Homework 4, 5, 6 Solutions

2.1.2(a) $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let $\epsilon > 0$. Then for $n \geq n^* = 2+\frac{1}{2\epsilon}$ we have $2n-3 \geq 4+\frac{1}{\epsilon}-3 > \frac{1}{\epsilon} > 0$, so $0 < \frac{1}{2n-3} < \epsilon$, and thus $|a_n - 0| = \frac{1}{2n-3} < \epsilon$. $\square$

2.1.2(g) $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Proof. Let $\epsilon > 0$. Then for $n \geq n^* = \frac{1}{4\epsilon^2}$ we have $2\sqrt{n} \geq \frac{1}{\epsilon} > 0$, and so $|a_n - 0| = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} \leq \epsilon$. $\square$

2.1.2(k) The sequence $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 1/n & \text{if } n \text{ is even} \end{cases}$ diverges.

Proof. Assume not. Then the sequence converges to some limit $A \in \mathbb{R}$. By definition of convergence (with $\epsilon = 1/4$) there exists n^* such that $|a_n - A| < 1/4$ for $n \geq n^*$. Choose an integer $k \geq n^*/2$. Then $2k \geq n^*$ and $2k+1 \geq n^*$, so $|a_{2k} - A| < 1/4$ and $|a_{2k+1} - A| < 1/4$. So $|1/(2k) - A| < 1/4$ and $|1 - A| < 1/4$. The second inequality implies $A > 3/4$, and the first one $A < 1/(2k) + 1/4 \leq 1/2 + 1/4 = 3/4$. This is a contradiction, so the statement is proved. $\square$

2.1.7 If $\lim\limits_{n\to\infty} a_{2n} = A$ and $\lim\limits_{n\to\infty} a_{2n-1} = A$, then $\lim\limits_{n\to\infty} a_n = A$.

Proof. Let $\epsilon > 0$. Then there exist n_1 and n_2 such that $|a_{2n} - A| < \epsilon$ for $n \geq n_1$ and $|a_{2n-1} - A| < \epsilon$ for $n \geq n_2$. Let $n^* = \max(2n_1, 2n_2 - 1)$, and let $n \geq n^*$ be arbitrary. If n is even, then there exists $k \in \mathbb{N}$ such that $n = 2k$. Since $n \geq n^* \geq 2n_1$ we get that $k \geq n_1$, and thus $|a_n - A| = |a_{2k} - A| < \epsilon$. If n is odd, then there exists $k \in \mathbb{N}$ such that $n = 2k-1$. Since $n \geq n^* \geq 2n_2 - 1$, we get $k \geq n_2$, and thus $|a_n - A| = |a_{2k-1} - A| < \epsilon$. Every number n is either even or odd, so we have proved the claim. $\square$

The converse is also true: If $\lim\limits_{n\to\infty} a_n = A$, then $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n-1} = A$.

Proof. Let $\epsilon > 0$. Then there exists n_1 such that $|a_n - A| < \epsilon$ for $n \geq n_1$. For $n \geq n_* = (n_1 + 1)/2$ we get $2n \geq 2n - 1 \geq n_1$ and thus $|a_{2n} - A| < \epsilon$ and $|a_{2n-1} - A| < \epsilon$. This shows the claim. $\square$

The first direction helps with 2(j), since $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} 1/(2n) = 0$ and $\lim\limits_{n\to\infty} a_{2n-1} = 0$, so the results implies that the sequence converges to 0.

The converse helps with 2(k), since $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} 1/(2n) = 0$ and $\lim_{n\to\infty} a_{2n-1} = 1$. If the sequence would converge, these two limits would have to be the same. Since they are different, the sequence itself diverges.

2.1.10 If $\{a_n\}$ converges to A, then the sequence $\{b_n\}$ defined by $b_n = (a_n + a_{n+1})/2$ converges to A, too.

Proof. Let $\epsilon > 0$. Then there exists n^* such that $|a_n - A| < \epsilon$ for $n \geq n^*$. Then $|b_n - A| = |(a_n + a_{n+1})/2 - A| = |(a_n - A) + (a_{n+1} - A)|/2 \leq |a_n - A|/2 + |a_{n+1} - A|/2 < \epsilon/2 + \epsilon/2 = \epsilon$ for $n \geq n^*$. □

2.1.20 Consider sequences $\{a_n\}$ and $\{b_n\}$, where $b_n = \sqrt[n]{a_n}$.

(a) If $\{b_n\}$ converges to 1, does the sequence $\{a_n\}$ necessarily converge?

No. Example is $a_n = n$, $b_n = \sqrt[n]{n}$.

(b) If $\{b_n\}$ converges to 1, does the sequence $\{a_n\}$ necessarily diverge?

No. Example is $a_n = b_n = 1$.

(c) Does $\{b_n\}$ have to converge to 1?

No. Example is $a_n = b_n = 0$.

2.2.11(c) $\lim_{n\to\infty} \dfrac{1}{2^n} = 0$ by Theorem 2.1.13.

2.2.11(d) $\lim_{n\to\infty} \dfrac{r^n}{n!} = 0$.

Proof. We will write $a_n = \frac{r^n}{n!}$. Choose $n_1 \in \mathbb{N}$ with $n_1 \geq |r|$. Let

$$M = |a_{n_1}| = \frac{|r|^{n_1}}{n_1!} = \frac{|r|}{1} \cdot \frac{|r|}{2} \cdots \frac{|r|}{n_1}.$$

We first claim that $|a_n| \leq M$ for all $n \geq n_1$. Proof by induction: The case $n = n_1$ is immediate by definition of M. Now if we already know the claim for some $n \geq n_1$, then

$$|a_{n+1}| = \frac{|r|^{n+1}}{(n+1)!} = \frac{|r|^n}{n!} \cdot \frac{|r|}{n+1} \leq M \frac{|r|}{n+1} \leq M.$$

(We used $n+1 \geq n_1 \geq |r|$ in the last inequality, and the induction hypothesis in the second-to-last inequality.) Now let $n > n_1$ be arbitrary. Then $n - 1 \geq n^*$, so $\frac{|r|^{n-1}}{(n-1)!} \leq M$, and thus

$$|a_n| = \frac{|r|^n}{n!} = \frac{|r|^{n-1}}{(n-1)!} \cdot \frac{|r|}{n} \leq \frac{M|r|}{n}.$$

We know that $\lim_{n\to\infty} \frac{M|r|}{n} = 0$. The squeeze theorem then implies that $\{a_n\}$ converges to 0, too. □

2.2.11(i) $\lim_{n\to\infty} \sqrt[n]{n + \sqrt{n}} = 1$.

Proof. This follows from the squeeze theorem, the estimate $1 \le \sqrt[n]{n+\sqrt{n}} \le \sqrt[n]{2n} = \sqrt[n]{2} \cdot \sqrt[n]{n}$, and $\lim\limits_{n\to\infty} \sqrt[n]{2} = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. □

2.2.13(a) Suppose that $\{a_n\}$ and $\{a_n b_n\}$ both converge, and $a_n \neq 0$ for large n. Is it true that $\{b_n\}$ must converge?

No. Example: $a_n = 1/n$, $b_n = n$.

2.2.13(b) Suppose that $\{a_n\}$ converges to a non-zero number and $\{a_n b_n\}$ converges. Prove that $\{b_n\}$ must also converge.

Proof. This follows immediately from the limit theorems. Let $A = \lim\limits_{n\to\infty} a_n$ and $C = \lim\limits_{n\to\infty} a_n b_n$. Then $b_n = \frac{a_n b_n}{a_n}$ is the quotient of two convergent sequences, where the denominator converges to a non-zero limit. From Theorem 2.2.1(c) we get that $\{b_n\}$ converges to C/A. □

2.2.18(a) Is it possible to have an unbounded sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty} a_n/n = 0$?

Yes. Example $a_n = \sqrt{n}$.

2.2.18(b) Prove that if the sequence $\{a_n\}$ satisfies $\lim\limits_{n\to\infty} a_n/n = L \neq 0$, then $\{a_n\}$ is unbounded.

Proof. Assume not. Then there exists M such that $|a_n| \le M$ for all n, and thus $|a_n/n| \le M/n$. We know $\lim\limits_{n\to\infty} M/n = 0$, and the squeeze theorem implies $\lim\limits_{n\to\infty} a_n/n = 0$, contradicting the assumption. □

2.2.21 If $0 \le \alpha \le \beta$, then $\lim \sqrt[n]{\alpha^n + \beta^n} = \beta$.

Proof. This is again the squeeze theorem. We know $\beta^n \le \alpha^n + \beta^n \le 2\beta^n$, so $\beta \le \sqrt[n]{\alpha^n + \beta^n} \le \sqrt[n]{2} \cdot \beta$. In class we proved $\lim\limits_{n\to\infty} \sqrt[n]{2} = 1$, so the sequence in the middle of the inequalities also has to converge to β. □

2.3.1 Prove the Comparison Theorem: If $\{a_n\}$ diverges to $+\infty$, and $a_n \le b_n$ for $n \ge n_1$, then $\{b_n\}$ also diverges to $+\infty$.

Proof. Let $M > 0$ be arbitrary. Then there exists n_2 such that $a_n > M$ for $n \ge n_2$. For $n \ge n^* = \max(n_1, n_2)$ we get $b_n \ge a_n > M$. □

2.3.3(a) Prove that $a_n = (n^2+1)/(n-2)$ diverges to $+\infty$.

Proof. For $n \ge 3$ we have $a_n \ge n^2/n = n$, and we already know that $\lim\limits_{n\to\infty} n = +\infty$. By comparison theorem $\{a_n\}$ diverges to $+\infty$, too. □

2.4.1 Give an example of a sequence that diverges to $+\infty$ but is not eventually increasing.

$a_n = n + (-1)^n$.

2.4.2 Give an example of a converging sequence that does not attain a maximum value.

$a_n = -1/n$.

2.4.11(a) Let the sequence $\{a_n\}$ be recursively defined by $a_1 = \sqrt{6}$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n \in \mathbb{N}$. Find the limit if it exists.

First claim: The sequence is increasing. Proof by induction: $a_2 = \sqrt{6+\sqrt{6}} \geq \sqrt{6} = a_1$. Assume that we know $a_{n+1} \geq a_n$. Then $a_{n+2} = \sqrt{6 + a_{n+1}} \geq \sqrt{6 + a_n} = a_{n+1}$.

Second claim: The sequence is bounded by 30. Proof by induction: $a_1 = \sqrt{6} \leq 30$. If we know $a_n \leq 30$, then $a_{n+1} = \sqrt{6 + a_n} \leq \sqrt{6 + 30} = 6 \leq 30$.

We know that monotone bounded sequences converge, so there exists some limit $A \in \mathbb{R}$. We can pass to the limit in the recursive equation to get $A = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + A}$ by limit theorems. From the equation $A = \sqrt{6 + A}$ we see that $A \geq 0$, and that $A^2 = 6 + A$. The two solutions of this equation are -2 and 3, and since $A \geq 0$, we get $A = 3$.

2.4.11(g) Same question as previous problem for $a_1 = 1$ and $a_{n+1} = 1 + a_n/2$.

First claim: The sequence is increasing. Proof by induction: $a_2 = 1 + 1/2 \geq 1 = a_1$. Assuming that we know $a_{n+1} \geq a_n$, we get $a_{n+2} = 1 + a_{n+1}/2 \geq 1 + a_n/2 = a_{n+1}$.

Second claim: The sequence is bounded by 30. Proof by induction: $a_1 = 1 \leq 30$. Assuming we know $a_n \leq 30$, we get $a_{n+1} = 1 + a_n/2 \leq 1 + 30/2 = 16 \leq 30$.

Again we know that the sequence converges to some limit $A \in \mathbb{R}$ because it is monotone and bounded. Passing to the limit in the recursive equation we get $A = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} (1 + a_n/2) = 1 + A/2$ by limit theorems. The equation $A = 1 + A/2$ has only one solution $A = 2$, so the limit is 2.

2.5.1 Let s_0 be an accumulation point of S. Prove that the following two statements are equivalent.

(a) Any neighborhood of s_0 contains at least one point of S different from s_0.

(b) Any neighborhood of s_0 contains infinitely many points of S.

The statement of this problem is unfortunately slightly screwed up. Kosmala assumes from the outset that s_0 is an accumulation point. Then (a) is always true, since it is the definition of accumulation points. So the direction "(b) implies (a)" is trivial, since (a) is true. It would also make both of these statements equivalent to completely unrelated true statements such as "(c) The angle sum in a Euclidean triangle is π."

We will show that (a) is equivalent to (b) for any $s_0 \in \mathbb{R}$, without any assumptions about s_0 being an accumulation point.

Proof. (b) $\implies$ (a): Let $\epsilon > 0$ be arbitrary. Then $(s_0 - \epsilon, s_0 + \epsilon)$ contains infinitely many points of S, so it contains at least two different points $s_1, s_2 \in S$. If $s_1 \neq s_0$, we have found a point of S in the neighborhood different from s_0. Otherwise we have $s_2 \neq s_1 = s_0$, so s_2 is the desired point.

(a) $\implies$ (b): Assume not. Then there exists $\epsilon > 0$ such that the neighborhood $(s_0 - \epsilon, s_0 + \epsilon)$ contains only finitely many points of S. Denote this finite set of points by T, and let $D = \{|s - s_0| : s \in T,\ s \neq s_0\} \cup \{\epsilon\}$. The set D is finite (since T is finite) and non-empty ($\epsilon \in D$), and all the numbers in D are positive. So $\epsilon_1 = \min D > 0$ exists.

Claim: There are no points of S different from s_0 in the neighborhood $(s_0 - \epsilon_1, s_0 + \epsilon_1)$.

Assume not. Then there exists $s \in S \cap (s_0 - \epsilon_1, s_0 + \epsilon_1)$ with $s \neq s_0$. This implies $0 < |s - s_0| < \epsilon_1 \leq \epsilon$, so $s \in T$. By definition $|s - s_0| \in D$, and thus $\epsilon_1 \leq |s - s_0|$. However, this contradicts $|s - s_0| < \epsilon_1$.

This contradiction proves the claim, and the claim itself contradicts (a), finishing the proof of this direction. $\square$

2.5.3 (a) Give an example of a sequence for which the set $S = \{a_n : n \in \mathbb{N}\}$ has exactly two accumulation points.

$a_n = (-1)^n(1 - 1/n)$, accumulation points 1 and -1.

2.5.3 (b) Give an example of a set S that contains infinitely many points but not every point of S is an accumulation point of S.

2.5.3 (c) Give an example of a set S where both $\sup S$ and exactly one accumulation point exist, but the values are not equal.

2.5.3 (d) Give an example of a set S where $\inf S$ and $\sup S$ are in S, but the accumulation point (points) is (are) not.

Example for (b), (c), and (d): $S = \{(-1)^n/n : n \in \mathbb{N}\}$. This set contains infinitely many points, the only accumulation point 0 is not in S, and both $\sup S = 1/2$ and $\inf S = -1$ are in S.