Final Preparation Key, Advanced Calculus, Fall 2008

1. True or false?

(a) If $g \circ f$ is one-to-one, then f is one-to-one.

True. If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$, and by assumption this implies $x = y$.

(b) If $g \circ f$ is one-to-one, then g is one-to-one.

False. Counterexample: $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is not one-to-one, but with $f : (0, +\infty) \to \mathbb{R}$, $f(x) = x$, the composition $g \circ f : (0, +\infty) \to \mathbb{R}$, $(g \circ f)(x) = x^2$ is one-to-one.

(c) If (a_n) is a bounded sequence, then $b_n = \frac{a_1+\ldots+a_n}{n}$ converges.

False. Counterexamples for this are slightly tricky to write down, one possibility is the following. Let (a_n) be a sequence of numbers ± 1 constructed inductively as follows: Let $a_1 = -1$. Now assume that $a_1, \ldots, a_n$ have already been determined. Define $a_{n+1} = \ldots = a_{2n} = 1$, and $a_{2n+1} = \ldots = a_{5n} = -1$. Then $b_{2n} \geq 0$ and $b_{5n} \leq -1/5$. Since $|b_{2n} - b_{5n}| \geq 1/5$, this shows that (b_n) is not a Cauchy sequence, hence divergent.

(d) If $b_n = \frac{a_1+\ldots+a_n}{n}$ converges, then (a_n) is bounded.

False. One possible counterexample is the unbounded sequence $a_{2k-1} = \sqrt{k}$, $a_{2k} = -\sqrt{k}$. Then $b_{2k} = 0$, and $b_{2k-1} = \frac{\sqrt{k}}{2k-1}$, so $\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} b_{2k-1} = 0$. This shows that $\lim_{n\to\infty} b_n = 0$.

(e) Every continuous function $f : [0, +\infty) \to \mathbb{R}$ is bounded.

False. Counterexample $f(x) = x$.

(f) Every continuous function $f : [0, +\infty) \to \mathbb{R}$ with $\lim_{x\to\infty} f(x) = 0$ is bounded.

True. The assumption implies that f is eventually bounded, i.e., there exists M and K_1 such that $|f(x)| \leq K_1$ for $x \geq M$. Continuous functions on closed bounded intervals are bounded, so there exists K_2 such that $|f(x)| \leq K_2$ for $x \in [0, M]$. Then $|f(x)| \leq K = \max(K_1, K_2)$ for $x \in [0, +\infty)$.

2. Find the limit of these sequences or show that it does not exist.

(a) $\lim_{n\to\infty} \frac{n}{\sqrt{n+1}} = \infty$

(b) $\lim_{n\to\infty} \frac{3^n-(-2)^n}{3^n+(-2)^n} = 1$.

(c) $\lim_{n\to\infty} \frac{2^n+(-3)^n}{2^n-3^n}$ does not exist, there are two subsequences converging to 1 and -1, respectively. (The sequence with the typo on the sheet I handed out converges to -1.)

(d) $d_1 = 0$, and $d_{n+1} = d_n^2 + 1/4$ for $n \geq 1$.

$\lim_{n\to\infty} d_n = 1/2$. The sequence satisfies $0 \leq d_n \leq 1/2$ for all n, and it is increasing. As a monotone and bounded sequence it converges to a limit $D \in [0, 1/2]$ satisfying $D = D^2 + 1/4$. The only solution is $D = 1/2$.

(e) $e_1 = 1$, and $e_{n+1} = e_n^2 + 1/4$ for $n \geq 1$.

$\lim_{n\to\infty} e_n = +\infty$. This sequence satisfies $e_n \geq 1$ for all n, and it is increasing. A limit E would have to satisfy $E = E^2 + 1/4$ and $E \geq 1$. Since there is no such real number, the sequence diverges to $+\infty$.

3. Find the limits or show that they do not exist.

(a) $\lim_{x\to\infty} \frac{1+x^2}{x^3-x^2} = 0$

(b) $\lim_{x\to 1} \frac{1+x^2}{x^3-x^2}$ does not exist. The limit from the left and from the right are $-\infty$ and ∞, respectively.

(c) $\lim_{x\to 0} \frac{1+x^2}{x^3-x^2} = -\infty$.

4. Where are the following functions continuous?

(a) $f(x) = [x]$ is continuous in $\mathbb{R} \setminus \mathbb{Z}$.

(b) $g(x) = x$ for $x \in \mathbb{Q}$, and $g(x) = 1/x$ for $x \notin \mathbb{Q}$, is continuous in ± 1.

5. (a) Show that the equation $r^x + x = 0$ has exactly one real solution$x$ for every $r > 0$.

First of all, the function $f_r(x) = r^x + x$ is strictly increasing, so there can be at most one solution. In order to show existence, we observe that $\lim_{x\to\infty} f_r(x) = +\infty$ and $\lim_{x\to-\infty} f_r(x) = -\infty$. So there exist $a, b \in \mathbb{R}$ with $f_r(a) < 0$ and $f_r(b) > 0$. The intermediate value theorem shows that there exists $x(r)$ between a and b with $f_r(x(r)) = 0$.

(b)* Denoting this solution by $x(r)$, show that this is a continuous function of r.

This is a little harder. Assume that $x(r)$ is not continuous at some point $r_0 > 0$. Then there exists $\epsilon > 0$ and a sequence (r_n) of positive numbers converging to r_0 with $|x(r_n) - x(r_0)| \geq \epsilon$ for all n. This implies that $x(r_n) \geq x(r_0) + \epsilon$ for infinitely many n, or that $x(r_n) \leq x(r_0) - \epsilon$ for infinitely many n. Let us first assume that the first case applies, and let $x_\epsilon = x(r_0) + \epsilon$. This implies $0 = r_n^{x(r_n)} + x(r_n) \geq r_n^{x_\epsilon} + x_\epsilon$ for infinitely many n. Passing to the limit along this subsequence we get $0 \geq r_0^{x_\epsilon} + x_\epsilon > r_0^{x_0} + x_0 = 0$, a contradiction. The other case leads to a contradiction in the same way.