First Practice Test Key, M221-01, Fall 2010

1. True or false? Justify your answers.

(a) The dot product of two vectors is 0 if they point in the same direction.

False, it is 0 if they are perpendicular.

(b) A linear system of equations $A\mathbf{x} = \mathbf{b}$ with coefficient matrix A and right-hand side $\mathbf{b}$ always has exactly one solution.

False, we have seen many examples with no solutions at all, or with infinitely many solutions.

(c) If the columns of A lie in a plane, then $A\mathbf{x} = \mathbf{b}$ only has a solution when $\mathbf{b}$ is in the same plane.

True, $A\mathbf{x}$ is a linear combination of the columns of A (with coefficients given by the entries in the vector $\mathbf{x}$).

(d) If a matrix A is invertible, then $A\mathbf{x} \neq \mathbf{0}$ for every non-zero vector $\mathbf{x}$.

True, a matrix is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$. If there was a non-zero vector $\mathbf{x}$ with $A\mathbf{x} = \mathbf{0}$, then we would have two solutions ($\mathbf{0}$ and $\mathbf{x}$) for $\mathbf{b} = \mathbf{0}$.

2. Find the angle between the vectors $\mathbf{v} = (3, 4, 0)$ and $\mathbf{w} = (2, 1, 2)$.

$$\alpha = \cos^{-1} \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \cos^{-1} \frac{10}{15} = 0.841 = 48.2°.$$

3. Which value of c will give linearly dependent columns?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & c \end{bmatrix}$$

A matrix is invertible if and only if it has all non-zero pivots. Performing elimination on A gives

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & c-2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & c \end{bmatrix},$$

where we first subtracted the first row from the third, then subtracted twice the second row from the third. Now the matrix is invertible if and only if the third pivot c is non-zero. The matrix has linearly dependent columns if and only if it is non-invertible, so if and only if $c = 0$.

4. Solve the following system of equations by elimination with matrices.

$$\begin{aligned} x - y &= 2 \\ -x + y + 2z &= 0 \\ 2x + y - z &= 1 \end{aligned}$$

Starting with the augmented matrix, the elimination proceeds as follows.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 0 \\ 2 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & -1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 3 & -1 & -3 \\ 0 & 0 & 2 & 2 \end{bmatrix},$$

where we added the first row to the second, subtracted twice the first row from the third, and then switched the second and third rows. From this upper triangular form, back substitution gives

$$\begin{array}{lcl} 2z = 2 & \Longrightarrow & z = 1 \\ 3y - z = -3 & \Longrightarrow & y = \frac{z-3}{3} = -\frac{2}{3} \\ x - y = 2 & \Longrightarrow & x = 2 + y = \frac{4}{3} \end{array}$$

5. Invert the matrix A by Gauss-Jordan elimination:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

We start with the same elimination steps as in the previous problem, only with a different augmented matrix.

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix},$$

Now we continue eliminating the elements above the main diagonal, and at the end we divide each row by the pivot.

$$\longrightarrow \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -3/2 & 1/2 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/6 & 1/3 \\ 0 & 3 & 0 & -3/2 & 1/2 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/6 & 1/3 \\ 0 & 1 & 0 & -1/2 & 1/6 & 1/3 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{bmatrix}.$$

So the inverse is

$$A^{-1} = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ -1/2 & 1/6 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

6. What is the inverse of the block matrix

$$A = \begin{bmatrix} I & U \\ 0 & I \end{bmatrix}$$

where the blocks are 3 by 3 matrices?

The solution is

$$A^{-1} = \begin{bmatrix} I & -U \\ 0 & I \end{bmatrix}$$