Homework 1 Solutions

1.1.4 (a) Prove that $A \subseteq B$ iff $A \cap B = A$.

Proof. First assume that $A \subseteq B$. If $x \in A \cap B$, then $x \in A$ and $x \in B$ by definition, so in particular $x \in A$. This proves $A \cap B \subseteq A$. Now if $x \in A$, then by assumption $x \in B$, too, so $x \in A \cap B$. This proves $A \subseteq A \cap B$. Together this implies $A = A \cap B$.

Now assume that $A \cap B = A$. If $x \in A$, then by assumption $x \in A \cap B$, so $x \in A$ and $x \in B$. In particular, $x \in B$. This proves $A \subseteq B$. □

1.1.4 (b) Prove $A \cap B = A \setminus (A \setminus B)$.

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular, $x \notin A \setminus B$ (because $x \in A \setminus B$ would imply $x \notin B$). So $x \in A \setminus (A \setminus B)$. This shows $A \cap B \subseteq A \setminus (A \setminus B)$. Now let $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. This means that $x \notin A$ or $x \in B$ (the negation of $x \in A$ and $x \notin B$). Since we know $x \in A$, this implies $x \in B$, so $x \in A \cap B$. This shows $A \setminus (A \setminus B) \subseteq A \cap B$. Together with the first part this shows $A \cap B = A \setminus (A \setminus B)$. □

1.1.4 (c) Prove $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Let $x \in (A \setminus B) \cup (B \setminus A)$. Then $x \in A \setminus B$ or $x \in B \setminus A$. In the first case, this implies $x \in A$ and $x \notin B$. From this we get $x \in A$ or $x \in B$ (since the first of those statements is true), so $x \in A \cup B$. We also get that $x \notin A \cap B$ (because $x \notin B$), so $x \in (A \cup B) \setminus (A \cap B)$. In the second case we get $x \in B$ and $x \notin A$, so by the same argument $x \in A \cup B$ and $x \notin A \cap B$. Again we conclude $x \in (A \cup B) \setminus (A \cap B)$. This shows $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

Now let $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in A$ or $x \in B$, and $x \notin A \cap B$. If $x \in A$, then $x \notin B$ (because otherwise $x \in A \cap B$), so $x \in A \setminus B$. If $x \notin A$, then by assumption $x \in B$, so $x \in B \setminus A$. In either case, $x \in (A \setminus B) \cup (B \setminus A)$. This shows $(A \cup B) \setminus (A \cap B) \subseteq x \in (A \setminus B) \cup (B \setminus A)$. Together with the first part this shows the claimed set equality. □

1.1.4 (d) Prove that $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Proof. If $p \in (A \cap B) \times C$, then $p = (x, y)$ with $x \in A \cap B$ and $y \in C$. This means $x \in A$, $x \in B$ and $y \in C$, and thus $(x, y) \in A \times C$ and $(x, y) \in B \times C$. This implies $p = (x, y) \in (A \times C) \cap (B \times C)$. This proves $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

If $p \in (A \times C) \cap (B \times C)$, then $p \in A \times C$ and $p \in B \times C$, so $p = (x, y)$ with $x \in A$ and $y \in C$, and $x \in B$ and $y \in C$. This implies $x \in A \cap B$ and $y \in C$, so $p = (x, y) \in (A \cap B) \times C$. This proves $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$. Together the two inclusions prove the claimed equality. □

1.1.4 (e) Prove that $A \cap B$ and $A \setminus B$ are disjoint, and that $A = (A \cap B) \cup (A \setminus B)$.

Proof. For the first part we have to prove that $(A \cap B) \cap (A \setminus B) = \emptyset$. Let $x \in (A \cap B) \cap (A \setminus B)$. Then $x \in A \cap B$ and $x \in A \setminus B$, so $x \in A$ and $x \in B$, and $x \in A$ and $x \notin B$. In particular, this implies $x \in B$ and $x \notin B$, which is a contradiction. I.e., there can be no such x and we proved that $(A \cap B) \cap (A \setminus B) = \emptyset$.

For the set equality, let $x \in A$ be arbitrary. Then either $x \in B$ or $x \notin B$. In the first case, $x \in A \cap B$, in the second case $x \in A \setminus B$. In either case, $x \in (A \cap B) \cup (A \setminus B)$. This shows $A \subseteq (A \cap B) \cup (A \setminus B)$.

Now let $x \in (A \cap B) \cup (A \setminus B)$. Then $x \in A \cap B$ or $x \in A \setminus B$. Either case implies $x \in A$ by definition. This shows $(A \cap B) \cup (A \setminus B) \subseteq A$. Together the two inclusions show the claimed set equality. □

1.2.5 Prove that if a function f has a maximum, then $\sup f$ exists and $\max f = \sup f$.

Proof. For the existence of the supremum we have to show that f is bounded above, and for the claimed equality we have to show that $\max f$ is the least upper bound for f.

By definition of the maximum, there exists $x_0 \in X$ with $f(x) \leq f(x_0) = \max f$ for all $x \in X$. This shows that $\max f$ is an upper bound for f, and that the supremum of f exists.

Now choose an arbitrary $M \in \mathbb{R}$ with $M < \max f$. Then $M < f(x_0)$, and thus M is not an upper bound. This shows that $\max f$ is the least upper bound, i.e., $\max f = \sup f$. □

1.2.22 Suppose that $f : X \to Y$.

For the following proofs we break down the "if and only if" into both directions. The symbol "$\Longrightarrow$" means that we show that the first assumption implies the second one, the symbol "$\Longleftarrow$" means that we are proving that the second assumption implies the first one. Similarly we break down the proof of set equalities into the two inclusions "$\subseteq$" and "$\supseteq$".

1.2.22 (a) Prove that $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$ iff f is injective.

Proof. We show the implications separately.

$\Longrightarrow$: Let $x_1, x_2 \in X$ be arbitrary with $f(x_1) = f(x_2)$. Let $A = \{x_1\}$ and $B = \{x_2\}$. By assumption, $f(A \cap B) = f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\}$. This implies that there exists an element $x \in A \cap B$ with $f(x) = f(x_1)$. Since $x \in A$ and $x \in B$ we have that $x = x_1$ and $x = x_2$, and hence $x_1 = x_2$. This shows that f is injective.

$\Longleftarrow$: This breaks down into two parts itself.

$\subseteq$: Let $y \in f(A \cap B)$. Then there exists $x \in A \cap B$ with $f(x) = y$. This implies that $x \in A$ and $x \in B$ with $f(x) = y$, thus $y \in f(A)$ and $y \in f(B)$. By definition, $y \in f(A) \cap f(B)$.

$\supseteq$: Let $y \in f(A) \cap f(B)$. Then $y \in f(A)$ and $y \in f(B)$. Thus there exists $x_1 \in A$ with $f(x_1) = y$ and there exists $x_2 \in B$ with $f(x_2) = y$. By injectivity of f we have $x_1 = x_2$, and thus $x_1 \in B$, too. So $x_1 \in A \cap B$ and hence $y = f(x_1) \in f(A \cap B)$. □

1.2.22 (b) Prove that $f(A \setminus B) = f(A) \setminus f(B)$ for all $A, B \subseteq X$ iff f is injective.

Proof. Set difference is intersection with the complement, so this proof mimicks the proof in (a).

$\Longrightarrow$: Let $x_1, x_2 \in X$ be arbitrary with $f(x_1) = f(x_2)$. Let $A = \{x_1\}$ and $B = \{x_2\}$. By assumption, $f(A \setminus B) = f(A) \setminus f(B) = \{f(x_1)\} \setminus \{f(x_2)\} = \emptyset$. This implies that $A \setminus B = \emptyset$, and hence $\{x_1\} \setminus \{x_2\} = \emptyset$. This means that $x_1 = x_2$ (because otherwise $\{x_1\} \setminus \{x_2\} = \{x_1\}$). This shows that f is injective.

$\Longleftarrow$: This breaks down into two parts itself.

$\subseteq$: Let $y \in f(A \setminus B)$. Then there exists $x \in A \setminus B$ with $f(x) = y$. This implies that $x \in A$ and $x \notin B$ with $f(x) = y$. We can immediately deduce $y \in f(A)$. Now we have to show that $y \notin f(B)$. Assume to the contrary that $y \in f(B)$. Then there exists $x_1 \in B$ with $f(x_1) = y$. By injectivity of f, we get $x = x_1$, and thus $x \in B$ and $x \notin B$, a contradiction. This shows that $y \notin f(B)$, and thus $y \in f(A) \setminus f(B)$.

$\supseteq$: Let $y \in f(A) \setminus f(B)$. Then $y \in f(A)$ and $y \notin f(B)$. Thus there exists $x \in A$ with $f(x) = y$. If $x \in B$, then $y \in f(B)$, which contradicts the previous statement, so we must have $x \notin B$. This implies $x \in A \setminus B$, and hence $y \in f(A \setminus B)$. □

1.2.22 (c) Prove that $f^{-1}(f(A)) = A$ for all $A \subseteq X$ iff f is injective.

Proof. $\Longrightarrow$: Let $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Let $A = \{x_1\}$. Then $f(A) = \{f(x_1)\}$, and since $f(x_1) = f(x_2)$ we have that $x_2 \in f^{-1}(f(A))$. By assumption $f^{-1}(f(A)) = A$, so $x_2 \in A = \{x_1\}$, and thus $x_1 = x_2$. This shows that f is injective.

$\Longleftarrow$:

$\subseteq$: Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, hence there exists $x_1 \in A$ with $f(x_1) = f(x)$. By injectivity, $x = x_1$, so $x \in A$.

$\supseteq$: Let $x \in A$. Then $f(x) \in f(A)$, and by definition this implies $x \in f^{-1}(f(A))$. □

1.2.22 (d) Prove that $f(f^{-1}(B)) = B$ for all $B \subseteq Y$ iff f is surjective.

Proof. $\Longrightarrow$: Let $y \in Y$ arbitrary. We have to show that there exists $x \in X$ with $f(x) = y$. Let $B = \{y\}$. By assumption, $f(f^{-1}(B)) = B = \{y\}$, so $y \in f(f^{-1}(B))$. By definition this means that there exists $x \in f^{-1}(B)$ with $f(x) = y$.

$\Longleftarrow$:

$\subseteq$: Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ with $f(x) = y$. By definition this means that $y = f(x) \in B$.

$\supseteq$: Let $y \in B$. By surjectivity of f there exists $x \in X$ with $f(x) = y$. This implies that $x \in f^{-1}(B)$. Then $y = f(x) \in f(f^{-1}(B))$. $\square$

For problems 23 and 24 we will choose $X = Y = \mathbb{R}$ and the functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. (Since f is neither injective nor surjective it is a good candidate for counterexamples.)

1.2.23 (a) Find an example for which $f^{-1}(f(A)) \neq A$.

$A = \{1\}$ gives $f(A) = \{1\}$ and $f^{-1}(f(A)) = \{-1, 1\} \neq A$.

1.2.23 (b) Find an example for which $f(f^{-1}(A)) \neq A$.

$A = \{-1\}$ gives $f^{-1}(A) = \emptyset$ and $f(f^{-1}(A)) = \emptyset \neq A$.

1.2.24 (a) Find an example for which $f(A \cap B) \neq f(A) \cap f(B)$.

$A = \{1\}$ and $B = \{-1\}$ give $A \cap B = \emptyset$, $f(A \cap B) = \emptyset$, $f(A) = \{1\} = f(B) = f(A) \cap f(B) \neq f(A \cap B)$.

1.2.24 (b) Find an example for which $f(A \setminus B) \neq f(A) \setminus f(B)$.

$A = \{1\}$ and $B = \{-1\}$ give $A \setminus B = \{1\}$, $f(A \setminus B) = \{1\}$, $f(A) = f(B) = \{1\}$, and $f(A) \setminus f(B) = \emptyset \neq f(A \setminus B)$.

As you can see, we could as well have chosen $X = Y = \{-1, 1\}$ and $f : X \to Y$ given by $f(1) = f(-1) = 1$ in all counterexamples.

1.4.1 (a) The negation of "there exists $p > 0$ such that for every x we have $f(x+p) = f(x)$" is "for all $p > 0$ there exists x with $f(x+p) \neq f(x)$".

In formal notation with quantifiers (using $\equiv$ for logical equivalence):

$$\sim (\exists p > 0 \, \forall x: \; f(x+p) = f(x)) \equiv \forall p > 0 \, \exists x: \; f(x+p) \neq f(x)$$

Mathematically, the original statement means that f is a periodic function.

1.4.1 (b) The negation of "for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever x and t are in D and satisfy $|x - t| < \delta$, then $|f(x) - f(t)| < \epsilon$" is "there exists $\epsilon > 0$ such that for every $\delta > 0$ there exist x and t in D with $|x - t| < \delta$, but $|f(x) - f(t)| \geq \epsilon$".

Again the same in formal notation:

$$\begin{aligned} &\sim (\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x, t \in D: \; |x - t| < \delta \Rightarrow |f(x) - f(t)| < \epsilon) \\ &\equiv \exists \epsilon > 0 \, \forall \delta > 0 \, \exists x, t \in D: \; (|x - t| < \delta) \wedge (|f(x) - f(t)| \geq \epsilon) \end{aligned}$$

Mathematically, the original statement means that the function f is uniformly continuous in D.

1.4.1 (c) The negation of "for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x \in D$ and $0 < |x - a| < \delta$, then $|f(x) - A| < \epsilon$" is "there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in D$ with $0 < |x - a| < \delta$, but $|f(x) - A| \geq \epsilon$".

In formal notation:

$$\begin{aligned} &\sim (\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in D: \; 0 < |x - a| < \delta \Rightarrow |f(x) - A| < \epsilon) \\ &\equiv \exists \epsilon > 0 \, \forall \delta > 0 \, \exists x \in D: \; (0 < |x - a| < \delta) \wedge (|f(x) - A| \geq \epsilon) \end{aligned}$$

Mathematically the original statement means that $\lim_{x \to a} f(x) = A$ (assuming that the domain of the function is D).

1.4.5 Consider the statement P: the sum of two irrational numbers is irrational.

1.4.5 (a) Give an example of a case in which P is true.

$\sqrt{2} + \sqrt{2} = 2\sqrt{2}$. (To show that $2\sqrt{2}$ is irrational, assume to the contrary that $2\sqrt{2} = p/q$ with integers p and q. Then $\sqrt{2} = p/(2q)$ would be rational as well. But we proved in class that this is not true, so this is a contradiction, and thus $2\sqrt{2}$ is irrational.)

1.4.5 (b) Prove or disprove P by giving a counterexample.

The statement is not true in general: $\sqrt{2}$ and $-\sqrt{2}$ are irrational, but their sum $\sqrt{2} + (-\sqrt{2}) = 0$ is rational. (If you want an example with positive numbers, choose $\sqrt{2}$ and $2 - \sqrt{2}$. Irrationality of $2 - \sqrt{2}$ follows in the same way as that of $2\sqrt{2}$.)