Homework 3 Solutions

Extra Problem. Prove that $(0,1)$, $(0,1]$, $[0,1]$, and $\mathbb{R}$ are equivalent sets.

Proof. The easiest equivalence is $(0,1) \sim \mathbb{R}$, one possible bijection is given by $f : (0,1) \to \mathbb{R}$,

$$f(x) = \begin{cases} 2 - \frac{1}{x} & \text{for } 0 < x < \frac{1}{2}, \\ \frac{1}{1-x} - 2 & \text{for } \frac{1}{2} \leq x < 1, \end{cases}$$

with inverse function

$$f^{-1}(y) = \begin{cases} \frac{1}{2-y} & \text{for } y < 0, \\ 1 - \frac{1}{2+y} & \text{for } y \geq 0. \end{cases}$$

To show $(0,1] \sim (0,1)$, one possible bijection $g : (0,1] \to (0,1)$ is given by

$$g(x) = \begin{cases} \frac{1}{n+1} & \text{for } x = \frac{1}{n},\ n \in \mathbb{N}, \\ x & \text{if } x \neq \frac{1}{n} \text{ for all } n \in \mathbb{N}, \end{cases}$$

with inverse

$$g^{-1}(y) = \begin{cases} \frac{1}{n-1} & \text{for } y = \frac{1}{n},\ n \in \mathbb{N},\ n \geq 2, \\ y & \text{if } y \neq \frac{1}{n} \text{ for all } n \in \mathbb{N},\ n \geq 2. \end{cases}$$

Then $h : [0,1] \to [0,1)$ defined by

$$h(x) = \begin{cases} g(x) & \text{for } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is again a bijection, so $[0,1] \sim [0,1)$. But $F : [0,1) \to (0,1]$, $F(x) = 1 - x$ is a bijection, too (with $F^{-1} = F$), so $[0,1) \sim (0,1]$. By transitivity the claimed equivalences follow. □

1.8.5. (d) Find all real values x such that

$$\frac{2x+1}{x-5} \leq 3.$$

We consider the cases $x \geq 5$ and $x < 5$ separately. If $x \geq 5$, the inequality becomes $2x + 1 \leq 3(x-5)$, which is equivalent to $x \geq 16$. If $x < 5$, then we get $2x + 1 \geq 3(x-5)$, leading to $x \leq 16$. From the first case we get all $x \in [16, \infty)$, from the second case we get all $x \in (-\infty, 5)$, so the set of all possible solutions is $[16, \infty) \cup (-\infty, 5)$.

1.8.13. If $a, b \in \mathbb{R}$ and $a - \epsilon < b$ for any $\epsilon > 0$, prove that $a \leq b$.

Proof. Assume not, i.e, $a > b$. Let $\epsilon = a - b > 0$. Then by assumption $a - \epsilon < b$, so $b = a - (a - b) =< b$. This is a contradiction. $\square$

1.8.14. (c) Prove that $|a| = \sqrt{a^2}$.

Proof. The square root of a number $y \geq 0$ is defined as the number $x \geq 0$ such that $x^2 = y$. So we have to show that $|a| \geq 0$ and that $|a|^2 = a^2$. If $a \geq 0$, we have $|a| = a \geq 0$ and $|a|^2 = a^2$. If $a < 0$, then we get $|a| = a > 0$ and $|a|^2 = (-a)^2 = a^2$. Thus we get the claim in both cases. $\square$

1.8.15. (b) Find all real values of x that satisfy $|2x - 5| \leq |x + 4|$.

We have to consider different cases depending on the signs of the expressions between absolute value signs.

If $x \geq 5/2$, then $2x - 5 \geq 0$ and $x + 4 \geq 0$, so the inequality is $2x - 5 \leq x + 4$, equivalent to $x \leq 9$. This yields the interval $[5/2, 9]$ as part of the solution.

If $-4 \leq x < 5/2$, then $2x - 5 < 0$ and $x + 4 \geq 0$, so the inequality is $-(2x - 5) \leq x + 4$, equivalent to $x \geq \frac{1}{3}$. This gives $[1/3, 5/2)$ as part of the solution.

If $x < -4$, then $2x - 5 < 0$, and $x + 4 < 0$, so the inequality is $-(2x - 5) \leq -(x + 4)$, equivalent to $x \geq 9$. However, there are no x which satisfy both $x < -4$ and $x \geq 9$, so this does not give any more solutions.

Putting everything together, the set of values of x for which the inequality is satisfied is $[5/2, 9] \cup [1/3, 5/2) = [1/3, 9]$.

1.8.15. (c) Find all real values of x such that $2|1 - 3x| > 5$.

This is equivalent to $|1 - 3x| > 5/2$, and this is satisfied if and only if $1 - 3x > 5/2$ or $1 - 3x < -5/2$. This is equivalent to $x < -1/2$ or $x > 7/6$, so the set of solutions is $(-\infty, -1/2) \cup (7/6, \infty)$.

1.8.15. (f) Find all real values of x such that $\left|\frac{2+x}{3-x}\right| \geq -1$.

Absolute values are always non-negative, so they are certainly greater than -1. This means that this is always true whenever it is well-defined, i.e., for all $x \in \mathbb{R}$, $x \neq 3$.

1.8.15. (g) Find all real values of x such that $|x + 6| = |2x + 1|$.

This true if and only if $x + 6 = 2x + 1$ or $x + 6 = -(2x + 1)$. This means that the only solutions are $x = 5$ and $x = -5/3$.