Second Practice Test Key, M221-01, Fall 2010

1. True or false? Justify your answers.

(a) The column space of AB is always contained in the column space of A.

True, because the columns of AB are linear combinations of the columns of A, so they are contained in the column space of A.

(b) The row space of AB is always contained in the row space of A.

False, e.g., the row space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the x-axis, but if we choose $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has the y-axis as its row space.

(c) The nullspace of a 3 by 4 matrix always has dimension at least 1.

True, the rank can be at most 3, and there are 4 variables total, so there is always at least one free variable, and the dimension of the nullspace is the number of free variables.

(d) The left nullspace of a 3 by 4 matrix always has dimension at least 1.

False, if the matrix has rank 3, then the left nullspace has dimension 0.

2. Let A be a matrix and R its row reduced form. Which of the four associated subspaces (column space, row space, nullspace and left nullspace) are the same for A and R? Justify your answer.

The row space and the nullspace are the same, the column space and left nullspace can be different.

3. If A is a 7 by 3 matrix, and its left nullspace has dimension two, find the dimensions of the other three associated subspaces.

The assumption implies that the rank of A is $r = 5$ which can't be true because it can be at most 3, so the assumption is impossible.

4. Find the row reduced form R for

$$A = \begin{bmatrix} -1 & 2 & 0 & -1 \\ 2 & -3 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Elimination leads to R (the last matrix in the following line):

$$\longrightarrow \begin{bmatrix} -1 & 2 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 2 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. For the same matrix as in 4, find the general solution to $A\mathbf{x} = \mathbf{0}$.

Nullspace matrix is

$$N = \begin{bmatrix} -2 & 3 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The general solution are all linear combinations of the two columns of N, i.e., $\mathbf{x_N} = x_3(-2, -1, 1, 0) + x_4(3, 2, 0, 1)$.

6. For the same matrix as in 4, find the complete solution to $A\mathbf{x} = (0, 1, 1)$.

Doing the elimination with the augmented matrix, we get

$$\begin{bmatrix}-1 & 2 & 0 & -1 & 0\\ 2 & -3 & 1 & 0 & 1\\ 1 & -1 & 1 & -1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 0 & -1 & 0\\ 0 & 1 & 1 & -2 & 1\\ 0 & 1 & 1 & -2 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 0 & -1 & 0\\ 0 & 1 & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$
$$\longrightarrow \begin{bmatrix}-1 & 0 & -2 & 3 & -2\\ 0 & 1 & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 2 & -3 & 2\\ 0 & 1 & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The particular solution is $\mathbf{x_P} = (2,1,0,0)$, so the complete solution is $\mathbf{x} = \mathbf{x_P} + \mathbf{x_N} = (2,1,0,0) + x_3(-2,-1,1,0) + x_4(3,2,0,1)$.

7. With the same matrix as in 4, for which vectors $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

For all vectors in the column space of A, i.e., all linear combinations of the first two columns. To get an explicit equation, we do elimination with the augmented matrix $\begin{bmatrix}A & \mathbf{b}\end{bmatrix}$ with a general vector (b_1, b_2, b_3).

$$\begin{bmatrix}-1 & 2 & 0 & -1 & b_1\\ 2 & -3 & 1 & 0 & b_2\\ 1 & -1 & 1 & -1 & b_3\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 0 & -1 & b_1\\ 0 & 1 & 1 & -2 & b_2+2b_1\\ 0 & 1 & 1 & -2 & b_3+b_1\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 0 & -1 & b_1\\ 0 & 1 & 1 & -2 & b_2+2b_1\\ 0 & 0 & 0 & 0 & b_3-b_2-b_1\end{bmatrix}.$$

Here we already see that the condition for solvability is $b_3 - b_2 - b_1 = 0$.

8. With the same matrix as in 4, find bases for the column space, row space, nullspace, and left nullspace of A, as well as for its reduced row form R.

For the column space we can take the pivot columns of A, i.e., $(-1,2,1)$ and $(2,-3,-1)$ form a basis for $C(A)$. For the row space we can take the pivot rows of either A or R, e.g., $(1,0,2,-3)$ and $(0,1,1,-2)$ form a basis for $C(A^T)$. The columns of the nullspace matrix N from problem 5 form a basis of the nullspace, i.e., $(-2,-1,1,0)$ and $(3,2,0,1)$ form a basis for $N(A)$. For the left nullspace, we can reduce A^T to reduced row form:

$$\begin{bmatrix}-1 & 2 & 1\\ 2 & -3 & -1\\ 0 & 1 & 1\\ -1 & 0 & -1\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 1\\ 0 & 1 & 1\\ 0 & 1 & 1\\ 0 & -2 & -2\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

There is only one special solution $(-1,-1,1)$, and it forms a basis for the left nullspace $N(A^T)$.

For the matrix R, the row space and nullspace are the same as for A, so we can take the same bases. A basis for the column space are the pivot columns of R, i.e., $(1,0,0)$ and $(0,1,0)$, and a basis for the left nullspace is the vector $(0,0,1)$.

The dimensions are the numbers of basis elements, i.e., $\dim C(A) = \dim C(A^T) = \dim N(A) = \dim C(R) = \dim C(R^T) = \dim N(R) = 2$, and $\dim N(A^T) = \dim N(R^T) = 1$.