M221: VECTORS WITH MUTUALLY OBTUSE ANGLES

Question: How many vectors with mutally obtuse angles (or equivalently, mutually negative dot products) can we have in the plane or in space?

This question is easy to answer in the plane. We can have three such vectors, but not four or more of them. (Three vectors at angles 0, $120°$, and $240°$ will do, for four vectors the total angle around them would be larger than $4 \cdot 90° = 360°$ which is obviously impossible.)

In three-dimensional space we can have four such vectors, but not five or more. One example of four such vectors is

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix},$$

(Check that this is indeed an example.)

On the other hand, we can not have five or more such vectors. The way we argue is called an "indirect proof". We assume that we have such a configuration and show that it leads to a contradiction. This establishes that the assumption is wrong, and that we can have at most four such vectors.

Let us assume that we have 5 vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}$ in space with mutually negative dot products. By rotating and scaling them we can arrange it so that $\mathbf{u} = (0, 0, 1)$. Now $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ must satisfy $v_3 = \mathbf{u} \cdot \mathbf{v} < 0$, and similarly $w_3 = \mathbf{u} \cdot \mathbf{w} < 0$, i.e., they must lie below the xy-plane. Now we look at the projections of these two vectors on the xy-plane, i.e., the vectors $\mathbf{v}^* = (v_1, v_2)$ and $\mathbf{w}^* = (w_1, w_2)$. Then $\mathbf{v}^* \cdot \mathbf{w}^* = v_1 w_1 + v_2 w_2 < v_1 w_1 + v_2 w_2 + v_3 w_3 = \mathbf{v} \cdot \mathbf{w} < 0$. (The inequality is true because $v_3 w_3 > 0$ as a product of two negative numbers.) In other words, the projections of $\mathbf{v}$ and $\mathbf{w}$ make an obtuse angle as well. Instead of $\mathbf{v}$ and $\mathbf{w}$ we could take any pair from $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}$, so the projections of these four vectors make mutually obtuse angles in the xy-plane. However, we saw that it was not possible to have four vectors like that. This means that our assumption in the beginning was wrong, and that we can only have four vectors with mutually obtuse angles in space.