Third Test Solutions, MATH 224, Spring 2007

1. (20 pts) Calculate $\iint_D (1+x^2)\,dA$, where D is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

Written as a type II integral, this is

$$\int_0^1\int_0^y (1+x^2)dx\,dy = \int_0^1 \left(y+\frac{y^3}{3}\right)dy = \frac{1}{2}+\frac{1}{12} = \frac{7}{12} \approx 0.583$$

2. (20 pts) Let E be the solid of constant density ρ bounded by $4z^2 = x^2+y^2$ and the plane $z=1$. Sketch E and find its moment of inertia about the z-axis.

The solid is a circular cone along the z-axis with its tip at the origin and its "base" (really situated at the top) being a disk of radius 2 at height $z=1$. In cylindrical coordinates E is given by $0 \le z \le 1$, $0 \le \theta \le 2\pi$, and $0 \le r \le 2z$. The moment of inertia is then

$$\begin{aligned} I_z &= \iiint_E (x^2+y^2)\rho\,dV = \rho\int_0^{2\pi}\int_0^1\int_0^{2z} r^2\cdot r\,dr\,dz\,d\theta = \rho\int_0^{2\pi}\int_0^1 \left[\frac{r^4}{4}\right]_{r=0}^{r=2z} dz\,d\theta \\ &= \rho\int_0^{2\pi}\int_0^1 4z^4\,dz\,d\theta = \rho\int_0^{2\pi}\frac{4}{5}d\theta = \frac{8\pi\rho}{5}. \end{aligned}$$

3. (20 pts) Find the work done by the force field $\mathbf{F}(x,y) = \langle y,-x\rangle$ on a particle that moves along the graph of $y = x^3 - x$ from $(-1,0)$ to $(1,0)$.

Since the path is part of a graph, we use x as a parameter, so $x=t$, $y=t^3-t$, and $dx = dt$, $dy = (3t^2-1)dt$, where $-1 \le t \le 1$. The work is

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx - x\,dy = \int_{-1}^1 [(t^3-t)-t(3t^2-1)]dt = \int_{-1}^1 (-2t^3)dt = 0.$$

4. (20 pts) One of the following vector fields is conservative. Find a potential for it, and use the potential to calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where the curve C is given by $\mathbf{r}(t) = \langle t^{3/2}, \cos(\pi t^2)\rangle$, $0 \le t \le 1$.

$$\mathbf{F}_1(x,y) = \langle e^{xy}, e^{xy}\rangle \qquad\qquad \mathbf{F}_2(x,y) = \langle ye^{xy}, xe^{xy}+1\rangle$$

We have $\dfrac{\partial P_1}{\partial y} = xe^{xy} \ne ye^{xy} = \dfrac{\partial Q_1}{\partial x}$, and $\dfrac{\partial P_2}{\partial y} = e^{xy}+xye^{xy} = \dfrac{\partial Q_2}{\partial x}$, so $\mathbf{F}_1$ is not conservative, whereas $\mathbf{F}_2$ is conservative. Finding a potential f for $\mathbf{F}_2$ is equivalent to solving $f_x = ye^{xy}$ and $f_y = xe^{xy}+1$. Integrating the first equation gives $f = e^{xy}+C(y)$, and plugging this into the second gives $xe^{xy}+C'(y) = xe^{xy}+1$, so $C'(y) = 1$. This means that $C(y) = y$ is one solution, and hence $f(x,y) = e^{xy}+y$ is a potential for $\mathbf{F}_2$.

The given path starts at $\mathbf{r}(0) = \langle 0,1\rangle$ and ends at $\mathbf{r}(1) = \langle 1,-1\rangle$, so $\int_C \mathbf{F}_2\cdot d\mathbf{r} = f(1,-1) - f(0,1) = e^{-1}+(-1)-(e^0+1) = e^{-1}-3 = \dfrac{1}{e}-3 \approx -2.632.$

5. (20 pts) Use Green's Theorem to evaluate $\int_C y(2x+1)\,dx + (x^2+3x)\,dy$, where C is the circle $x^2+y^2=9$ with counterclockwise orientation.

If D denotes the disk $x^2+y^2 \le 9$, Green's Theorem gives

$$\begin{aligned}\int_C y(2x+1)\,dx + (x^2+3x)\,dy &= \iint_D ((2x+3)-(2x+1))dA \\ &= \iint 2\,dA = 2\,\text{area}(D) = 2\cdot 9\pi = 18\pi,\end{aligned}$$

since the area of a disk of radius 3 is 9π.