Homework Key, Advanced Calculus, Fall 2008

3.1.2 (a) $\lim_{x\to\infty} f(x) = 2$.

Proof. Let $\epsilon > 0$. Then for $x \in D$ with $x > 3 + 6/\epsilon$ we have $x - 3 > 6/\epsilon > 0$ and thus

$$|f(x) - 2| = \left|\frac{2x}{x-3} - 2\right| = \left|\frac{6}{x-3}\right| = \frac{6}{x-3} < \frac{6}{6/\epsilon} = \epsilon.$$

□

3.1.2 (b) $\lim_{x\to\infty} \frac{1-x^2}{x-2} = -\infty$.

Proof. Let $K > 0$. For $x > 2$ we have $x^2 - 1 > \frac{x^2}{2}$ and $x - 2 > x > 0$, so $\frac{x^2-1}{x-2} > \frac{x^2/2}{x} = x/2$. So for $x > \max(2, 2K)$ we get

$$\frac{1-x^2}{x-2} < -\frac{x}{2} < -K.$$

□

3.1.2 (c) $\lim_{x\to-\infty} f(x) = -\infty$.

Proof. Let $K > 0$. For $x < -2$ we have $x^2+1 > x^2 > 0$ and $0 < 2-x < -2x$, so $\frac{x^2+1}{2-x} > \frac{x^2}{-2x} = -x/2$. So for $x \in \mathbb{Q}$ with $x < -\max(2, 2K)$ we get

$$\frac{x^2+1}{x-2} < \frac{x}{2} < -K.$$

□

3.1.2 (d) $\lim_{x\to-\infty} \frac{-1}{x+1} = 0$.

Proof. Let $\epsilon > 0$. Then for $x < -1 - 1/\epsilon$ we have $x + 1 < -1/\epsilon < 0$ and thus $|x + 1| > 1/\epsilon > 0$. This implies $|f(x) - 0| = \left|\frac{-1}{x+1}\right| = \frac{1}{|x+1|} < \epsilon$. □

3.1.5 (c) $\lim_{n\to\infty} \frac{-2n}{3\sqrt{n^2-1}} = -2/3$.

Proof.

$$\frac{-2n}{3\sqrt{n^2-1}} = \frac{-2}{3\sqrt{1-1/n^2}},$$

and we know $\lim_{n\to\infty} 1/n^2 = 0$. Applying limit theorems yields the result. □

3.1.5 (g) $\lim_{x\to\infty} \sqrt{x} = \infty$.

Proof. Let $K > 0$. Then for $x > K^2$ we have $\sqrt{x} > K$. $\square$

3.1.5 (i) $\lim_{x\to-\infty} \frac{x-3}{|x-3|} = -1$.

Proof. For $x < 3$ we have $|x-3| = -(x-3)$, so $\frac{x-3}{|x-3|} = -1$. In particular, $\left|\frac{x-3}{|x-3|} - (-1)\right| = 0 < \epsilon$ for any $\epsilon > 0$. $\square$

3.1.6 (a) E.g., the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

is unbounded on $\mathbb{R}$, yet $\lim_{x\to\infty} f(x) = 0$ is finite.

3.1.6 (b) If $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$, then for every $\epsilon > 0$ there exists $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in D$ with $x \geq M$. Now if t is a number with $-t \in D$ and $t \leq -M$, then $-t \geq M$, and thus $|f(-t) - L| < \epsilon$. This shows that $\lim_{t\to-\infty} f(-t) = L$.

The other direction works exactly the same way, and the cases where $L = \pm\infty$ are simple modifications.

3.1.6 (c) $\lim_{x\to-\infty} 2^x e^{-x} = \lim_{x\to\infty} 2^{-x} e^x = \lim_{x\to\infty} (e/2)^x = \infty$ since $e/2 > 1$.

3.2.1 (a) $\lim_{x\to 0} (x+1)^3 = 1$.

Proof. Let $\epsilon > 0$. Then for $|x| < \delta := \min(1, \epsilon/7)$ we get $|x^2 + 3x + 3| \leq |x|^2 + 3|x| + 3 < 1 + 3 + 3 = 7$, because $|x| < 1$. This implies $|(x+1)^3 - 1| = |x^3 + 3x^2 + 3x| = |x||x^2 + 3x + 3| \leq 7|x| < \epsilon$, since $|x| < \epsilon/7$. $\square$

3.2.1 (d) $\lim_{x\to 0} \frac{x^2}{|x|} = 0$.

Proof. Let $\epsilon > 0$. Then for $0 < |x| < \delta := \epsilon$ we get $\left|\frac{x^2}{|x|} - 0\right| = |x| < \delta = \epsilon$. $\square$

3.2.1 (f) $\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} = 2$.

Proof. Let $\epsilon > 0$, and choose $\delta = \epsilon$. Let $x \geq 0$ with $|x-1| < \delta$. We have

$$\begin{aligned}\left|\frac{1-x}{1-\sqrt{x}} - 2\right| &= \left|\frac{1-x-2+2\sqrt{x}}{1-\sqrt{x}}\right| = \left|\frac{-1+2\sqrt{x}-x}{1-\sqrt{x}}\right| \\ &= \left|\frac{-(1-\sqrt{x})^2}{1-\sqrt{x}}\right| = |-1+\sqrt{x}| = \left|\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}+1}\right| \\ &= \frac{|x-1|}{\sqrt{x}+1} \leq |x-1| < \epsilon.\end{aligned}$$

□

3.2.8 (a) $\lim_{x\to 3/8} f(x) = 1$.

Proof. Let $\epsilon > 0$, and choose $\delta = 1/24$. If $x \in \mathbb{R}$ with $|x - 3/8| < 1/24$, then $1/3 = 3/8 - 1/24 < x < 3/8 + 1/24 < 1/2$, so $2 < 1/x < 3$. In particular, x can not be the reciprocal of an integer, and thus $f(x) = 1$, and $|f(x) - 1| = 0 < \epsilon$. □

3.2.8 (b) $\lim_{x\to -1/3} f(x) = 1$.

Proof. Let $\epsilon > 0$, and choose $\delta = 1/12$. If $x \in \mathbb{R}$ with $0 < |x - (-1/3)| < 1/12$, then $-1/2 < -1/3 - 1/12 < x < -1/3 + 1/12 = -1/4$, so $-4 < 1/x < -2$. In particular, the only way that x can be the reciprocal of an integer is $x = -1/3$. However, this contradicts $0 < |x - (-1/3)|$, and thus $f(x) = 1$, and $|f(x) - 1| = 0 < \epsilon$. □

3.2.8 (c) $\lim_{x\to 0} f(x)$ does not exist.

Proof. Let $x_n = 1/n$. Then $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) = 0$. Let $y_n = \sqrt{2}/n$. Then y_n is irrational for all n, and hence not the reciprocal of an integer. This implies $f(y_n) = 1$, and thus $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} f(y_n) = 1$. Since the limits of $\{f(x_n)\}$ and $\{f(y_n)\}$ are different, the limit of $f(x)$ as x tends to 0 does not exist. □

Squeeze Theorem If $f, g, h : D \to \mathbb{R}$ are functions with $\lim_{x\to\infty} f(x) = A = \lim_{x\to\infty} h(x)$, and $f(x) \leq g(x) \leq h(x)$ eventually, then $\lim_{x\to\infty} g(x) = A$.