Practice Midterm Exam, Math 547, Fall 2010

1. Are the following statements true or false for a general measure space $(X, \mathbf{X}, \mu)$? Justify your answers.

(a) Every measurable function is the pointwise limit of a sequence of simple functions (ϕ_n).

(b) If $f \geq 0$ is integrable, then $\sqrt{f}$ is integrable.

(c) If $f \leq 0$ is integrable, then $\int f \, d\mu = \inf \int \phi \, d\mu$, where the infimum is taken over all simple functions ϕ satisfying $\phi \geq f$.

(d) If f is integrable and $\int_E f \, d\mu = 0$ for all $E \in \mathbb{X}$, then $f = 0$ μ-a.e.

2. Let $(X, \mathbb{X})$ be a measurable space, and let $f : X \to \mathbb{R}$ be a function which assumes only rational values, i.e. $f(x) \in \mathbb{Q}$ for all $x \in X$. Show that f is measurable if and only if $f^{-1}(a) \in \mathbb{X}$ for all $a \in \mathbb{R}$. If we drop the assumption of rational values, which direction fails? Give an example.

3. Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} e^{-nx}$ for $x \geq 0$. Show that $\displaystyle\int_0^{\infty} f(x) \, dx = -\frac{1}{3}$.

4. Let $p, q > 1$ be numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that (f_n) and (g_n) are sequences in L_p and L_q, respectively, with $f_n \to f$ in L_p, and $g_n \to g$ in L_q. Show that $f_n g_n \to fg$ in L_1.

5. Let $p \in [1, \infty)$ and $f \in L_p = L_p(X, \mathbf{X}, \mu)$. Show that there exists a constant $C \in \mathbb{R}$ with $\mu(\{|f| \geq \alpha\}) \leq \dfrac{C}{\alpha^p}$ for all $\alpha > 0$. Show that the converse is not true, i.e., there are measurable functions which satisfy this inequality for some C and all $\alpha > 0$, but are not in L_p.