Final Preparation, MATH 234, Fall 2008

1. Find the acute angle between two diagonals of a cube.

This is the angle between the vectors $\mathbf{v} = \langle 1,1,1\rangle$ and $\mathbf{w} = \langle 1,1,-1\rangle$, so $\alpha = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \cos^{-1}\left(\frac{1}{3}\right) = 1.23 = 70.5°$.

2. Find a parametric equation for the line through $(-2,2,4)$ which is perpendicular to the plane $2x - y + 5z = 12$.

$\mathbf{r}(t) = \langle -2+2t, 2-t, 4+5t\rangle$.

3. Identify and sketch the surface $-4x^2 + y^2 - 4z^2 = 4$.

This is a hyperboloid of two sheets.

4. Sketch and find the length of the curve $\mathbf{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t\rangle$, $0 \le t \le \pi$. (**NOTE**: The version I handed out contains a typo leading to a harder integral.)

This curve looks somewhat like a deformed part of a helix along the x-axis. Its length is $L = \int_0^\pi \|\mathbf{r}'(t)\|\,dt = \int_0^\pi \sqrt{(3t^{1/2})^2 + (-2\sin 2t)^2 + (2\cos 2t)^2}\,dt = \int_0^\pi \sqrt{9t+4}\,dt = \left[\frac{2}{27}(9t+4)^{3/2}\right]_0^\pi = \frac{2}{27}((9\pi+4)^{3/2} - 4^{3/2}) = 12.99$

5. A particle starts at the origin with initial velocity $\langle 1,-1,3\rangle$. Its acceleration is $\mathbf{a}(t) = \langle 6t, 12t^2, -6t\rangle$. Find its position function.

Velocity is $\mathbf{v}(t) = \langle 1+3t^2, -1+4t^3, 3-3t^2\rangle$, position vector is $\mathbf{r}(t) = \langle t+t^3, -t+t^4, 3t-t^3\rangle$.

6. Find the directions in which the directional derivative of $f(x,y) = ye^{-xy}$ at the point $(0,2)$ has the value 1.

$\nabla f(x,y) = \langle -y^2e^{-xy}, e^{-xy} - xye^{-xy}\rangle$, so $\nabla f(0,2) = \langle -4, 1\rangle$. If $\mathbf{u}$ is a unit vector which makes an angle of α with $\langle -4,1\rangle$, then $D_{\mathbf{u}}f(0,2) = \|\nabla f(0,2)\|\cos\alpha = \sqrt{17}\cos\alpha$. This is equal to 1 if $\alpha = \pm\cos^{-1}\left(\frac{1}{\sqrt{17}}\right) = \pm 1.33$. The direction of $\langle -4,1\rangle$ is $\beta = \pi + \tan^{-1}\left(\frac{1}{-4}\right) = 2.90$, so the directions in which the directional derivative is 1 are $\beta \pm \alpha$, i.e., 1.57 and 4.22. The corresponding unit vectors are $\langle 0,1\rangle$ and $\langle -.47, -.88\rangle$.

7. Find equations of the tangent plane and the normal line to the surface $\sin(xyz) = x + 2y + 3z$ at the point $(2,-1,0)$.

The normal direction is given by the gradient of $F(x,y,z) = x + 2y + 3z - \sin(xyz)$. We get $\nabla F(x,y,z) = \langle 1 - yz\cos(xyz), 2 - xz\cos(xyz), 3 - xy\cos(xyz)\rangle$, so $\nabla F(2,-1,0) = \langle 1,2,5\rangle$. The normal line is given by $\mathbf{r}(t) = \langle 2+t, -1+2t, 5t\rangle$, and the tangent plane is given by $(x-2) + 2(y+1) + 5z = 0$.

8. Find the critical points and classify them for $f(x,y) = (x^2+y)e^{y/2}$.

$\nabla f(x,y) = \langle 2xe^{y/2}, e^{y/2} + \frac{1}{2}(x^2+y)e^{y/2}\rangle = e^{y/2}\langle 2x, 1 + \frac{1}{2}(x^2+y)\rangle$. Setting this equal to zero we get $x = 0$ and $1 + \frac{1}{2}y = 0$, so $y = -2$. Now $f_{xx} = 2e^{y/2}$, $f_{xy} = f_{yx} = xe^{y/2}$, and $f_{yy} = \frac{1}{2}e^{y/2}(1 + \frac{1}{2}(x^2+y)) + \frac{1}{2}e^{y/2}$, so at $x = 0$ and $y = -2$ we get $f_{xx} = 2 > 0$,

$f_{xy} = f_{yx} = 0$, and $f_{yy} = \frac{1}{2}$, and $D = f_{xx}f_{yy} - (f_{xy})^2 = 1 > 0$, so we have a local minimum.

9. Find the absolute maximum and minimum of $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the disk $x^2+y^2 \le 4$.

By the Extreme Value Theorem the absolute maximum and minimum exist. In order to find all possible candidates inside the disk we need all zeros of the gradient $\nabla f(x,y) = e^{-x^2-y^2}\langle -2x(x^2+2y^2)+2x, -2y(x^2+2y^2)+4y\rangle = e^{-x^2-y^2}\langle 2x(-x^2-2y^2+1), 2y(-x^2-2y^2+2)\rangle$. The first component is zero if $x=0$ or $x^2+2y^2=1$. The second component is zero if $y=0$ or $x^2+2y^2=2$. From this we get the five solutions $(0,0)$, $(0,\pm1)$, and $(\pm1,0)$. Plugging all of these into f we get $f(0,0)=0$, $f(0,\pm1)=2e^{-1}$, and $f(\pm1,0)=e^{-1}$.

In order to find possible candidates on the boundary it is probably easiest to parameterize it as $x = 2\cos t$ and $y = 2\sin t$. Then $f(2\cos t, 2\sin t) = e^{-4}(4+4\sin^2 t)$, which has minimum and maximum values at $t=0$ and $t=\pi/2$, respectively. (The same values repeat at $t=\pi$ and $t=3\pi/2$, but this is irrelevant for the question.) The values are $f(2,0)=4e^{-4}$ and $f(0,2)=8e^{-4}$, respectively.

Comparing all possible candidates, the maximum is the largest value $f(0,\pm1) = 2e^{-1} = .74$, and the minimum is the smallest value $f(0,0)=0$.

10. Find $\iint_D \frac{1}{1+x^2} dA$ where D is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

We get $\int_0^1 \int_0^y (1+x^2)\,dx\,dy = \int_0^1 (y+\frac{1}{3}y^3)dy = \frac{1}{2}+\frac{1}{12} = \frac{7}{12} = .58$.

11. Find $\iiint_H z^3\sqrt{x^2+y^2+z^2}dV$ where H is the solid hemisphere that lies above the xy-plane and has center the origin and radius 1.

Using spherical coordinates, we get

$$\int_0^{2\pi}\int_0^1\int_0^{\pi/2} (r\cos\phi)^3 \cdot r \cdot r^2 \sin\phi\, d\phi\, dr\, d\theta = \int_0^{2\pi}\int_0^1\int_0^{\pi/2} r^6\cos^3\phi\sin\phi\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 r^6\left[-\frac{1}{4}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/2} dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{4}r^6\, dr d\theta$$
$$= \frac{2\pi}{4\cdot 7} = \frac{\pi}{14} = .22.$$

12. Evaluate the line integral $\int_C x\,ds$ where C is the arc of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$.

Using the parameterization $x(t)=t$, $y(t)=t^2$, $0\le t\le 1$, we get $\int_0^1 x(t)\sqrt{x'(t)^2+y'(t)^2}\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt = \left[\frac{1}{12}(1+4t^2)^{3/2}\right]_0^1 = \frac{5^{3/2}-1}{12} = .85$.

13. Show that $\mathbf{F}(x,y) = \langle 4x^3y^2-2xy^3, 2x^4y-3x^2y^2+4y^3\rangle$ is conservative, and use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve $\mathbf{r}(t) = \langle t+\sin\pi t, 2t+\cos\pi t\rangle$, $0\le t\le 1$.

Denoting the components of $\mathbf{F}$ by P and Q, one easily checks $P_y = Q_x$, so $\mathbf{F}$ is conservative. Integration leads to the potential $f(x,y) = x^4y^2 - x^2y^3 + y^4$. The curve has initial point $\mathbf{r}(0) = \langle 0, 1\rangle$ and endpoint $\mathbf{r}(1) = \langle 1, 1\rangle$, so the value of the integral is $f(1,1) - f(0,1) = 1 - 1 = 0$.

14. Use Green's Theorem to evaluate $\int_C \sqrt{1+x^3}\, dx + 2xy\, dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$. (I.e., it is the boundary of the triangle, parameterized in positive orientation.)

If D denotes the inside of the triangle, we get $\iint_D (2y - 0) dA = \int_0^1 \int_0^{3x} 2y\, dy\, dx = \int_0^1 9x^2\, dx = 3$.