16.1 Vector Fields

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Vector Fields

Definition

An n-dimensional vector field is a function assigning to each point P in an n-dimensional domain an n-dimensional vector $\mathbf{F}(P)$.

2-D Vector Fields

$\mathbf{F}(x, y) = \langle \mathbf{F}_1(x, y), \mathbf{F}_2(x, y) \rangle$.

3-D Vector Fields

$\mathbf{F}(x, y, z) = \langle \mathbf{F}_1(x, y, z), \mathbf{F}_2(x, y, z), \mathbf{F}_3(x, y, z) \rangle$.

Remarks

- The dimensions of the domain and the vector have to match.
- We will assume that all vector fields are smooth.

Real-Life Examples of Vector Fields

- Velocity fields
  - Flow around an airfoil
  - Flow of ocean currents
  - Wind velocity on the surface of the earth
- Force fields
  - Magnetic fields
  - Gravitational fields
  - Electric fields

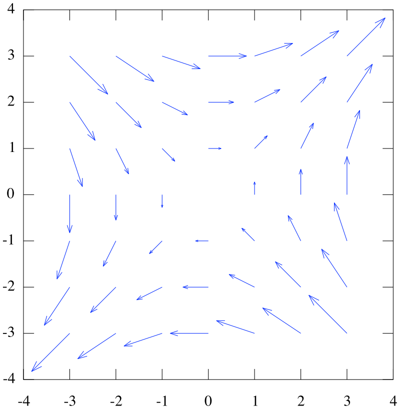
Mathematical examples of vector fields

- 2-D vector fields
  - $\mathbf{F}(x, y) = \langle y, x \rangle$
  - Unit radial vector field:
    $$\mathbf{e}_r(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$$
- 3-D vector fields
  - $\mathbf{F}(x, y, z) = \langle 1, x + z, 2y \rangle$
  - Unit radial vector field:
    $$\mathbf{e}_r(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$$
  - Gravitational force of a mass M at the origin on a mass m at $\mathbf{x}$:
    $$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^2}\mathbf{e}_r(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^3}\mathbf{x}$$

Sketching a 2-D Vector Field

$$F(x, y) = \langle y, x \rangle$$

$$F(0,0) = \langle 0, 0 \rangle$$
$$F(1,0) = \langle 0, 1 \rangle$$
$$F(1,1) = \langle 1, 1 \rangle$$
$$F(0,1) = \langle 1, 0 \rangle$$
$$F(-1,1) = \langle 1, -1 \rangle$$
$$F(-1,0) = \langle 0, -1 \rangle$$
$$F(-1,-1) = \langle -1, -1 \rangle$$
$$F(0,-1) = \langle -1, 0 \rangle$$
$$F(1,-1) = \langle -1, 1 \rangle$$
$$F(2,0) = \langle 0, 2 \rangle$$
$$F(2,1) = \langle 1, 2 \rangle$$



Unit Vector Fields

Definition

A vector field $\mathbf{F}(\mathbf{x})$ is a unit vector field if $\|\mathbf{F}(\mathbf{x})\| = 1$ for all $\mathbf{x}$.

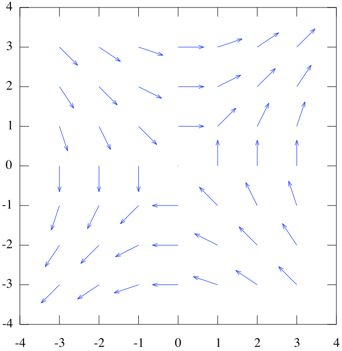
Examples

- $\mathbf{F}(x, y) = \langle 1, 0 \rangle$
- Unit radial vector field:
$$\mathbf{e}_r(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$$
- $\mathbf{G}(x, y) = \langle y, x \rangle$ is not a unit vector field, $\|\mathbf{G}(x, y)\| = \sqrt{x^2 + y^2} \neq 1$.
- $\mathbf{H}(x, y) = \frac{1}{\sqrt{x^2+y^2}} \langle y, x \rangle$ is a unit vector field parallel to $\mathbf{G}(x, y)$.

Sketch of a Unit Vector Field

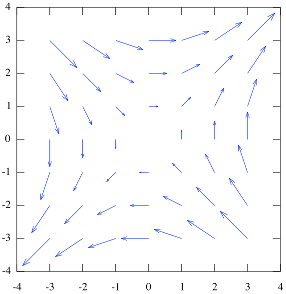
$$\mathbf{H}(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \langle y, x \rangle = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right\rangle$$

$$\mathbf{H}(1, 0) = \langle 0, 1 \rangle$$
$$\mathbf{H}(1, 1) = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$$
$$\mathbf{H}(0, 1) = \langle 1, 0 \rangle$$
$$\mathbf{H}(-1, 1) = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$$
$$\mathbf{H}(-1, 0) = \langle 0, -1 \rangle$$
$$\mathbf{H}(-1, -1) = \langle -1/\sqrt{2}, -1/\sqrt{2} \rangle$$
$$\mathbf{H}(0, -1) = \langle -1, 0 \rangle$$
$$\mathbf{H}(1, -1) = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$$
$$\mathbf{H}(2, 0) = \langle 0, 1 \rangle$$
$$\mathbf{H}(2, 1) = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$$

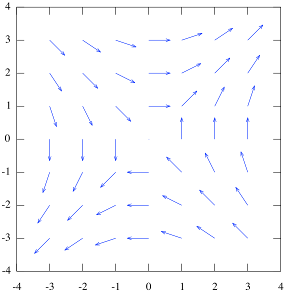


Unit vs. Non-unit Vector Field

$$\mathbf{F}(x,y) = \langle y, x \rangle$$



$$\mathbf{H}(x,y) = \frac{1}{\sqrt{x^2+y^2}}\langle y, x \rangle$$



Radial Vector Fields

Definition

A vector field $\mathbf{F}(\mathbf{x})$ is a radial vector field if $\mathbf{F}(\mathbf{x}) = f(\|\mathbf{x}\|)\mathbf{x}$ with some function $f(r)$.

Remarks

- A radial vector field is a vector field where all the vectors point straight towards ($f(r) < 0$) or away ($f(r) > 0$) from the origin, and which is rotationally symmetric.
- The definition in the textbook is wrong.

Radial Vector Field Examples

Definition

A vector field $\mathbf{F}(\mathbf{x})$ is a radial vector field if $\mathbf{F}(\mathbf{x}) = f(\|\mathbf{x}\|)\mathbf{x}$ with some function $f(r)$.

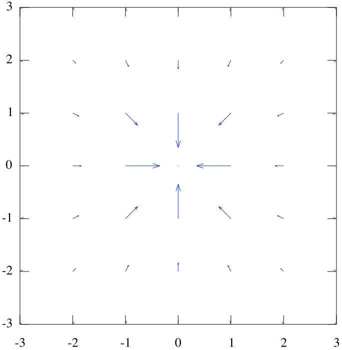
Examples

- $\mathbf{F}(\mathbf{x}) = \mathbf{x}$
- Unit radial vector field:
$$\mathbf{e}_r(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$$
- Gravitational field:
$$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^2}\mathbf{e}_r(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^3}\mathbf{x}$$

Sketching a Radial Vector Field

$$\mathbf{F}(\mathbf{x}) = -\frac{1}{\|\mathbf{x}\|^2}\mathbf{e}_r(\mathbf{x}) = -\frac{1}{\|\mathbf{x}\|^3}\mathbf{x}$$

$$\mathbf{F}(1,0) = \langle -1, 0\rangle$$
$$\mathbf{F}(1,1) = \langle -1/\sqrt{8}, -1/\sqrt{8}\rangle$$
$$\mathbf{F}(0,1) = \langle 0, -1\rangle$$
$$\mathbf{F}(-1,1) = \langle 1/\sqrt{8}, -1/\sqrt{8}\rangle$$
$$\mathbf{F}(-1,0) = \langle 1, 0\rangle$$
$$\mathbf{F}(-1,-1) = \langle 1/\sqrt{8}, 1/\sqrt{8}\rangle$$
$$\mathbf{F}(0,-1) = \langle 0, 1\rangle$$
$$\mathbf{F}(1,-1) = \langle -1/\sqrt{8}, 1/\sqrt{8}\rangle$$
$$\mathbf{F}(2,0) = \langle -1/4, 0\rangle$$
$$\mathbf{F}(2,1) = \langle -2/\sqrt{125}, 1/\sqrt{125}\rangle$$



Conservative Vector Fields

Definition

A vector field $\mathbf{F}(\mathbf{x})$ is conservative if $\mathbf{F}(\mathbf{x}) = \nabla V(\mathbf{x})$ for some smooth scalar function $V(\mathbf{x})$.

The function $V(\mathbf{x})$ is the (scalar) potential of the vector field.

Examples

- $\mathbf{F}(x, y) = \langle y, x \rangle$ is conservative with potential $V(x, y) = xy$, because $\nabla V(x, y) = \langle y, x \rangle$.
- $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ is conservative with potential $V(x, y, z) = \frac{x^2+y^2+z^2}{2}$.
- $\mathbf{F}(x, y) = \langle x, x \rangle$ is not conservative. Why? If there was a potential with $\nabla V = \langle x, x \rangle$, this would imply $V_x = x$ and $V_y = x$, so $V_{xy} = 0$ and $V_{yx} = 1$. By Clairaut's Theorem, $V_{xy} = V_{yx}$, but $0 \neq 1$. This proves that no such potential exists.

A Criterion for Conservative Vector Fields

$$\langle F_1, F_2 \rangle = \nabla V = \langle V_x, V_y \rangle \implies \frac{\partial \mathbf{F}_1}{\partial y} = V_{xy} = V_{yx} = \frac{\partial \mathbf{F}_2}{\partial x}$$

Theorem

Every conservative vector field $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle$ *satisfies*

$$\frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}$$

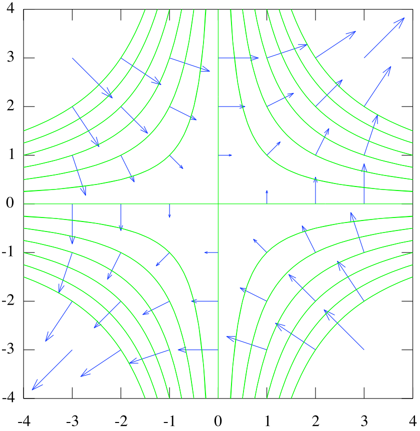
Theorem

Every conservative vector field
$\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$ *satisfies*

$$\frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}, \quad \frac{\partial \mathbf{F}_1}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial x}, \quad \frac{\partial \mathbf{F}_2}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial y},$$

A Conservative Vector Field

$$F(x,y) = \langle y, x \rangle \text{, with potential } V(x,y) = xy$$



Vector field (blue) and contour map of the potential (green)

More on Conservative Vector Fields

Theorem

Conservative vector fields are perpendicular to the contour lines of the potential function.

Theorem

If $\mathbf{F}$ *is a conservative vector field in a connected domain, then any two potentials differ by a constant.*

In other words, potentials are unique up to an additive constant.

More Examples

Which of the following vector fields are conservative? Can you find a potential?

- $\mathbf{F}(x, y) = \langle 1, 2 \rangle$
- Conservative, $V(x, y) = x + 2y$
- $\mathbf{F}(x, y) = \langle x^2, y \rangle$
- Conservative, $V(x, y) = \frac{1}{3}x^3 + \frac{1}{2}y^2$
- $\mathbf{F}(x, y) = \langle y, x^2 \rangle$
- Not conservative, $\frac{\partial \mathbf{F}_1}{\partial y} = 1 \neq 2x = \frac{\partial \mathbf{F}_2}{\partial x}$
- $\mathbf{F}(x, y, z) = \langle x, 2, x \rangle$
- Not conservative, $\frac{\partial \mathbf{F}_1}{\partial z} = 0 \neq 1 = \frac{\partial \mathbf{F}_3}{\partial x}$
- $\mathbf{F}(x, y, z) = \langle z, 2, x \rangle$
- Conservative, $V(x, y, z) = xz + 2y$