Homework 2 Solutions

1.3.2 (d) Prove $\sum_{k=1}^{n}(2k-1) = n^2$ by mathematical induction.

Proof. For $n = 1$ we have $\sum_{k=1}^{1}(2k-1) = 2\cdot 1 - 1 = 1 = 1^2$. Assuming now that the statement is true for n, we get

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2.$$

□

1.3.2 (s) Prove $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ for all $n \geq 2$ by mathematical induction.

Proof. For $n = 2$ we have $\sum_{k=1}^{2} \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{2}{\sqrt{2}} = \sqrt{2}$. Assuming that the statement is true for n, we get

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \\ &> \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}.\end{aligned}$$

□

1.3.6 (c) Use mathematical induction to prove the following: If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 1}$ for all $n \in \mathbb{N}$, then $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Proof. $a_2 = \sqrt{3a_1 + 1} = \sqrt{3+1} = 2$, so $a_1 = 1 < 2 = a_2$. Assuming that we already know $a_n < a_{n+1}$, we know that $3a_n + 1 < 3a_{n+1} + 1$, so $a_{n+1} = \sqrt{3a_n + 1} < \sqrt{3a_{n+1} + 1} = a_{n+2}$. □

1.3.6 (d) Use mathematical induction to prove the following: If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 1}$ for all $n \in \mathbb{N}$, then $a_n < \frac{7}{2}$ for all $n \in \mathbb{N}$.

Proof. $a_1 = 1 < \frac{7}{2}$ is immediate. Now assume that we already know $a_n < \frac{7}{2}$. Then $3a_n + 1 < 3 \cdot \frac{7}{2} + 1 = \frac{23}{2} < \frac{49}{4}$, so $a_{n+1} = \sqrt{3a_n + 1} < \sqrt{\frac{49}{4}} = \frac{7}{2}$. □

1.5.12 Suppose that $f : A \to B$ and $g : B \to C$ are two bijections. Prove that $g \circ f : A \to C$ has an inverse function $f^{-1} \circ g^{-1} : C \to A$. (This verifies $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. We use Theorem 1.5.6 (c) to check that $g \circ f$ is a bijection with inverse $f^{-1} \circ g^{-1}$. We know that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1}(g^{-1}(g(f(x)))) = f^{-1}(f(x)) = x$$

for all $x \in A$, and

$$(g \circ f) \circ (f^{-1} \circ g^{-1})(z) = g(f(f^{-1}(g^{-1}(z)))) = g(g^{-1}(z)) = z$$

for all $z \in C$. By Theorem 1.5.6 (c) this implies that $g \circ f$ is bijective with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(We used the facts that $f^{-1}(f(x)) = x$ for all $x \in A$; $f(f^{-1}(y)) = y$ for all $y \in B$; $g^{-1}(g(y)) = y$ for all $y \in B$; and $g(g^{-1}(z)) = z$ for all $z \in C$. All of these directly follow from Theorem 1.5.4.) □

1.5.15 (a) $\arctan\left(\tan \frac{3\pi}{4}\right) = -\frac{\pi}{4}$.

1.5.15 (b) $\arctan(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

1.5.15 (c) $\tan(\arctan x) = x$ for all $x \in \mathbb{R}$.