Key to Final Practice Problems, M221-01, Fall 2010

1. True or false? Justify your answers.

(a) If a linear system of equations $A\mathbf{x} = \mathbf{b}$ has more than one solution, then it has infinitely many solutions.

True, because the number of solutions to linear systems is always either 0, 1, or ∞.

(b) If A and B are two invertible n by n matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

False, the correct equation is $(AB)^{-1} = B^{-1}A^{-1}$. For a simple example, use $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, so $(AB)^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$. Also, $A^{-1} = A$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, so $A^{-1}B^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \neq (AB)^{-1}$. (You don't really need to find an example here, but it might be instructive to see how you come up with something like this. Both A and B are matrices corresponding to row operations. A multiplies the first row with -1, and B adds the first row to the second row. Now AB first adds the first row to the second row and then multiplies the first row by -1, whereas BA first multiplies the first row by -1 and then adds it to the second row. You can see that the outcome of these two is not the same, so $(AB)^{-1}$ is not the same as $(BA)^{-1} = A^{-1}B^{-1}$ either. As an exercise, try to see what the inverse matrices do as row operations.)

(c) Given an n by n matrix A, the matrix $B = A + A^T$ is always symmetric.

True, $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

(d) The solutions to $A\mathbf{x} = \mathbf{b}$ form a subspace.

False, unless $\mathbf{b} = \mathbf{0}$ the space of solutions does not contain the zero vector, and thus can not be a subspace. (In the case $\mathbf{b} = \mathbf{0}$ it is a subspace, namely the nullspace of A.)

(e) If A is a singular n by n matrix, then A^2 is also singular.

True, for several different reasons. E.g., singular matrices have determinant zero, and determinants of products of matrices are the products of the determinants, so $|A^2| = |A|^2 = 0^2 = 0$, and so A^2 is singular.

(f) If λ is not an eigenvalue of A, then $A - \lambda I$ is invertible.

True, the eigenvalues are the solutions of $|A - \lambda I| = 0$, so if λ is not an eigenvalue, $|A - \lambda I| \neq 0$, and any matrix with a non-zero determinant is invertible.

2. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are unit vectors such that $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 45°, find the length of the sum $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

Solution: This can be found by elementary geometry, but dot products make it pretty easy: $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|^2 = (\mathbf{u} + \mathbf{v} + \mathbf{w}) \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 + 2\mathbf{u} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{w} + 2\mathbf{v} \cdot \mathbf{w}$. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are unit vectors, $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 = 1$, and since $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, we get $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0$. Lastly, the angle between $\mathbf{v}$ and $\mathbf{w}$ being 45° gives us $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos 45° = \frac{1}{\sqrt{2}}$, so putting everything together we get $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|^2 = 3 + \frac{2}{\sqrt{2}} = 3 + \sqrt{2}$, so $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \sqrt{3 + \sqrt{2}}$.

3. What 3 by 3 matrix multiplies (x, y, z) to give $(x + 2z, x + y - z, 2x + z)$?

Solution: $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

4. Solve the following system of equations by elimination with matrices.

$$\begin{aligned} x + 2z &= -1 \\ x + y - z &= 2 \\ 2x + z &= 2 \end{aligned}$$

Solution: We already have the matrix A from problem 3, now we just have to perform Gauss elimination on the augmented matrix and then solve by back substitution:

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & -3 & 3 \end{bmatrix}.$$

Now back substitution gives $-3z = 3$, so $z = -1$, $y - 3z = 3$, so $y = 3 + 3z = 0$, and $x + 2z = -1$, so $x = -1 - 2z = 1$.

5. Find the determinant and inverse of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}.$$

Solution: For the determinant we can use cofactor expansion with respect to the second row and get $|A| = 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$. For the inverse we use Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & 3 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1 & -1/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & -1/3 & 2 & 2/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1 & -1/3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1/3 & 1 & 2/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1 & -1/3 \end{bmatrix},$$

so

$$A^{-1} = \begin{bmatrix} -1/3 & 1 & 2/3 \\ 0 & 1 & 0 \\ 2/3 & -1 & -1/3 \end{bmatrix}.$$

6. Find the rank, and dimensions and bases for all four subspaces of

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 2 & 0 & -2 \end{bmatrix}.$$

Gauss elimination gives

$$\longrightarrow \begin{bmatrix} 2 & -2 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -2 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This show that the rank is 2, and thus $\dim C(A) = \dim C(A^T) = 2$, $\dim N(A) = 3 - 2 = 1$, $\dim N(A^T) = 4 - 2 = 2$. A basis for $C(A)$ is given by the first two columns of A, i.e, $(2, 1, 1, 2)$ and

$(-2,1,-1,0)$, a basis for $C(A^T)$ is given by the first two rows of either A or U, i.e., $(2,-2,-2)$ and $(1,1,-1)$, or $(2,-2,-2)$ and $(0,2,0)$. For the nullspace we have a basis formed by the one special solution $(1,0,1)$. The left nullspace we get by elimination on the transpose A^T:

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ -2 & 1 & -1 & 0 \\ -2 & -1 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives us a basis for $N(A^T)$, the two special solutions $(-1/2,0,1,0)$ and $(-1/2,-1,0,1)$.

7. With the same matrix A as in problem 6, find conditions on (b_1,b_2,b_3,b_4) for solvability of $A\mathbf{x}=\mathbf{b}$. Find the complete solution for $\mathbf{b}=(0,2,0,2)$.

Solution: Typically, one would solve this by elimination on the augmented matrix and read off the conditions on the right-hand side of the zero rows at the end. However, we can also use the result for problem 6 to answer these questions. The system is solvable whenever $\mathbf{b}$ is in the column space. Since the column space and the left nullspace are orthogonal complements, $\mathbf{b}$ is in the column space if and only if it is perpendicular to all elements in the left nullspace. This is true if and only if $\mathbf{b}$ is perpendicular to the two vectors $(-1/2,0,1,0)$ and $(-1/2,-1,0,1)$ which form the basis of $N(A^T)$. So we get the two conditions $-b_1/2+b_3=0$ and $-b_1/2-b_2+b_4=0$ for solvability.

In order to find a particular solution, we eliminate on the augmented matrix first:

$$\begin{bmatrix} 2 & -2 & -2 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -2 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -2 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we set the free variable $x_3=0$, and solve by back substitution: $2x_2=2$, so $x_2=1$, and $2x_1-2x_2-2x_3=0$, so $x_1=x_2+x_3=1$, giving the particular solution $\mathbf{x}_p=(1,1,0)$. We already know the nullspace solution $\mathbf{x}_n=x_3(1,0,1)$ from problem 6, so the complete solution is $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_n=(1,1,0)+x_3(1,0,1)=(1+x_3,1,x_3)$.

8. Let A be an invertible 3 by 3 matrix. Find the four subspaces, rank, and dimension of the block matrices

$$B=\begin{bmatrix} A & A \end{bmatrix} \text{ and } C=\begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.$$

Solution: The columns of B are the same as the columns of A, only repeated, so the column spaces of A and B are the same. Since A is invertible, $C(A)=C(B)=\mathbb{R}^3$. This also shows that B has rank $3=\dim C(B)=\dim C(B^T)$, and thus $\dim N(B^T)=3-3=0$ and $\dim N(B)=6-3=3$. Performing Gauss-Jordan elimination on an invertible matrix always produces the identity matrix, so the same elimination performed on B gives $\begin{bmatrix} I & I \end{bmatrix}$. A basis for the nullspace is given by the special solutions $(-1,0,0,1,0,0)$, $(0,-1,0,0,1,0)$, and $(0,0,-1,0,0,1)$, a basis for the row space is given by the three rows $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$, and $(0,0,1,0,0,1)$. Performing elimination on C gives $\begin{bmatrix} A & A \\ 0 & -A \end{bmatrix} \longrightarrow \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \longrightarrow \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. This shows that the rank of C is 6, and thus $\dim C(C)=\dim C(C^T)=6$, and $\dim N(C)=\dim N(C^T)=0$. So $C(C)=C(C^T)=\mathbb{R}^6$, and $N(C)=N(C^T)=\{\mathbf{0}\}$.

9. If $A\mathbf{x}=\mathbf{b}$ has infinitely many solutions, and $\mathbf{c}$ is another vector, how many solutions can $A\mathbf{x}=\mathbf{c}$ possibly have?

Solution: None at all or infinitely many. The assumptions imply that $\mathbf{b}$ is in the column space $C(A)$, and that the matrix A has at least one free variable. If $\mathbf{c}$ is also in $C(A)$, there is a particular solution, and then the complete solution has at least one free parameter, so there are infinitely many solutions. If $\mathbf{c}$ is not in $C(A)$, then $A\mathbf{x} = \mathbf{c}$ has no solutions at all.

10. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Solution:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda^2+4)$$

has roots $\lambda_1 = 3$, $\lambda_{2/3} = \pm 2i$. We get the eigenvector for $\lambda_1 = 3$ by elimination on $A - 3I$:

$$\begin{bmatrix} -3 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -3-4/3 \end{bmatrix},$$

giving the eigenvector $(0, 1, 0)$. For $\lambda_2 = 2i$ we get

$$\begin{bmatrix} -2i & 0 & -2 \\ 0 & 3-2i & 0 \\ 2 & 0 & -2i \end{bmatrix} \longrightarrow \begin{bmatrix} -2i & 0 & -2 \\ 0 & 3-2i & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

giving the eigenvector $(i, 0, 1)$. For $\lambda_3 = -2i$ we get the complex conjugate eigenvector $(-i, 0, 1)$.

11. A 2 by 2 matrix has eigenvalues 1 and 2 with corresponding eigenvectors $\mathbf{x}_1 = (1, 1)$ and $\mathbf{x}_2 = (-1, 1)$. Find A. (Hint: Diagonalization.)

Solution: The assumptions give us $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, so

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$