First Practice Test, Advanced Calculus, Fall 2009

1. State the definitions of injectivity (one-to-one), surjectivity (onto), and bijectivity. (Be as rigorous as possible.)

2. Which of the following statements are true for arbitrary functions $f : X \to Y$ and sets $A, B \subseteq X$? Give a proof or counterexample.

(a) $A \subseteq B \implies f(A) \subseteq f(B)$.

(b) $f(A) \subseteq f(B) \implies A \subseteq B$.

(c) A and B are disjoint if and only if $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint.

3. Find all $x \in \mathbb{R}$ that satisfy

$$\left| 1 - \left| \frac{x-1}{x+1} \right| \right| < \frac{1}{2}.$$

4. Prove by induction that $2^n < n!$ for all natural numbers $n \geq 4$.

5. (a) Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be bounded functions with $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in A$. Show that

$$\sup_{x \in A} (fg)(x) \leq \sup_{x \in A} f(x) \cdot \sup_{x \in A} g(x).$$

(b) Show that the assumption that f and g be non-negative is essential, i.e., give an example where the inequality fails for bounded $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$.