Third Practice Test Key, M221-01, Fall 2010

1. True or false? Justify your answers.

(a) $\det(2I) = 2$

False, the value is 2^n for an n by n matrix.

(b) If an n by n matrix A has n different eigenvalues, then A is diagonalizable.

True, then it has n linearly independent eigenvectors.

(c) If an n by n matrix A has n different eigenvalues, then A is invertible.

False, this is only the case if all eigenvalues are non-zero.

(d) If a matrix A has only real eigenvalues, then $A^2 + I$ is invertible.

True, the eigenvalues of $A^2 + I$ are of the form $\lambda^2 + 1$, where λ are the eigenvalues of A. If λ is real, then $\lambda^2 + 1$ is never zero, and so $A^2 + I$ does not have zero as an eigenvalue, and is thus invertible.

2. Find the determinant of

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \\ 5 & 1 & 2 & 3 \\ 1 & 6 & 4 & 3 \end{bmatrix}.$$

Cofactor expansion along the third row, then the first row gives

$$|A| = -3 \begin{vmatrix} 1 & 0 & 2 \\ 5 & 1 & 3 \\ 1 & 6 & 3 \end{vmatrix} = -3 \left(\begin{vmatrix} 1 & 3 \\ 6 & 3 \end{vmatrix} + 2 \begin{vmatrix} 5 & 1 \\ 1 & 6 \end{vmatrix} \right) = -3(-15 + 58) = 129.$$

3. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$

For a triangular matrix the eigenvalues are on the diagonal, so they are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

Eigenvectors are found by elimination, subtracting λ from the diagonal and finding the special nullspace solutions.

$$\lambda_1 = 1: \quad \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{special solution} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\lambda_2 = 2: \quad \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{special solution} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

$$\lambda_3 = 3: \quad \begin{bmatrix} -2 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{special solution} \quad \mathbf{x}_3 = \begin{bmatrix} 9/2 \\ 3 \\ 1 \end{bmatrix}.$$

4. (a) For which parameters a is the following matrix diagonalizable?

$$A = \begin{bmatrix} 1 & 0 \\ 1+a & a^2 \end{bmatrix}.$$

The matrix is lower triangular, so the eigenvalues are the diagonal elements 1 and a^2. Unless $a = \pm 1$, these two eigenvalues are different, and so the matrix is diagonalizable. If $a = -1$, then A is the identity matrix, in particular it is already diagonal, and so it is clearly diagonalizable. If $a = 1$, then A is not diagonalizable, since $A - I$ in that case has rank 1, so it has a one-dimensional nullspace, and so there do not exist two linearly independent eigenvectors.

(b) Find the diagonalization $A = S\Lambda S^{-1}$ for the case $a = 2$.

For $a = 2$ we have eigenvalues 1 and 4, so we find the eigenvectors as before:

$$\lambda_1 = 1: \qquad \begin{bmatrix} 1-1 & 0 \\ 3 & 4-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix}, \qquad \text{special solution} \qquad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = 4: \qquad \begin{bmatrix} 1-4 & 0 \\ 3 & 4-4 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 3 & 0 \end{bmatrix}, \qquad \text{special solution} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$S = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

so the diagonalization has the form

$$\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

5. (a) The matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$$

has an eigenvector $\mathbf{x}_1 = (1, 1, 0)$. Find the corresponding eigenvalue λ_1.

$$A\mathbf{x}_1 = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{x}_2,$$

so $\lambda_1 = 2$.

(b) Knowing that another eigenvalue is $\lambda_2 = 4$ (you don't have to verify this), what is the third eigenvalue λ_3?

Since the trace of A is the sum of diagonal elements $3 + 0 + 1 = 4$, and this is also the sum of the eigenvalues, we get $\lambda_3 = 4 - \lambda_1 - \lambda_2 = 4 - 2 - 4 = -2$.

6. Let R be a 3 by 3 matrix corresponding to a $120°$ rotation about the line $x = y = z$ in space.

(a) Find one eigenvalue and corresponding eigenvector of R. (Hint: Try the direction of the axis of rotation.)

Since the axis stays fixed under a rotation, any vector in the direction of the axis is an eigenvector with eigenvalue $\lambda_1 = 1$. As an eigenvector we can take $\mathbf{x}_1 = (1, 1, 1)$.

(b) What are the other two eigenvalues? (Hint: If you apply R three times you get a rotation by $360°$ which is the same as not rotating at all, so $R^3 = I$.)

Since $R^3 = I$ has 1 as a triple eigenvalue, any eigenvalue λ of R must satisfy $\lambda^3 = 1$. So the only possible eigenvalues are 1 and $e^{\pm 2\pi i/3} = \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Complex eigenvalues always come in complex conjugate pairs, so there are only two possibilities. Either R has 1 as a triple eigenvalue, or R has all these three numbers as eigenvalues. Deciding between these two alternatives is a little tricky, but if you guessed that R has three different eigenvalues, so the other ones are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$, then you were right.

Here is the argument: Let $\mathbf{u}$ be a unit vector perpendicular to the axis direction $(1,1,1)$, and let $\mathbf{v}$ be another unit vector perpendicular to both $\mathbf{u}$, and $(1,1,1)$. There are two possible choices for $\mathbf{v}$, and we can choose the one in the direction of the rotation, i.e., the one closer to $R\mathbf{u}$. (Draw a picture of the $\mathbf{u}$-$\mathbf{v}$ plane to visualize this and the following.) Then $R\mathbf{u} = -\frac{1}{2}\mathbf{u} + \frac{\sqrt{3}}{2}\mathbf{v}$ and $R\mathbf{v} = -\frac{1}{2}\mathbf{v} - \frac{\sqrt{3}}{2}\mathbf{u}$. So if we form the complex vector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$, we get $R\mathbf{w} = -\frac{1}{2}\mathbf{u} + \frac{\sqrt{3}}{2}\mathbf{v} + i\left(-\frac{1}{2}\mathbf{v} - \frac{\sqrt{3}}{2}\mathbf{u}\right) = -\frac{1}{2}(\mathbf{u} + i\mathbf{v}) - i\frac{\sqrt{3}}{2}(\mathbf{u} + i\mathbf{v}) = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)(\mathbf{u} + i\mathbf{v})$.

(Alright, there are other possible arguments to show this, but I have to admit that none of them would be suitable for a M221 test. So in the unlikely event that you found this or some other correct argument on your own, you should really consider math grad school...)