, we give some notation and definitions.

Notation 301. Let I be a finite subset of $\mathbb{Z}_{\geq 0}$. Denote I_0 denote the subset of I obtained by removing the maximal element of I.

Definition 302. Let $\mathsf{I} \hookrightarrow \mathsf{J}$ be the inclusion between two finite linearly ordered sets. Provided $\mathsf{Min}(I) = \mathsf{Min}(J)$, this functor admits a left adjoint, which is the ***floor function***

$$\begin{aligned} \lfloor \mathsf{I} \subset \mathsf{J} \rfloor : \mathsf{J} &\longrightarrow \mathsf{I} \\ j &\longmapsto \mathsf{max}\{i \mid i \leq j\} \end{aligned} \ .$$

We denote $\lfloor \mathsf{I} \subset \mathsf{J} \rfloor_i$ as the fiber of the floor function over $i \in \mathsf{I}$

$$\begin{tikzcd} \lfloor \mathsf{I} \subset \mathsf{J} \rfloor_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{J} \arrow[d, "\lfloor \mathsf{I} \subset \mathsf{J} \rfloor"] \\ \{i\} \arrow[r] & \mathsf{I} \end{tikzcd} .$$

Definition 303. Let $\mathsf{I} \hookrightarrow \mathsf{J}$ be the inclusion of two linearly ordered posets. Define the map $\mu_{\lfloor \mathsf{I} \subset \mathsf{J} \rfloor_i}$ to be the multiplication map

$$\begin{aligned} \mu_{\lfloor \mathsf{I} \subset \mathsf{J} \rfloor_i} : \mathbb{T}^{\lfloor \mathsf{I} \subset \mathsf{J} \rfloor_i} &\longrightarrow \mathbb{T} \\ (x_0, \ldots, x_n) &\longmapsto (x_0 \ldots x_n) \end{aligned} .$$

Observation 304. We now unpack the right-lax functor

$$\check{\mathscr{G}} \left(\mathcal{V}^{\mathsf{E}_\infty} \right) : (\mathbb{Z}_{\geq 0})^{\mathsf{r.lax}} \to \mathsf{Pr}_{\mathsf{st}} \ .$$

(1) For each $i \in (\mathbb{Z}_{\geq 0})^{\mathsf{r.lax}}$, the value of the right-lax functor on i is sent to the ith-strata, which in this case is the ∞-category $\mathcal{V}$.

(2) For each $\{i < j\}$ in $\mathbb{Z}_{\geq 0}$, the value of the right-lax functor on the morphism $\{i < j\}$ is the endomorphim of $\mathcal{V}$ given by tensoring with $\mathbb{T}$

$$\mathcal{V} \xrightarrow{\mathbb{T} \odot -} \mathcal{V} \ .$$

More generally, for each morphism from i to j, which is a subset $\mathsf{I} \subset \mathbb{Z}_{\geq 0}$ with minimum value i and maximum value j, the right-lax functor evaluated on I is the functor

$$\mathcal{V} \xrightarrow{\mathbb{T}^{|\mathsf{I}_0|} \odot -} \mathcal{V} \ .$$

(3) The value of a 2 morphism $\{i \leq j \leq k\} \to \{i,k\}$ is the natural transformation between endofunctors

$$\mathbb{T}^{\{i,j\}} \odot - \xrightarrow{\mu_\odot} \mathbb{T}^{\{i\}} \odot -$$

implimented by the multiplication map $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$. More generally, the value of a 2-morphism from $\mathsf{I} \to \mathsf{J}$, which is an inclusion $\mathsf{J} \subset \mathsf{I}$ preserving the minimum and maximum values, is the natural transformation

$$\prod_{i \in \mathsf{I}_0} \mu_{\lfloor \mathsf{I} \subset \mathsf{J} \rfloor_i} : \mathbb{T}^{\mathsf{J}_0} \odot - \longrightarrow \mathbb{T}^{\mathsf{I}_0} \odot - \quad .$$

Furthermore, the reflected reconstruction theorem (Theorem 238) gives an equivalence of ∞-categories

$$\mathcal{V}^{\mathsf{E}_\infty} \simeq \mathsf{lim}^{\mathsf{l.lax}} \left(\mathbb{Z}_{\geq 0} \xrightarrow{\mathscr{G}(\mathcal{V}^{\mathsf{E}_\infty})} \mathsf{Pr}_{\mathsf{st}} \right) \ .$$

Lemma 289 and Lemma 294 identify tensoring with $\mathbb{T}$ and the multiplication map μ in the stable setting. Therefore we seek to unpack the right-lax limit with the identifications

$$\mathbb{T} \odot - \underset{\text{Lemma 289}}{\simeq} (1 \oplus \Sigma)$$

$$\left(\mathbb{T} \times \mathbb{T} \odot - \xrightarrow{\mu} \mathbb{T} \odot - \right) \underset{\text{Lemma 294}}{\simeq} \left(V \oplus \Sigma V \oplus \Sigma V \oplus \Sigma^2 V \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix}} V \oplus \Sigma V \right) \ .$$

Notation 305. Let P be a finite linearly ordered poset, and let F be an endofunctor of $\mathcal{V}$. Define

$$\bigotimes_{p \in \mathsf{P}} \mathsf{F} := \mathsf{F}^{\mathsf{P}}$$

s to be the iterated composition of the endofunctor F, once for each element $p \in \mathsf{P}$.

Observation 306. Note by Lemma 289, there is an equivalence of endofunctors

$$\mathbb{T}^{\mathsf{P}} \odot - \;\simeq\; \bigotimes_{\mathsf{P}} (1 \oplus \Sigma) \ .$$

Lemma 307. *There is an equivalence of endofunctors*

$$\bigotimes_{\mathsf{P}} (1 + \Sigma) \simeq \bigoplus_{S \in \mathcal{P}(\mathsf{P})} \Sigma^S \ ,$$

where $\Sigma^{\emptyset}$ is understood to be the identity functor 1 on $\mathcal{V}$.

Proof. Using that finite-fold direct sums are finite coproducts, this follows immediately from the binomial theorem

$$(1+x)^n = \sum_{k \in [n]} \tbinom{n}{k} x^n$$

after noting that exact functors[1] distribute over direct sums of exact functors. □

Lemma 308. *Through the identification*

$$\mathbb{T} \odot - \underset{\textit{Lemma 289}}{\simeq} (1 \oplus \Sigma) \ ,$$

together with the identification

$$\bigotimes_{\mathsf{P}} (1 + \Sigma) \underset{\textit{Lemma 307}}{\simeq} \bigoplus_{S \in \mathcal{P}(\mathsf{P})} \Sigma^S \ ,$$

the multiplication map

$$\mu_k : \mathbb{T}^k \odot - \to \mathbb{T} \odot -$$

corresponds to the following matrix, where the columns are indexed by $S \in \mathcal{P}([k-1])$ and

[1] Exact functors are those that preserve preserve both finite limits and colimits

$T \in \mathcal{P}([0])$

$$[\mu_k]^S_T = \begin{cases} \delta^S_\emptyset & T = \emptyset \\ \eta^{|S|-|T|} = \eta^{|S|-1} & T = \{0\} \end{cases} \tag{5.6}$$

where η^{-1} is understood to be 0, and η^0 is understood to be the identity.

Proof. We prove this by induction on k. The base case is the case $\mu_1 : \mathbb{T}^2 \odot - \to \mathbb{T} \odot -$ which is the matrix

$$\mu_1 = \begin{array}{cc} & \begin{array}{cccc} \Sigma^\emptyset & \Sigma^{\{0\}} & \Sigma^{\{1\}} & \Sigma^{\{0,1\}} \end{array} \\ \begin{array}{c} 1 \\ \Sigma^{\{0\}} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix} \end{array}$$

This matrix satisfies the formula for $S \subset \mathcal{P}([1])$ and $T \subset [0]$

$$[\mu_1]^S_T = \begin{cases} \delta^S_\emptyset & T = \emptyset \\ \eta^{|S|-|T|} = \eta^{|S|-1} & T = \{0\} \end{cases}.$$

The map μ_k can be factored as

$$\begin{tikzcd} \mathbb{T}^{k+1} \odot - \arrow[r] \arrow[d, "\simeq"] & \mathbb{T}^k \odot - \arrow[r] \arrow[d, "\simeq"] & \mathbb{T} \odot - \arrow[d, "\simeq"] \\ \bigoplus_{S \in \mathcal{P}[k]} \Sigma^S \arrow[r] & \bigoplus_{S \in \mathcal{P}([k-1])} \Sigma^S \arrow[r, "\mu_k"] & (1+\Sigma) \end{tikzcd}.$$

We assume that μ_{k-1} satisfies the formula (5.6) by induction. The matrix given by tensoring μ_1 with $(1 \oplus \Sigma)$ is a diagonal matrix, given by the following formula with rows indexed by $S \subset \mathcal{P}([k])$ and rows indexed by $T \subset \mathcal{P}([k-1])$.

$$[\mathbb{T}^{k-2} \odot \mu_1]^S_T = \begin{cases} [\mu_1]^{S_{\leq 1}}_{T_{\leq 0}} = \eta^{|S|-|T|} & (S_{>1}) - 1 = T > 0 \\ 0 & \text{else} \end{cases}.$$

Here the set $(S > 1) - 1$ is the set obtained by shifting the elements of $S_{>1}$ down by 1. The formula for the matrix $\mathbb{T}^{k-2} \odot \mu_1$ is seen by inspecting

$$\bigoplus_{U \in \mathcal{P}([k-2])} \Sigma^U \left(\bigoplus_{S \in \mathcal{P}([1])} \Sigma^S \xrightarrow{\mu_1} \bigoplus_{T \in \mathcal{P}([0])} \Sigma^T \right) .$$

Therefore consider the composite

$$[\mu_k]_T^S = \left([\mu_{k-1} \circ \mathbb{T}^{k-2} \odot \mu_1] \right)_T^S .$$

By the formula for $\mathbb{T}^{k-2} \odot \mu_1$, there is a unique $U \subset \mathcal{P}([k-1])$ such that the composite

$$\Sigma^S \xrightarrow{\mathbb{T}^{k-2} \odot \mu_1} \Sigma^U \xrightarrow{\mu_{k-1}} \Sigma^T$$

is nonzero. This seen by noting that the column of each matrix has precisely one nonzero entry. The composite is given then by the formula

$$[\mu_k]_T^S = \left([\mu_{k-1} \circ \mathbb{T}^{k-2} \odot \mu_1] \right)_T^S = [\mu_{k-1}]_T^U [\mathbb{T}^{k-2} \odot \mu_1]_U^S$$

which if $T \neq \emptyset$ is given by

$$[\mu_k]_T^S = \eta^{|Y|-|T|} \circ \eta^{|S|-|U|} = \eta^{|S|-|U|+|U|-|T|} = \eta^{|U|-|T|}$$

and if $T = \emptyset$

$$[\mu_k]_T^S = \delta_\emptyset^S .$$

$\square$

Example 309. The matrix μ_1 is the matrix of Lemma 290

$$\mu_1 = \begin{array}{c c} & \begin{array}{cccc} \Sigma^{\emptyset} & \Sigma^{\{0\}} & \Sigma^{\{1\}} & \Sigma^{\{0,1\}} \end{array} \\ \begin{array}{c} 1 \\ \Sigma^{\{0\}} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix} \end{array}$$

The matrix μ_2 is the matrix

$$\mu_2 := \begin{array}{c c} & \begin{array}{cccccccc} \Sigma^{\emptyset} & \Sigma^{\{0\}} & \Sigma^{\{1\}} & \Sigma^{\{0,1\}} & \Sigma^{\{2\}} & \Sigma^{\{0,2\}} & \Sigma^{\{1,2\}} & \Sigma^{\{0,1,2\}} \end{array} \\ \begin{array}{c} 1 \\ \Sigma^{\{0\}} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta & 1 & \eta & \eta & \eta^2 \end{bmatrix} \end{array}$$

The matrix μ_3 is the matrix

$$\mu_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta & 1 & \eta & \eta & \eta^2 & 1 & \eta & \eta & \eta^2 & \eta & \eta^2 & \eta^2 & \eta^3 \end{bmatrix}$$

where the columns are ordered by the maximum element of $S \subset \mathcal{P}([3])$.

In summary, by unpacking Observation 304, we have identified E_∞-modules as follows.

Theorem 310. *The right-lax functor*

$$\mathbb{Z}_{\geq 0} \xrightarrow{\check{\mathscr{G}}(\mathcal{V}^{\mathsf{E}_\infty})} \mathsf{Pr}_{\mathsf{st}}$$

evaluates as follows:

(1) The value on an object $i \in \mathbb{Z}_{\geq 0}$ is $\mathcal{V}$.

(2) The value on a 1-morphism from i to j, which is a subset $\mathsf{I} \subset \mathbb{Z}_{\geq 0}$ with $\mathsf{Min}(I) = i$ and $\mathsf{Max}(I) = j$, is the functor

$$\left(\mathcal{V} \xrightarrow{(1 \oplus \Sigma)^{|\mathsf{I}_0|}} \mathcal{V}\right) \simeq \left(\mathcal{V} \xrightarrow{\bigoplus_{S \subset \mathcal{P}(\mathsf{I}_0)} \Sigma^S} \mathcal{V}\right) .$$

which is the iterated suspension of the endofunctor of $(1\oplus\Sigma)$ of $\mathcal{V}$, iterated once for each non maximal element of $\mathcal{V}$.

(3) The value on a 2-morphism from J to I, which is an inclusion $\mathsf{I}\subset\mathsf{J}$, is given by

$$\bigotimes_{i\in\mathsf{I}_0}\left(\bigoplus_{S\subset\lfloor\mathsf{I}\subset\mathsf{J}\rfloor_i}\Sigma^S\right)\xrightarrow{\bigotimes_{i\in\mathsf{I}_0}\mu_{|\lfloor\mathsf{I}\subset\mathsf{J}\rfloor_i|}}\bigotimes_{i\in I_0}(1\oplus\Sigma)\ .$$

The left lax limit of this functor defines an equivalence of ∞-categories

$$\mathsf{Mod}_{\mathsf{E}_\infty}(\mathcal{V})\simeq\lim{}^{\mathsf{l.lax}}\left(\mathbb{Z}_{\geq 0}\xrightarrow{\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right)}\mathsf{Pr}_{\mathsf{st}}\right)\ .$$

Remark 311. Note that the value on a two morphism $\mathsf{I}\subset\mathsf{J}$ is a matrix which only depends on the matrices μ_k for various k in $\mathbb{Z}_{\geq 0}$.

Remark 312. The data of an object of the left-lax limit of the right-lax functor $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right)$ is the data of

(1) A functor

$$V_\bullet:\mathbb{Z}_{\geq 0}\xrightarrow{\left(V_0\xrightarrow{f_0}\to V_1\xrightarrow{f_1}V_2\xrightarrow{f_2}\dots\right)}\mathcal{V}\ .$$

(2) A natural transformation

$$\partial_\bullet:\Sigma V_\bullet\to V_{\bullet+1}\ ,$$

which is the data of a diagram in $\mathcal{V}$

$$\begin{tikzcd}
\Sigma V_0 \arrow[r, "f_0"] \arrow[d, "\partial_0"'] & \Sigma V_1 \arrow[r, "f_1"] \arrow[d, "\partial_1"] & \Sigma V_2 \arrow[r, "f_2"] \arrow[d, "\partial_2"] & \dots \\
V_1 \arrow[r, "f_1"'] & V_2 \arrow[r, "f_2"'] & V_3 \arrow[r, "f_3"'] & \dots
\end{tikzcd}$$

(3) A coherently compatible family of identification between natural transformations

$$\partial^r \simeq \eta^{r-1}\partial \qquad (r \geq 2) \ .$$

Observation 313. For a functor $\mathsf{sd}([i])^{\mathsf{op}} \to \mathcal{V}^{\mathsf{E}_i}$ with image in $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_i}\right)$, there exists a factorization

$$\begin{tikzcd}
\mathsf{sd}([i-1])^{\mathsf{op}} \arrow[r, hook] \arrow[d, dashed] & \mathsf{sd}([i])^{\mathsf{op}} \arrow[d] \\
\mathcal{V}^{\mathsf{E}_{i-1}} \times [i-1] \arrow[r] & \mathcal{V}^{\mathsf{E}_i} \times [i]
\end{tikzcd}$$

with image in $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_i}\right)$.

This observation allows for the identification of $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_\infty}\right)$ inductively, in terms of $\check{\mathscr{G}}\left(\mathcal{V}^{\mathsf{E}_i}\right)$.

Example 314. A E_0-module is the data of a functor

$$[0] \to \mathcal{V} \ .$$

The functor $[0] \to \mathcal{V}$ is the data of an object $V_0 \in \mathcal{V}$.

Example 315. A E_1-module is the data of:

(1) Two objects V_0 and V_1 in $\mathcal{V}$.

(2) A morphism

$$V_0 \oplus \Sigma V_0 \xrightarrow{[f_0,\partial_0]} V_1$$

in $\mathcal{V}$.

Example 316. An E_2 module

$$\mathsf{E}_2 \to \mathcal{V}$$

is the data of:

(1) For each $i \in [2]$, an object $V_i \in \mathcal{V}$.

(2) For each $\{i < j\} \in [2]$, a morphism in $\mathcal{V}$

$$V_i \oplus \Sigma V_i \xrightarrow{\begin{bmatrix} f_j^i & \partial_j^i \end{bmatrix}} V_j \ .$$

(3) A commutative square in $\mathcal{V}$

$$\begin{tikzcd}
V_0 \oplus \Sigma V_0 \oplus \Sigma V_0 \oplus \Sigma^2 V_0 \arrow[r, "{\begin{bmatrix} f_1^0 & \partial_1^0 & 0 & 0 \\ 0 & 0 & f_1^0 & \partial_1^0 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix}}"'] & V_1 \oplus \Sigma V_1 \arrow[d, "{\begin{bmatrix} f_2^1 & \partial_2^1 \end{bmatrix}}"] \\
V_0 \oplus \Sigma V_0 \arrow[r, "{\begin{bmatrix} f_2^0 & \partial_2^0 \end{bmatrix}}"'] & V_2
\end{tikzcd}$$

Therefore the data of this square commuting is an identification of the two matrices

$$[f_2^0 \quad \partial_2^0 \quad \partial_2^0 \quad \eta\partial_2^0] = [f_2^1 f_1^0 \quad f_2^1 \partial_1^0 \quad \partial_2^1 f_1^0 \quad \partial_2^1 \partial_1^0]$$

Recall, that a $\mathbb{CP}^i$ is the groupoid completion of the ∞-category E_i. Therefore we also unpack several E_i-modules that also satisfy the gropuoid completion condition.

Example 317. A $\mathbb{CP}^0$-module is the data of a an object $V \in \mathcal{V}$.

Example 318. A $\mathbb{CP}^1$-module is the data of:

(1) An object $V \in \mathcal{V}$.

(2) A morphism

$$V \oplus \Sigma V \xrightarrow{[1,\partial]} V \ .$$

Equivalently, this is the data of a morphism

$$\Sigma V \xrightarrow{\partial} V \ .$$

Example 319. We example the data of a $\mathbb{CP}^2$-module. A $\mathbb{CP}^2$-module determines:

(1) An object $V \in \mathcal{V}$.

(2) For each $\{i \leq k\} \in [2]$, a morphism in $\mathcal{V}$

$$V \oplus \Sigma V \xrightarrow{[1,\partial_k^i]} V \oplus V \ .$$

(3) A commutative square in $\mathcal{V}$

$$\begin{tikzcd}
V \oplus \Sigma V \oplus \Sigma V \oplus \Sigma^2 V \arrow[r, "{\begin{bmatrix} 1 & \partial_1^0 & 0 & 0 \\ 0 & 0 & 1 & \partial_1^0 \end{bmatrix}}"] \arrow[d, swap, "{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix}}"] & V \oplus \Sigma V \arrow[d, "{[1 \quad \partial_2^1]}"] \\
V \oplus \Sigma V \arrow[r, swap, "{[1 \quad \partial_2^0]}"] & V
\end{tikzcd}$$

Multiplying the matrices of each composite gives identifications

$$[1 \quad \partial_2^0 \quad \partial_2^0 \quad \eta\partial_2^0] = [1 \quad \partial_1^0 \quad \partial_2^1 \quad \partial_2^1\partial_1^0]$$

Therefore it must be the case that

$$\partial_1^0 \simeq \partial_2^1 \simeq \partial_2^0 \ .$$

Therefore, the data of a $\mathbb{CP}^2$-module is:

(1) An object $V \in \mathcal{V}$.

(2) A morphism $\Sigma V \to V$

(3) An identification of $\eta\partial \simeq \partial^2$

$$\begin{tikzcd} \Sigma^2 V \arrow[r, "\partial"] \arrow[d, "\eta"'] & \Sigma V \arrow[d, "\partial"] \\ \Sigma V \arrow[r, "\partial"'] & V \end{tikzcd}.$$

Example 320. A $\mathbb{CP}^3$ module is the data of:

(1) An object $V \in \mathcal{V}$.

(2) A morphism $\Sigma V \to \Sigma V$.

(3) The data of a commutative square

$$\begin{tikzcd} V \oplus \Sigma V \oplus \Sigma V \oplus \Sigma^2 V \arrow[r, "{\begin{bmatrix} 1 & \partial & 0 & 0 \\ 0 & 0 & 1 & \partial \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{bmatrix}}"'] & V \oplus \Sigma V \arrow[d, "{\begin{bmatrix} 1 & \partial \end{bmatrix}}"] \\ V \oplus \Sigma V \arrow[r, "{\begin{bmatrix} 1 & \partial \end{bmatrix}}"'] & V \end{tikzcd}$$

which is equivalent with the data of an identification $\eta\partial \simeq \partial^2$

$$\begin{tikzcd} \Sigma^2 V \arrow[r, "\partial"] \arrow[d, "\eta"'] & \Sigma V \arrow[d, "\partial"] \\ \Sigma V \arrow[r, "\partial"'] & V \end{tikzcd}.$$

(4) The data of a cube in $\mathcal{V}$

$$
\begin{tikzcd}
(V \oplus \Sigma V)^{\{0,1,2\}} \arrow[rr, "\mathbb{T}^2 \odot [1 \quad \partial]"] \arrow[dr, "\mu \otimes \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes \mu"'] & & (V \oplus \Sigma V)^{\{1,2\}} \arrow[dr, "\mu"] \arrow[dd, "\mathbb{T} \oplus [1 \quad \partial]" near end] & \\
& (V \oplus \Sigma V)^{\{1\}} \arrow[rr, "\mathbb{T} \odot [1 \quad \partial]" near start] \arrow[dd, "\mu" near end] & & (V \oplus \Sigma V)^{\{2\}} \arrow[dd, "{[1 \quad \partial]}"] \\
(V \oplus \Sigma V)^{\{0,1\}} \arrow[rr, "\mathbb{T} \odot [1 \quad \partial]" near start] \arrow[dr, "\mu"'] & & (V \oplus \Sigma V)^{\{1\}} \arrow[dr, "{[1 \quad \partial]}"] & \\
& (V \oplus \Sigma V)^{\{0\}} \arrow[rr, "{[1 \quad \partial]}"'] & & V
\end{tikzcd}
.
$$

Note each face is determined by the previous data. Namely, 4 of the faces are the 4 $\mathbb{CP}^2$-modules induced by each of the four conservative functors $[1] \to [2]$, and one of the faces is multiplication of $\mathbb{T}$. This is equivalent with an identification of $\eta^2\partial = \partial^3$

$$
\begin{tikzcd}
\Sigma^3 V \arrow[rr, "\partial"] \arrow[dr, "\eta"] \arrow[dd, "\eta"'] & & \Sigma^2 V \arrow[dr, "\eta"] \arrow[dd, "\partial" near end] & \\
& \Sigma^2 V \arrow[rr, "\partial" near start] \arrow[dd, "\eta" near end] & & \Sigma V \arrow[dd, "\partial"] \\
\Sigma^2 V \arrow[rr, "\partial" near end] \arrow[dr, "\eta"'] & & \Sigma V \arrow[dr, "\partial"] & \\
& \Sigma V \arrow[rr, "\partial"'] & & V
\end{tikzcd}
.
$$

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APPENDIX

MISCELLANEOUS CATEGORY THEORY RESULTS

The proof of Theorem 276 relied on many facts from ∞-category theory. The important ones are given here for the readers reference. This appendix read on its own may feel disjointed, and the reader is encouraged to use it as a reference for Sections 5.1-5.4.

We record the following result from §4 of [10].

Lemma 321. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and $\mathcal{V}$ be a presentable stable ∞-category. Then the left and right Kan extensions exits, and they define left and right adjoints to the functor $F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{V}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{V})$.*

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C}, \mathcal{V}) \arrow[r, bend left, "F_!"] \arrow[r, bend right, "F_*"'] & \mathsf{Fun}(\mathcal{D}, \mathcal{V}) \arrow[l, "F^*" description] \end{tikzcd}$$

Lemma 322. *Let $\mathcal{V}$ be a presentable stable ∞-category.*

(1) Let $\pi : \mathcal{E} \to \mathcal{B}$ be a fully faithful right fibration. The right Kan extension functor

$$\pi_* : \mathsf{Fun}(\mathcal{E}, \mathcal{V}) \to \mathsf{Fun}(\mathcal{B}, \mathcal{V})$$

is given by extension by 0.

(2) Let $\pi : \mathcal{E} \to \mathcal{B}$ be a fully faithful left fibration. The left Kan extension functor

$$\pi_! : \mathsf{Fun}(\mathcal{E}, \mathcal{V}) \to \mathsf{Fun}(\mathcal{B}, \mathcal{V})$$

is given by extension by 0.

Proof. By taking opposites of $\mathcal{E}$ and $\mathcal{B}$, the two statements imply one another. So we only prove the first statement.

Using that $\mathcal{V}$ is assumed presentable, the value of the right Kan extension π_* on a functor $\mathcal{E} \xrightarrow{F} \mathcal{V}$ evaluated on an object $b \in \mathcal{B}$ is given by the limit indexed by the ∞-undercategory:

$$\pi_*(F)(b) \ \simeq \ \mathsf{lim}\left(\mathcal{E}^{b/} \to \mathcal{E} \xrightarrow{F} \mathcal{V}\right) \ .$$

Using that π is a fully faithful right fibration, the ∞-undercategory $\mathcal{E}^{b/}$ is a (-1)-type: it is empty if $b \notin \pi(\mathcal{E})$ and it is a contractible ∞-groupoid if $b \in \pi(\mathcal{E})$. The result then follows from the fact that the limit indexed by the empty ∞-category is a final object. $\square$

Observation 323. The functor

$$\mathsf{Fun}\,(-, \mathcal{V}) : \mathsf{Cat}^{\mathsf{op}} \to \mathsf{Cat}$$

is an endofunctor of Cat that takes values in presentable stable ∞-categories. Moreover it takes values in ${\mathsf{Pr}_{\mathsf{st}}}^R$, where an object is a presentable stable ∞-category, and a morphism is a functor that is a right adjoint. There is an isomorphim of categories

$${\mathsf{Pr}_{\mathsf{st}}}^R \to {\mathsf{Pr}_{\mathsf{st}}}^{L^{\mathsf{op}}}$$

that sends a functor to its left adjoint. All together then there is a composite functor

$$\mathsf{Cat}^{\mathsf{op}} \to \mathsf{Pr_{st}}^{L\,\mathsf{op}} .$$

which we can consider as a functor

$$\mathsf{Cat} \to \mathsf{Pr_{st}}^{L}$$

by taking opposites.

The following is an immediate consequence of the universal property of colimits.

Lemma 324. *Let*

$$\mathsf{F} : \mathcal{K}^{\triangleright} \to \mathsf{Cat}$$

be a colimit diagram, and

$$\mathsf{G} : \mathsf{F}(+\infty) \to \mathcal{V}$$

be a functor. Then the functor

$$\begin{aligned} \mathcal{K}^{\triangleright} &\longrightarrow \mathcal{V} \\ k &\longmapsto \mathsf{colim}\left(\mathsf{F}(k) \xrightarrow{!} \mathsf{F}(+\infty) \to \mathcal{V}\right) \end{aligned}$$

is a colimit diagram.

We record the following result from §5 of [10].

Lemma 325. *Let* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *be a functor.*

(1) If $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *is a right adjoint, then the functor preserves limits.*

(2) If $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *is a left adjoint, then the functor preserves limits.*

We record the following result from [2].

Lemma 326. *Let* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *be a functor.*

(1) If $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *is a right adjoint, then* F *is a final functor.*

(2) If $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *is a left adjoint, then* F *is an initial functor.*

The definition of homotopy colimits is designed so the following two results are true.

Lemma 327. *Consider a pair of functors* $\mathsf{F} : \mathcal{C} \to \mathsf{Top}$ *and* $\mathsf{G} : \mathcal{C} \to \mathsf{Top}$*, with a natural transformation* $\mathsf{F} \underset{\alpha}{\Rightarrow} \mathsf{G}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathsf{F}"] \arrow[r, bend right, "\mathsf{G}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{Top} \end{tikzcd} .$$

If the components of α *are given by homotopy equivalences, then*

$$\mathsf{hocolim}(\mathcal{C} \xrightarrow{\mathsf{F}} \mathsf{Top}) \simeq \mathsf{hocolim}\left(\mathcal{C} \xrightarrow{\mathsf{G}} \mathsf{Top}\right)$$

Lemma 328. *Let* $\mathsf{C} \xrightarrow{\mathsf{F}} \mathsf{Top}$ *be a functor that is cofibrant. Then the canonical map*

$$\mathsf{hocolim}(\mathcal{C} \xrightarrow{\mathsf{F}} \mathsf{Top}) \xrightarrow{\simeq} \mathsf{colim}(\mathcal{C} \xrightarrow{\mathsf{F}} \mathsf{Top})$$

is an equivalence.

Definition 329. Let $\mathcal{C}$ be an ∞-category. The ***classifying space*** of $\mathcal{C}$ is colimit of the constant functor

$$\mathsf{B}\mathcal{C} := \mathsf{colim}\left(\mathcal{C} \xrightarrow{\mathsf{const}_*} \mathcal{S}\mathsf{paces}\right) .$$

Definition 330. Let G be a group. Define the category $\mathcal{B}G$ to be the category with a single object, with hom space

$$\mathsf{Hom}_{\mathcal{B}G}(*, *) = G .$$

The composition rule of $\mathcal{B}G$ is determined by multiplication in G.

The following result follows from the univalence-completeness axiom, in the presentation of $(\infty, 1)$-categories as complete Segal spaces.

Lemma 331. *Let* G *be a group object in* $\mathcal{S}\mathsf{paces}$*, then the canonical functor*

$$\mathcal{B}G \to \mathsf{B}G$$

is an equivalence in $\mathsf{Cat}_{(\infty,1)}$*.*

Theorem 332 (Local Immersion Theorem)**.** *Let* $M_0 \hookrightarrow M$ *be an immersion of smooth manifolds, and let* x *be a point in* M_0*. Then there exists charts* (U, ϕ) *about* x *in* M_0 *and a chart* (V, φ) *such that*

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, "\phi"', "\cong"] & V \arrow[d, "\varphi", "\cong"'] \\
\mathbb{R}^{n_0} \arrow[r, hook, "i"] & \mathbb{R}^{n_1}
\end{tikzcd}$$

the inclusion of U *into* V *is the canonical local embedding of* $\mathbb{R}^{n_0}$ *into* $\mathbb{R}^{n_1}$*.*

Definition 333. Let $\mathsf{L} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ and $\mathsf{R} : \mathcal{C} \to \mathcal{D}$ be two functors between categories. The ***coend*** is the colimit of the diagram

$$\coprod_{f:s\to t\in \mathsf{Ar}(\mathcal{C})} \mathsf{L}(t) \times \mathsf{R}(s) \underset{\mathsf{id}\times\mathsf{R}(f)}{\overset{\mathsf{L}(f)\times\mathsf{id}}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} \mathsf{L}(c) \times \mathsf{R}(t) \quad .$$

Definition 334. Let $\mathsf{L} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ and $\mathsf{R} : \mathcal{C} \to \mathcal{D}$ be two functors between ∞-categories, where $\mathcal{D}$ admits small limits and colimits. The ∞-***coend***[1] is the colimit of the functor

$$\mathsf{L} \otimes \mathsf{R} := \mathsf{colim}\left(\mathsf{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathsf{L}\times\mathsf{R}} \mathcal{D} \times \mathcal{D} \xrightarrow{-\times-} \mathcal{D}\right) .$$

Observation 335. Recall that the ∞-groupoid-completion functor

$$|-|: \mathsf{Cat}_{(\infty,1)} \longrightarrow \mathcal{S}\mathsf{paces} ,$$

is, by definition, left adjoint to the canonical inclusion. Therefore, $|-|$ preserves colimits: for $\mathcal{K} \xrightarrow{F} \mathsf{Cat}_\infty$ a functor, the canonical map between spaces

$$\underset{x\in\mathcal{K}}{\mathsf{colim}}|F(x)| \xrightarrow{\simeq} \left|\underset{x\in\mathcal{K}}{\mathsf{colim}}F(x)\right|$$

is an equivalence.

Lemma 336. *Let $\mathsf{F} : \mathcal{K}_0 \hookrightarrow \mathcal{K}$ be a fully faithful functor between ∞-categories, and let $\mathcal{V}$ be a presentable stable ∞-category.*

(1) The left Kan extension functor

$$\mathsf{F}_! : \mathsf{Fun}\left(\mathcal{K}_0, \mathcal{V}\right) \to \mathsf{Fun}\left(\mathcal{K}, \mathcal{V}\right)$$

is fully faithful.

(2) The right Kan extension functor

$$\mathsf{F}_* : \mathsf{Fun}\left(\mathcal{K}_0, \mathcal{V}\right) \to \mathsf{Fun}\left(\mathcal{K}, \mathcal{V}\right)$$

is fully faithful.

Proof. (1) follows since the left Kan extension can be computed pointwise due to $\mathcal{V}$ admiting limits and colimits. (2) follows as the dual statement to (1). $\square$

Definition 337. Let $\mathcal{W} \subset \mathcal{K}$ be an ∞-subcategory of an ∞-category $\mathcal{K}$. The localization of $\mathcal{K}$ with respect to $\mathcal{W}$ is defined to be the ∞-category determined by the pushout in $\mathsf{Cat}_{(\infty,1)}$

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{K} \arrow[d] \\ |W| \arrow[r] & \mathcal{K}[\mathcal{W}^{-1}] \end{tikzcd} .$$

The definition of the twisted arrow ∞-category is such that the following is true.

[1]We will frequently just say coend, and rely on context to differentiate between the two

Lemma 338. *Let* F *and* G *be functors from an* ∞*-category* $\mathcal{C}$ *to a presentable stable* ∞*-category* $\mathcal{V}$*. Then the hom space* $\mathsf{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{V})}(\mathsf{F},\mathsf{G})$ *is presented by the following limit*

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{V})}(\mathsf{F},\mathsf{G}) \simeq \mathsf{lim}(\mathsf{TwAr}(\mathcal{C}) \xrightarrow{(\mathsf{min},\mathsf{max})} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathsf{F}^{\mathsf{op}}\times\mathsf{G}} \mathcal{V} \times \mathcal{V} \xrightarrow{\mathsf{Hom}_{\mathcal{V}}(-,-)} \mathcal{S}\mathsf{paces} \ .$$

Lemma 339. *The functor* $\mathcal{S}\mathsf{paces} \xrightarrow{(+)} \mathcal{S}\mathsf{paces}^{*/}$ *preserves colimits.*

Proof. The functor is left adjoint to the forgetful functor which forgets the basepoint. □

Notation 340. Let X and Y be spaces. Define

$$Y^X := \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(X,Y) \ .$$

Lemma 341. *Let* X *and* Y *be spaces. The canonically commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d] & X \amalg Y \arrow[d] \\ * \arrow[r, "\langle + \rangle"'] & Y_+ \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram in $\mathcal{S}\mathsf{paces}$

Proof. The diagram is a pushout if for each space Z, the diagram

$$\begin{tikzcd} Z^{Y_+} \arrow[r] \arrow[d] & Z^{X\amalg Y} \arrow[d] \\ Z^* \arrow[r] & Z^X \end{tikzcd}$$

is a limit diagram. Using the fact that $Z^* \simeq Z$ and $X \amalg Y$ is the coproduct, this is equivalent to checking

$$\begin{tikzcd} Z \times Z^Y \arrow[r] \arrow[d] & Z^X \times Z^Y \arrow[d] \\ Z \arrow[r] & Z^X \end{tikzcd}$$

is a limit diagram. Since the diagram is in $\mathcal{S}\mathsf{paces}$ it suffices to check for all $z \in Z$ if the map between the fibers

$$
\begin{tikzcd}
Z^Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & Z \times Z^Y \arrow[dr] \arrow[dd, "\pi_Z"] & \\
& Z^Y \arrow[rr] \arrow[dd] & & Z^X \times Z^Y \arrow[dd, "\pi_{Z^X}"] \\
* \arrow[rr, "\langle z \rangle"] \arrow[dr] & & Z \arrow[dr] & \\
& * \arrow[rr, "\langle \mathsf{const}_z \rangle"] & & Z^X
\end{tikzcd}
$$

is an equivalence. This is clearly the case since it is just the identity morphism.

$\square$

Lemma 342. *Let*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\
* \arrow[r, "\langle + \rangle"'] \arrow[ur, phantom, "\ulcorner", very near end] & Z_+
\end{tikzcd}
$$

be a pushout diagram in Spaces*. Then the diagram*

$$
\begin{tikzcd}
X_+ \arrow[r, "f_+"] \arrow[d] & Y_+ \arrow[d, "g_+"] \\
* \arrow[r, "\langle + \rangle"'] \arrow[ur, phantom, "\ulcorner", very near end] & Z_+
\end{tikzcd}
$$

is a pushout in Spaces*.*

Proof. The diagram is a pushout if after applying for each space $A \in \mathsf{Spaces}$, applying $\mathsf{Hom}_{\mathsf{Spaces}}(-, A)$ gives a limit diagram in Spaces

$$
\begin{tikzcd}
A^{Z_+} \arrow[r] \arrow[d] & A^{Y_+} \arrow[d] \\
A^{*} \arrow[r] & Z^{X_+}
\end{tikzcd}
.
$$

Using the universal property of the coproducts gives an equivalent diagram

$$
\begin{tikzcd}
A^{Z_+} \times * \arrow[r] \arrow[d] & A^{Y} \times A^{*} \arrow[d] \\
A^{*} \times * \arrow[r] & Z^{X} \times A^{*}
\end{tikzcd}
.
$$

This diagram is a limit diagram if the map on fibers is an equivalence

$$\begin{tikzcd}
\mathsf{fib}(\langle + \rangle) \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & A^{Z_+} \arrow[dr, "g_+^\circ"] \arrow[dd] & \\
& \mathsf{fib}(f^\circ \times \mathsf{id}) \arrow[rr] \arrow[dd] & & A^Y \times A^* \arrow[dd, "(f^\circ{,}\mathsf{id})"] \\
* \arrow[rr] \arrow[dr] & & A^* \arrow[dr] & \\
& * \arrow[rr] & & A^X \times A^*
\end{tikzcd}.$$

This is immediate after noting the equivalence

$$\mathsf{fib}(f^\circ) \simeq \mathsf{fib}(f^\circ \times \mathsf{id})$$

since the identity map $\mathsf{id} : \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(*, A) \to \mathsf{Hom}_{\mathcal{S}\mathsf{paces}}(*, A)$ is an equivalence. $\square$

Lemma 343.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
* \arrow[r, "\langle + \rangle"'] & Z_+ \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

be a pushout diagram in $\mathcal{S}\mathsf{paces}$*. Then*

$$\begin{tikzcd}
X_+ \arrow[r, "f_+"] \arrow[d] & Y_+ \arrow[d] \\
* \arrow[r, "\langle + \rangle"'] & Z_+ \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{paces}^{*/}$

Proof. By Lemma 342, the diagram

$$\begin{tikzcd}
X_+ \arrow[r, "f"] \arrow[d] & Y_+ \arrow[d] \\
* \arrow[r, "\langle + \rangle"'] & Z_+ \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout diagram in $\mathcal{S}\mathsf{paces}$. Using Lemma 344, where $\mathcal{K}$ is the cospan diagram and $\mathcal{X}$ is

$\mathcal{S}\mathsf{paces}$, the diagram

$$\begin{tikzcd} {|\mathcal{K}| \odot * \simeq *} \arrow[r, "\langle + \rangle"] \arrow[d, "\simeq"'] & \mathsf{colim}(\mathcal{K} \to \mathcal{S}\mathsf{paces}^{*/} \to \mathcal{S}\mathsf{paces}) \arrow[d] \\ * \arrow[r, "\langle + \rangle"'] & \mathsf{fgt}(\mathsf{colim}(\mathcal{K} \to \mathcal{S}\mathsf{paces}^{*/})) \end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{paces}$. The colimit

$$\mathsf{colim}(\mathcal{K} \to \mathcal{S}\mathsf{paces}^{*/} \to \mathcal{S}\mathsf{paces}) = Z_+$$

by assumption, and therefore

$$Z_+ \xrightarrow{\simeq} \mathsf{fgt}(\mathsf{colim}(\mathcal{K} \to \mathcal{S}\mathsf{paces}^{*}))$$

is an equivalence in $\mathcal{S}\mathsf{paces}$. Furthermore Lemma 344 gives the equivalence

$$(* \xrightarrow{\langle + \rangle} Z_+) \simeq \mathsf{colim}(\mathcal{K} \to \mathcal{S}\mathsf{paces}^{*/})$$

Therefore the diagram

$$\begin{tikzcd} X_+ \arrow[r, "f_+"] \arrow[d] & Y_+ \arrow[d] \\ * \arrow[r, "\langle + \rangle"'] & Z_+ \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram in $\mathcal{S}\mathsf{paces}^{*/}$. $\square$

The following result is an immediate application of the mapping-out property of colimits.

Lemma 344. *Let $c \in \mathcal{C}$ be an object in a cocomplete ∞-category. Let $\mathcal{K} \to \mathcal{C}^{c/}$ be a functor. Then the diagram*

$$\begin{tikzcd} {|\mathcal{K}| \odot c} \arrow[r, "f"] \arrow[d] & \mathsf{colim}\left(\mathcal{K} \to \mathcal{C}^{c/} \to \mathcal{C}\right) \arrow[d] \\ c \arrow[r, "\langle + \rangle"'] & \mathsf{colim}\left(\mathcal{K} \to \mathcal{C}^{c/}\right) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout.

Lemma 345. *Let $f : x \to y$ be a morphisms in an ∞-groupoid $\mathcal{G}$. Then the functors*

$$\mathcal{G}_{/x} \to \mathcal{G}_{/y}$$

and

$$\mathcal{G}^{y/} \to \mathcal{G}^{x/}$$

induced by composition with f are equivalences.

Proof. We prove that the functor $\mathcal{G}_{x/} \to \mathcal{G}_{y/}$ is an equivalence, and the case $\mathcal{G}^{y/} \to \mathcal{G}^{x/}$ follows by duality. Note that since $\mathcal{G}$ is a groupoid, there exists a morphisms $f^1 : y \to x$ and a triangle in $\mathcal{G}$

$$\begin{tikzcd} x \arrow[rr, "\mathsf{id}"] \arrow[dr, "f"'] & & x \\ & y \arrow[ur, "f^{-1}"'] & \end{tikzcd} .$$

This induces a diagram

$$\begin{tikzcd} \mathcal{G}_{/x} \arrow[rr, "\mathsf{id}"] \arrow[dr, "-\circ f"'] & & \mathcal{G}_{/x} \\ & \mathcal{G}_{/y} \arrow[ur, "-\circ f^{-1}"'] & \end{tikzcd} .$$

Therefore this witnesses $- \circ f$ as the right inverse to $- \circ f^{-1}$. A similar argument shows that $- \circ f$ is also the left inverse to $- \circ f^{-1}$. Therefore $\mathcal{G}_{/y} \simeq \mathcal{G}_{/x}$ are equivalent. ☐

Definition 346. Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be a functor, and let $d \in \mathcal{D}$. The category $\mathcal{C}_{/d}$ is defined to be the pullback

$$\begin{tikzcd} \mathcal{C}_{/d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}_{/d} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathsf{F}"'] & \mathcal{D} \end{tikzcd} .$$

Similarly, the category $\mathcal{C}^{d/}$ is defined to be the pullback

$$\begin{tikzcd} \mathcal{C}^{d/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}^{d/} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathsf{F}"'] & \mathcal{D} \end{tikzcd} .$$

Lemma 347. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{G}$ an ∞-groupoid, and $\mathsf{F} : \mathcal{C} \to \mathcal{G}$ a functor. Then the two squares are each pullbacks*

$$\begin{tikzcd} \mathcal{C}_{/g} \arrow[d] & \mathcal{C}_{|g} \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{g/} \arrow[d] \\ \mathcal{G}_{/g} & * \arrow[l] \arrow[r] & \mathcal{G}^{g/} \end{tikzcd} . \tag{A.1}$$

Moreover, since the bottom two arrows

$$\mathcal{G}_{/g} \xleftarrow{\simeq} * \xrightarrow{\simeq} \mathcal{G}^{g/}$$

are equivalences, the top arrows are also equivalences.

Proof. We first show that the left square is a pullback. Consider the diagram

$$\begin{tikzcd} \mathcal{C}_{|g} \arrow[r] \arrow[d] & \mathcal{C}_{/g} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ * \arrow[r, "\langle g \xrightarrow{\mathrm{id}} g \rangle"'] & \mathcal{G}_{/g} \arrow[r] & \mathcal{G} \end{tikzcd} . \tag{A.2}$$

The right square of (A.2) is a pullback by definition. The outer square of (A.2) is also a pullback, since it is just the definition of the fiber over g. Therefore the left square of (A.2) is a pullback, as well as the left square of (A.1). Dually, the right square of $A.1$ is also a pullback. $\square$

Proposition 348 (Quillen Theorem A)**.** *Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. The functor F is final if and only if for each object $d \in \mathcal{D}$, the classifying space*

$$|\mathcal{C}^{d/}| \simeq *$$

is contractible. Dually, the functor F is initial if and only if the classifying space

$$|\mathcal{C}_{/d}| \simeq *$$

is contractible.

Proposition 349 (Quillen Theorem B)**.** *Let $\mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories such that for each morphism $f : d \to d'$ in $\mathcal{D}$, the functor*

$$\mathcal{C}_{/d} \to \mathcal{C}_{/d'}$$

induced by composition with f is an equivalence on classifying spaces

$$|\mathcal{C}_{/d}| \xrightarrow{\simeq} |\mathcal{C}_{/d'}| .$$

Then for each $d \in \mathcal{D}$ the diagram

$$\begin{tikzcd} {|\mathcal{C}_{/d}|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {|\mathcal{C}|} \arrow[d] \\ * \arrow[r] & {|\mathcal{D}|} \end{tikzcd}$$

is a pullback diagram in Spaces.

Proof. This is Theorem 5.16 of [2]. □

Lemma 350. *Let* $\mathsf{R} : \mathcal{E} \to \mathcal{B}$ *be a fully faithful right fibration, and* $\mathsf{F} : \mathcal{E} \to \mathcal{S}\mathsf{p}$ *be a functor. Then the Right Kan extension* F_* *along* R *is given by extension given by extension by zero*

$$\begin{tikzcd} & & \mathsf{Fun}(\mathcal{B}\backslash\mathcal{E}, \mathcal{V}) \\ \mathsf{Fun}(\mathcal{E}, \mathcal{V}) \arrow[r, "\mathsf{F}_*"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "\mathsf{id}"] & \mathsf{Fun}(\mathcal{B}, \mathcal{V}) \arrow[ur] \arrow[dr] & \\ & & \mathsf{Fun}(\mathcal{E}, \mathcal{V}) \end{tikzcd}$$

Proof. The right Kan extension of a functor $\mathsf{F} : \mathcal{E} \to \mathcal{V}$ along $\mathcal{E} \to \mathcal{B}$ evaluated on a an object $b \in \mathcal{B}$ is given by the limit

$$\mathsf{F}_!(b) = \lim \left(\mathcal{E}^{b/} \to \mathcal{E} \xrightarrow{\mathsf{F}} \mathcal{V} \right)$$

There are two cases, either $b \in \mathcal{E}$, or $b \in \mathcal{B}\backslash\mathcal{E}$. If $b \in \mathcal{E}$, then

$$\mathcal{E}^{b/} \simeq \mathcal{B}^{b/} \ .$$

The category $\mathcal{B}^{b/}$ has an initial object $b \xrightarrow{\mathsf{id}} b$. Therefore

$$\lim \left(\mathcal{E}^{b/} \to \mathcal{E} \xrightarrow{\mathsf{F}} \mathcal{V} \right) \simeq \lim \left(* \xrightarrow{\langle b \to b \rangle} \mathcal{E}^{b/} \to \mathcal{E} \xrightarrow{\mathsf{F}} \mathcal{V} \right) = \mathsf{F}(b) \ .$$

For the second case, assume that $b \in \mathcal{B}\backslash\mathcal{E}$. An object in $\mathcal{E}^{b/}$ is the data

$$\begin{tikzcd} \{1\} \arrow[r, "\langle b \rangle"] \arrow[d] & \mathcal{E} \arrow[d] \\ {[1]} \arrow[r, "\langle a \to b \rangle"] & \mathcal{B} \\ \{0\} \arrow[u] \arrow[ur, "\langle a \rangle"'] & \end{tikzcd} \ .$$

However, by the lifting property of $\mathcal{E} \to \mathcal{B}$ being a right fibration, then $b \in \mathcal{E}$.

$\square$

The following result follows from the observation that the unstraightening of the identity functor $\mathsf{Spaces} \xrightarrow{\mathsf{id}} \mathsf{Spaces}$ is the left fibration $\mathsf{Spaces}^{*/} \to \mathsf{Spaces}$.

Lemma 351. *Let $\mathsf{F} : \mathcal{C} \to \mathsf{Spaces}$ be a functor. The unstraightening $\mathsf{Un}(\mathsf{F})$ is the pullback pullback*

$$\begin{tikzcd} \mathsf{Un}(\mathsf{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Spaces}^{*/} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathsf{F}"'] & \mathsf{Spaces} \end{tikzcd} .$$

The following result is extracted from §2 of [10].

Lemma 352. *Let $\mathsf{F} : \mathcal{C} \to \mathsf{Spaces}$ be a functor. The colimit of F is the classifying space of the unstraightening of F.*

Lemma 353. *Let $\mathsf{F} : \mathcal{C} \hookrightarrow \mathcal{D}$ be a monomorphism. Then $\mathsf{TwAr}(\mathsf{F}) : \mathsf{TwAr}(\mathcal{C}) \to \mathsf{TwAr}(\mathcal{D})$ is a monomorphism.*

Proof. First, recall that $\mathcal{C} \to \mathcal{D}$ is a monomorphism if and only if $\forall [p] \in \Delta$,

$$\mathsf{Hom}_{\mathsf{Cat}}([p], \mathcal{C}) \to \mathsf{Hom}_{\mathsf{Cat}}([p], \mathcal{D})$$

is a monomorphism between spaces. Therefore we seek to show

$$\mathsf{Hom}_{\mathsf{Cat}}([p], \mathsf{TwAr}(\mathcal{C})) \to \mathsf{Hom}_{\mathsf{Cat}}([p], \mathsf{TwAr}(\mathcal{A}))$$

is a monomorphism of spaces. By the yoneda lemma and the definition of $\mathsf{TwAr}(\mathcal{C})$, there are the equivalences

$$\mathsf{Hom}_{\mathsf{Cat}}([p], \mathsf{TwAr}(\mathcal{C})) \simeq \mathsf{TwAr}(\mathcal{C})([p]) := \mathsf{Hom}_{\mathsf{Cat}}([p]^{op} \bigstar [p], \mathcal{C}) \ .$$

Therefore it suffices to check if

$$\mathsf{Hom}_{\mathsf{Cat}}([p]^{\mathsf{op}} \bigstar [p], \mathcal{C}) \to \mathsf{Hom}_{\mathsf{Cat}}([p]^{\mathsf{op}} \bigstar [p], \mathcal{D})$$

is a monomorphism of spaces. Using that $[p]^{\circ} \bigstar [p] \simeq [2p+1]$, the monomorphism condition gives the desired result

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{Cat}}([p], \mathsf{TwAr}(\mathcal{C})) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathsf{Cat}}([p], \mathsf{TwAr}(\mathcal{D})) \arrow[d, "\simeq"] \\ \mathsf{Hom}_{\mathsf{Cat}}([p]^{\circ} \bigstar [p], \mathcal{C}) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathsf{Cat}}([p]^{\circ} \bigstar [p], \mathcal{D}) \arrow[d, "\simeq"] \\ \mathsf{Hom}_{\mathsf{Cat}}([2p+1], \mathcal{C}) \arrow[r, hook] & \mathsf{Hom}_{\mathsf{Cat}}([2p+1], \mathcal{D}) \end{tikzcd} .$$

□

Lemma 354. *Consider a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathsf{F}"] \arrow[d] & \mathcal{V} \\ \mathcal{D} \arrow[ur, bend right, "\mathsf{F}_!"'] & \end{tikzcd}$$

where $\mathsf{F}_!$ *witnesses a right Kan extension. Then*

$$\mathsf{lim}(\mathsf{F}) \to \mathsf{lim}(\mathsf{F}_!)$$

is an equivalence.

Proof. Recall that the limit of a functor is the value of the right Kan extension along a point. Consider the right Kan extension of $\mathsf{F}_!$ along $*$

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\mathsf{F}_!"] \arrow[d] & \mathcal{V} \\ * \arrow[ur, bend right, "\mathsf{lim}(\mathsf{F}_!)"'] & \end{tikzcd} .$$

Since the right Kan extension of a right Kan extension witnesses a right Kan extension, the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathsf{F}"] \arrow[d] & \mathcal{V} \\ \mathcal{D} \arrow[ur, bend right, "\mathsf{F}"] \arrow[d] & \\ * \arrow[uur, bend right, "\mathsf{lim}(\mathsf{F}_!)"'] & \end{tikzcd}$$

witnesses $\mathsf{lim}(\mathsf{F}_!)$ as the right Kan extension of F along $\mathcal{C} \to \mathcal{D} \to *$. Since the category $*$ is terminal in $\mathsf{Cat}_{(\infty,1)}$, this implies $\mathsf{lim}(\mathsf{F}_!)$ witnesses the left Kan extension of F along $\mathcal{C} \xrightarrow{!} *$, which is by definition $\mathsf{lim}(\mathsf{F})$. Therefore

$$\mathsf{lim}(\mathsf{F}) \simeq \mathsf{lim}(\mathsf{F}_!)$$

. □

Observation 355. Let $\mathcal{C}$ be an ∞-category and c be an object of $\mathcal{C}$. Then the ∞-categories

$$\mathcal{C}_{/c} \simeq \left(\mathcal{C}^{\mathsf{op}c^\circ/}\right)$$

Lemma 356. *Let* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *be a functor between* $(\infty, 1)$*-categories.*

(1) The following are equivalent.

(1) *The functor* F *is a right adjoint.*

(2) *For each object* $d \in \mathcal{D}$*, the* ∞*-category* $\mathcal{C}^{d/}$ *has an initial object*

(2) *The following are equivalent*

(1) F *is a left adjoint*

(2) *For each* $d \in \mathcal{D}$*, the* ∞*-category* $\mathcal{C}_{d/}$ *has a terminal object.*

Proof. This Lemma 2.17 of [2]. □

Definition 357. Let $\mathcal{C}$ and $\mathcal{D}$ and $\mathcal{B}$ and be in Pr^L. A ***bi-colimit preserving*** functor is a functor

$$\mathsf{F} : \mathcal{C} \times \mathcal{D} \to \mathcal{B}$$

such that for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, the restriction functors

$$\mathsf{F}_{|c} : c \times \mathcal{D} \hookrightarrow \mathcal{C} \times \mathcal{D} \to \mathcal{B}$$

$$\mathsf{F}_{|d} : \mathcal{C} \times d \hookrightarrow \mathcal{C} \times \mathcal{D} \to \mathcal{B}$$

are colimit preserving.

Definition 358. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. The ***tensor product of presentable*** ∞***-categories*** is the ∞-category that satisfies the universal property

$$\begin{tikzcd} \mathcal{C} \times \mathcal{D} \arrow[r] \arrow[d] & \mathcal{V} \\ \mathcal{C} \otimes \mathcal{D} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

with respect to bicolimit preserving functors $\mathcal{C} \times \mathcal{D} \to \mathcal{V}$.

The following result is an immediate consequence of the fact that colimits in a functor ∞-category are computed value-wise.

Lemma 359. *Let* $\mathcal{K}$ *be a (small)* ∞*-category. Then* $\mathsf{Fun}(\mathcal{K}, \mathcal{V})$ *is a presentable* ∞*-category.*

The following result is a direct consequence of the fact that colimits commute with one another.

Lemma 360. *The functor*

$$\mathsf{Fun}(\mathcal{K}, \mathsf{Spaces}) \times \mathcal{V} \to \mathsf{Fun}(\mathcal{K}, \mathcal{V})$$

that is adjoint to the functor

$$\mathcal{K} \times \mathsf{Fun}(\mathcal{K}, \mathcal{V}) \times \mathcal{V} \to \mathsf{Spaces} \times \mathcal{V} \xrightarrow{\odot} \mathcal{V}$$

is a bi-colimit preserving functor.

Lemma 361. *The inclusion* $\mathsf{zCat}_{(\infty,1)} \to \mathsf{Cat}_{(\infty,1)}$ *admits a left adjoint*

$$\mathsf{Cat}_{(\infty,1)} \xrightarrow{(-)_\diamond} \mathsf{zCat}_{(\infty,1)}$$

that sends a category $\mathcal{C}$ *to the category* $\mathcal{C}_\diamond$ *that adjoins a zero object to the category* $\mathcal{C}$.

Proof. Notice that the forget functor $\mathsf{zCat}_{(\infty,1)} \to \mathsf{Cat}_{(\infty,1)}$ preserves limits. Using that $\mathsf{Cat}_{(\infty,1)}$ is presentable, this forgetful functor admits a left adjoint. $\square$

Lemma 362. *The tensor product restricts to* StPr *with the category of Spectra* $\mathcal{S}\mathsf{p}$ *as the unit.*

Proof. This follows since the category of $\mathcal{S}\mathsf{p}$ is freely generated by colimits by the sphere spectrum $\mathbb{S}$. $\square$