First Test, Advanced Calculus, Fall 2008

1. State the definitions of injectivity (one-to-one), surjectivity (onto), and bijectivity. (Be as rigorous as possible.)

A function $f : A \to B$ is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in A$.

A function $f : A \to B$ is surjective if $f(A) = B$.

A function is bijective if it is both surjective and injective.

2. Which of the following statements are true for arbitrary functions $f : X \to Y$ and sets $A, B \subseteq X$? Give a proof or counterexample.

(a) $A \subseteq B \implies f(A) \subseteq f(B)$.

True. If $y \in f(A)$, then there exists $x \in A$ with $f(x) = y$. By assumption $x \in B$, so $y \in f(B)$.

(b) $f(A) \subseteq f(B) \implies A \subseteq B$.

False. Counterexample: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $A = \{1\}$, $B = \{-1\}$. Then $f(A) = f(B) = \{1\}$, but $A \not\subseteq B$.

(c) A and B are disjoint if and only if $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint.

False. Counterexample: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $A = B = \{-1\}$. Then $f^{-1}(A) = f^{-1}(B) = \emptyset$ are disjoint (the empty set is disjoint from any other set), but A and B are not.

3. Find all $x \in \mathbb{R}$ that satisfy

$$\left|1 - \left|\frac{x-1}{x+1}\right|\right| < \frac{1}{2}.$$

The inequality is equivalent to

$$-\frac{1}{2} < 1 - \left|\frac{x-1}{x+1}\right| < \frac{1}{2},$$

which is equivalent to

$$-\frac{3}{2} < -\left|\frac{x-1}{x+1}\right| < -\frac{1}{2}.$$

If $x = -1$, this is undefined. Otherwise $-2|x+1| < 0$, so multiplying with it gives the equivalent inequalities

$$3|x+1| > 2|x-1| > |x+1|.$$

Now we consider the cases $x < -1$, $-1 < x < 1$, and $x \geq 1$ separately.

If $x \geq 1$, the inequalities are

$$3(x+1) > 2(x-1) > x+1.$$

The left inequality is equivalent to $x > -5$, the right inequality to $x > 3$. Both of these have to be satisfied, as well as $x \geq 1$, so we get $x \in (3, \infty)$ in this case.

If $-1 < x < 1$, we get

$$3(x+1) > -2(x-1) > x+1.$$

Here the left inequality is satisfied for $x > -\frac{1}{5}$, the right inequality for $x < \frac{1}{3}$, so we get $x \in \left(-\frac{1}{5}, \frac{1}{3}\right)$ in this case.

If $x < -1$, we get

$$-3(x+1) > -2(x-1) > -(x+1).$$

These are the same inequalities as in the case $x \geq 1$, except that the signs are reversed. So instead of $x > -5$ and $x > 3$ we get $x < -5$ and $x < 3$. Both of these have to be satisfied, as well as $x < -1$, so the solution set in this case is $x \in (-\infty, -5)$.

Combining all three cases, the inequality is satisfied for $x \in (-\infty, -5) \cup \left(-\frac{1}{5}, \frac{1}{3}\right) \cup (3, \infty)$.

4. Prove by induction that

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

for all $n \in \mathbb{N}$.

Proof. For $n = 1$ the statement is $1 \leq 1$, so it is true. We now need to show

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n} \implies \sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}.$$

We start with the left side and use the assumption.

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{1}{k^2} &= \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \\ &\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ &= 2 - \frac{n^2+n+1}{n(n+1)^2} \\ &< 2 - \frac{n^2+n}{n(n+1)^2} \\ &= 2 - \frac{1}{n+1}. \end{aligned}$$

□

5. (a) Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be bounded functions with $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in A$. Show that

$$\sup_{x\in A}(fg)(x) \leq \sup_{x\in A} f(x) \cdot \sup_{x\in A} g(x).$$

Proof. The supremum is an upper bound, so

$$0 \leq f(x) \leq \sup_{y\in A} f(y)$$

and

$$0 \leq g(x) \leq \sup_{y\in A} g(y)$$

for all $x \in A$. Multiplying both inequalities (this is allowed because all numbers are non-negative) we get

$$0 \leq f(x)g(x) \leq \sup_{y\in A} f(y) \cdot \sup_{y\in A} g(y)$$

for all $x \in A$. This shows that the right side of this inequality is an upper bound for fg on A. Since the supremum is the least upper bound, we get

$$\sup_{x\in A}(fg)(x) \leq \sup_{y\in A} f(y) \cdot \sup_{y\in A} g(y).$$

(At this point we could rename the variables from y to x, but this is not necessary. Don't get too attached to variable names.) □

(b) Show that the assumption that f and g be non-negative is essential, i.e., give an example where the inequality fails for bounded $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$.

One example is given by $A = [-1, 0]$ and $f(x) = g(x) = x$ for all $x \in [0, 1]$. Then $\sup_{x\in A} f(x) = \sup_{x\in A} g(x) = 0$, but $\sup_{x\in A}(fg)(x) = \sup_{-1\leq x\leq 0} x^2 = 1 \not\leq 0 \cdot 0$.