## Section 3.1

**2.** Suppose the multiplication $c\mathbf{x}$ is defined to produce $(cx_1, 0)$ instead of $(cx_1, cx_2)$, which of the eight conditions are not satisfied?

Solution: 3, 4, 5

**10.** Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors $(b_1, b_2, b_3)$ with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$.

Solution: a, d, e

**19.** Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution: $C(A)$ is a line (the $x$-axis), $C(B)$ is a plane (the $xy$-plane), $C(C)$ is a line (through $\mathbf{0}$ and $(1, 2, 0)$).

## Section 3.2

**1.** Reduce these matrices to their ordinary echelon forms $U$:

$$\text{(a) } A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{(b) } B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Solution:

$$\text{(a) } U_A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(b) } U_B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

**3.** By combining the special solutions in Problem 2, describe every solution to $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when there are no ___________.

Solution: All solutions to $A\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = x_2(-2, 1, 0, 0, 0) + x_4(0, 0, -2, 1, 0) + x_5(0, 0, -3, 0, 1)$, all solutions to $B\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = x_3(1, -1, 1)$. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when there are no <u>free variables</u>.

**9.** True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An $m$ by $n$ matrix has no more than $n$ pivot variables.

(d) An $m$ by $n$ matrix has no more than $m$ pivot variables.

Solution: (a) False, e.g., $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has two free variables. (b) True, because an invertible matrix has only $\mathbf{0}$ in its nullspace. (c) True because there are only $n$ variables total. (d) True because every row contains at most one pivot.

**31.** If the nullspace of $A$ consists of all multiples of $\mathbf{x} = (2, 1, 0, 1)$, how many pivots appear in $U$? What is $R$?

Solution: 3 pivots appear in $U$, and $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ in block form, where $I$ is the 3 by 3 identity matrix, and $F = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$. The 0 blocks have 3 and 1 columns, respectively, and they could have any number of rows.

## Section 3.3

**2.** Find the reduced row echelon form $R$ and the rank of these matrices:

(a) The 3 by 4 matrix with all entries equal to 4.

(b) The 3 by 4 matrix with $a_{ij} = i + j - 1$.

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

Solution: (a) Rank 1,

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Rank 2,

$$R = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Rank 1,

$$R = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**8.** Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \text{ and } B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \text{ and } M = \begin{bmatrix} a & b \\ c & \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix} \text{ and } M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}.$$

**12.** If $A$ has rank $r$, then it has an $r$ by $r$ submatrix $S$ that is invertible. Remove $m - r$ rows and $n - r$ columns to find an invertible submatrix $S$ inside $A$, $B$, and $C$. You could keep the pivot rows and pivot columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$S_A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad S_B = \begin{bmatrix} 1 \end{bmatrix} \qquad S_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**24.** What is the nullspace matrix $N$ (containing the special solutions) for $A$, $B$, $C$?

$$A = \begin{bmatrix} I & I \end{bmatrix} \text{ and } B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} I & I & I \end{bmatrix}.$$

Solution:

$$N_A = \begin{bmatrix} -I \\ I \end{bmatrix} \text{ and } N_B = \begin{bmatrix} -I \\ I \end{bmatrix} \text{ and } N_C = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}.$$

## Section 3.4

**1.** Execute the six steps of Worked Example 3.4A to describe the column space and nullspace of $A$ and the complete solution to $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

Solution: First we reduce the augmented matrix to row echelon form (step 1):

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}.$$

This shows that the condition for the existence of a solution is $b_3 + b_2 - 2b_1 = 0$ (step 2). The column space of of $A$ is the plane containing all linear combinations of the pivot columns $(2, 2, 2)$ and $(4, 5, 3)$, or equivalently all vectors $\mathbf{b}$ satisfying the condition $b_3 + b_2 - 2b_1 = 0$ (step 3). The free variables are $x_3$ and $x_4$, so the first special solution is $x_3 = 1$, $x_4 = 0$, and by back substitution with $U$ we get $x_2 = -x_3 - 2x_4 = -1$ and $x_1 = \frac{1}{2}(-4x_2 - 6x_3 - 4x_4) = -1$, so $\mathbf{s_1} = (-1, -1, 1, 0)$. Similarly, the second special solution has $x_3 = 0$, $x_4 = 1$, and back substitution gives $x_2 = -2$ and $x_1 = 2$, so $\mathbf{s_2} = (2, -2, 0, 1)$. The nullspace consists of all linear combinations $\mathbf{x_N} = c_1\mathbf{s_1} + c_2\mathbf{s_2}$ (step 4). The particular solution to $\mathbf{b} = (4, 3, 5)$ is obtained by setting $x_3 = x_4 = 0$ and solving by back substitution with $U$, so $x_2 = b_2 - b_1 - x_3 - x_4 = -1$ and $x_1 = \frac{1}{2}(b_1 - 4x_2 - 6x_3 - 4x_4) = 4$, so $\mathbf{x_P} = (4, -1, 0, 0)$. The complete solution is $\mathbf{x} = \mathbf{x_P} + \mathbf{x_N}$ (step 5). Continuing the elimination to the reduced form gives

$$\begin{bmatrix} 2 & 0 & 2 & -4 & 5b_1 - 4b_2 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & -2 & \frac{5}{2}b_1 - 2b_2 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}.$$

This shows directly from the matrix that the particular solution is always $(\frac{5}{2}b_1 - 2b_2, b_2 - b_1, 0, 0)$, and that the special solutions are $(-1, -1, 1, 0)$ and $(2, -2, 0, 1)$ (step 6).

**5.** Under what conditions on $b_1$, $b_2$, $b_3$ is this system solvable? Include $\mathbf{b}$ as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3 \end{aligned}$$

Solution: Again we reduce the augmented matrix by elimination:

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{bmatrix}.$$

The condition is $b_3 - b_2 - 2b_1 = 0$. Nullspace solutions are multiples of the special solution $\mathbf{s} = (2, 0, 1)$, particular solution is $(5b_1 - 2b_2, b_2 - 2b_1, 0)$, so the complete solution when the condition holds is $(5b_1 - 2b_2, b_2 - 2b_1, 0) + x_3(2, 0, 1)$.

**17.** Solution: The largest possible rank of a 6 by 4 matrix is <u>4</u>. Then there is a pivot in every <u>column</u> of $U$ and $R$. The solution to $A\mathbf{x} = \mathbf{b}$ <u>is unique</u>. The nullspace of $A$ is <u>zero</u>. An example is $A = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (where $I$ is the 4 by 4 identity matrix, and 0 is a 2 by 4 matrix of zeroes).

**24.** Give examples of matrices $A$ for which the number of solutions to $A\mathbf{x} = \mathbf{b}$ is

(a) 0 or 1, depending on $\mathbf{b}$

(b) $\infty$, regardless of $\mathbf{b}$

(c) 0 or $\infty$, depending on $\mathbf{b}$

(d) 1, regardless of $\mathbf{b}$

Solution: (a) $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**31.** Find matrices $A$ and $B$ with the given property or explain why you can't:

(a) The only solution of $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) The only solution of $B\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution: (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, (b) Impossible because $B$ would have to be a 2 by 3 matrix without free variables, i.e., of rank 3. However, the rank of a matrix can not be greater than the number of rows, in this example it could be at most 2.

**34.** Suppose you know that the 3 by 4 matrix $A$ has the vector $\mathbf{s} = (2, 3, 1, 0)$ as the only special solution to $A\mathbf{x} = \mathbf{0}$.

(a) What is the rank of A and the complete solution to $A\mathbf{x} = \mathbf{0}$?

(b) What is the exact row reduced echelon form $R$ of $A$?

(c) How do you know that $A\mathbf{x} = \mathbf{b}$ can be solved for all $\mathbf{b}$?

Solution: (a) The rank is 3, and the complete solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = x_3\mathbf{s} = x_3(2, 3, 1, 0)$.

(b)

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) The matrix has full row rank, so there always is at least one solution (and since there is a free variable, really infinitely many).

## Section 3.5

**1.** Show that $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ are independent but $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, $\mathbf{v_4}$ are dependent:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v_4} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + c_4\mathbf{v_4} = 0$ or $A\mathbf{x} = \mathbf{0}$. The $\mathbf{v}$'s go in the columns of $A$.

Solution: Any four vectors in $\mathbf{R}^3$ are dependent (since the dimension of $\mathbf{R}^3$ is 3), and the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ are independent because they are columns of an invertible 3 by 3 matrix. Solving the system $A\mathbf{x} = \mathbf{0}$ is finding the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Special solution here is $x_4 = 1$, $x_3 = -4$, $x_2 = 1$, $x_1 = 1$, so the nullspace consists of all $(c_1, c_2, c_3, c_4) = x_4(1, 1, -4, 1)$.

**5.** Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

Solution: Elimination on the matrices formed from these column vectors gives for (a):

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & \frac{18}{5} \end{bmatrix}$$

So this matrix has rank 3 and the columns are independent.

For (b) we get

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix},$$

so this matrix has rank 2 and the columns are dependent.

**13.** Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of $A$, (b) column space of $U$, (c) row space of $A$, (d) row space of $U$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: All dimensions are 2, (c) and (d) are the same space.

**16.** Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

(d) The column space and the nullspace of $I$ (4 by 4)

Solution: (a) $(1, 1, 1, 1)$; (b) $(1, 0, 0, -1)$, $(0, 1, 0, -1)$, $(0, 0, 1, -1)$; (c) $(-1, 1, 0, 1)$, $(-1, 1, 1, 0)$; (d) The columns of $I$ for the column space, empty basis for the nullspace.

**24.** True or false (give a good reason):

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

Solution: (a) False, e.g., $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$

(b) False, e.g., $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) True, both equal the rank of $A$.

(d) False, the columns might not be linearly independent, e.g., take either example from (a) or (b).

## Section 3.6

**2.** Find bases and dimensions for the four subspaces associated with $A$ and $B$:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

Solution: $\dim C(A) = \dim C(A^T) = \dim N(A^T) = 1$, $\dim N(A) = 2$, bases are $(1, 2)$ for $C(A)$, $(1, 2, 4)$ for $C(A^T)$, $(-2, 1, 0)$ and $(-4, 0, 1)$ for $N(A)$, and $(-2, 1)$ for $N(A^T)$.

For the matrix $B$ we have $\dim C(B) = \dim C(B^T) = 2$, $\dim N(B) = 1$, $\dim N(B^T) = 0$, bases are $(1, 2)$ and $(2, 5)$ for $C(B)$, $(1, 2, 4)$ and $(2, 5, 8)$ for $C(B^T)$, $(-4, 0, 1)$ for $N(B)$, and nothing for $N(B^T)$.

**3.** Find a basis for each of the four subspaces associated with $A$:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: $(1, 1, 0)$ and $(3, 4, 1)$ for $C(A)$, $(0, 1, 2, 3, 4)$ and $(0, 1, 2, 4, 6)$ for $C(A^T)$, $(1, 0, 0, 0, 0)$, $(0, -2, 1, 0, 0)$ and $(0, 2, 0, -2, 1)$ for $N(A)$, and $(1, -1, 1)$ for $N(A^T)$.

**7.** Suppose the 3 by 3 matrix $A$ is invertible. Write down bases for the four subspaces for $A$, and also for the 3 by 6 matrix $B = \begin{bmatrix} A & A \end{bmatrix}$.

Solution: The row and column space of $A$ are both $\mathbf{R}^3$, so a basis for those is $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The nullspace and the left nullspace of $A$ are both zero, so the bases for those are empty. The column space of $B$ is the same as the column space of $A$, so the same basis works there, the row space has basis $(1, 0, 0, 1, 0, 0)$, $(0, 1, 0, 0, 1, 0)$, and $(0, 0, 1, 0, 0, 1)$. The left nullspace of $B$ is zero, so its basis is empty, and the nullspace of $B$ has basis $(-1,0,0,1,0,0)$, $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$.

**13.** True or false (with a reason or counterexample):

(a) If $m = n$, then the row space of $A$ equals the column space.

(b) The matrices $A$ and $-A$ share the same four subspaces.

(c) If $A$ and $B$ share the same four subspaces, then $A$ is a multiple of $B$.

Solution: (a) False, e.g., $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(b) True. Every column of $-A$ is a linear combination of columns of $A$ (in fact, just a multiple of one column), and vice versa, so the column spaces are the same. Every solution to $A\mathbf{x} = \mathbf{0}$ also solves $-A\mathbf{x} = \mathbf{0}$ and vice versa, so the nullspaces are the same. Since $(-A)^T = -(A^T)$, the same is true for the column space and nullspace of $A^T$ and $(-A)^T$, so the four subspaces are the same for $A$ and $-A$.

(c) False, any two $n$ by $n$ invertible matrices share the same subspaces, and they need not be multiples of each other, e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**25.** True or false (with a reason or counterexample):

(a) $A$ and $A^T$ have the same number of pivots.

(b) $A$ and $A^T$ have the same left nullspace.

(c) If the row space equals the column space then $A^T = A$.

(d) If $A^T = -A$ then the row space equals the column space.

Solution: (a) True, both equal the rank of the matrix.

(b) False, any non-square matrix gives a counterexample, and it is not even true for square matrices, e.g., $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) False, any non-symmetric invertible matrix is a counterexample, e.g., $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(d) True, because $C(A^T) = C(-A^T)$ by problem 13b, and $-A^T = A$ by assumption, so $C(-A^T) = C(A)$.