16.2 Line Integrals

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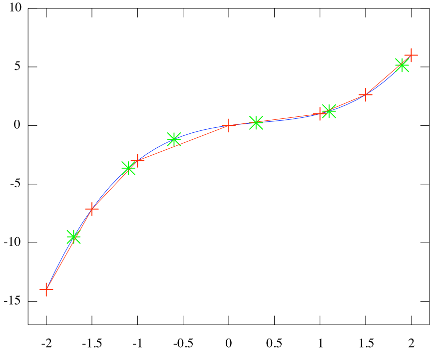
Scalar Line Integrals

Definition

$$\int_{\mathcal{C}} f(\mathbf{x})\, ds = \lim_{\{\Delta s_i\}\to 0} \sum_{i=1}^{N} f(P_i)\Delta s_i$$

Riemann sum illustration. Subdivision into arcs by red crosses, sample points P_i as green stars, the Δs_i are the length of the red line segments.



Applications of Scalar Line Integrals

- Length

$$\text{length}(\mathcal{C}) = \int_{\mathcal{C}} 1\, ds$$

- Mass of a wire of linear density $\rho(\mathbf{x})$

$$\text{mass} = \int_{\mathcal{C}} \rho(\mathbf{x}) ds$$

- Surface area of (one side of) a fence of height $h(\mathbf{x})$

$$\text{area} = \int_{\mathcal{C}} h(\mathbf{x}) ds$$

- Electric Potential of a charged wire with linear charge density $\rho(\mathbf{x})$

$$V(\mathbf{y}) = k \int_{\mathcal{C}} \frac{\rho(\mathbf{x})}{\|\mathbf{x} - \mathbf{y}\|} ds, \text{ where } k \text{ is } \textit{Coulomb's constant}.$$

Calculation of Scalar Line Integrals

Theorem

Let $\mathbf{c}(t)$ *be a parametrization of a curve* $\mathcal{C}$ *for* $a \leq t \leq b$*. If* $f(\mathbf{x})$ *and* $\mathbf{c}'(t)$ *are continuous, then*

$$\int_{\mathcal{C}} f(\mathbf{x})\, ds = \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| dt.$$

Differential version

$$ds = \|\mathbf{c}'(t)\| dt.$$

Scalar Line Integral Example I

Example

Calculate $\int_{\mathcal{C}} xy + z\,ds$ over the helix $\mathcal{C}$ parametrized by $\mathbf{c}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$ for $0 \leq t \leq \pi$.

Compute *ds*

$$\begin{aligned} \mathbf{c}'(t) &= \langle -3\sin 3t, 3\cos 3t, 4 \rangle \\ \|\mathbf{c}'(t)\| &= \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = \sqrt{25} = 5 \\ ds &= \|\mathbf{c}'(t)\| dt = 5dt. \end{aligned}$$

Scalar Line Integral Example II

Example

Calculate $\int_{\mathcal{C}} xy + z\, ds$ over the helix $\mathcal{C}$ parametrized by $\mathbf{c}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$ for $0 \leq t \leq \pi$.

Compute ds

$$ds = \|\mathbf{c}'(t)\| dt = 5dt.$$

Write out integrand and evaluate

$$\begin{aligned}
\int_{\mathcal{C}} xy + z\, ds &= \int_0^\pi (\cos 3t \sin 3t + 4t) 5\, dt \\
&= 5 \int_0^\pi \cos 3t \sin 3t\, dt + 20 \int_0^\pi t\, dt \\
&= 5 \left[\frac{1}{6} \sin^2 3t \right]_0^\pi + 20 \left[\frac{1}{2} t^2 \right]_0^\pi = 10\pi^2
\end{aligned}$$

Scalar Line Integral Applied Example I

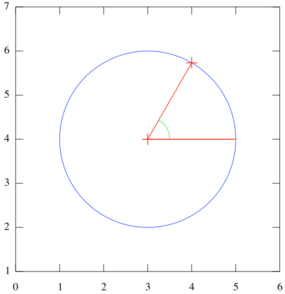
Example

Find the mass of a wire in the shape of a circle of radius 2 centered at (3,4) with linear mass density $\rho(x, y) = y^2$.

Parametrize $\mathcal{C}$

$$\begin{aligned}\mathbf{c}(t) &= \langle 3, 4\rangle + 2\langle \cos t, \sin t\rangle \\ &= \langle 3 + 2\cos t, 4 + 2\sin t\rangle,\end{aligned}$$

where $0 \leq t \leq 2\pi$.



Scalar Line Integral Applied Example II

Example

Find the mass of a wire in the shape of a circle of radius 2 centered at (3,4) with linear mass density $\rho(x, y) = y^2$.

Parametrize $\mathcal{C}$

$$\mathbf{c}(t) = \langle 3 + 2\cos t, 4 + 2\sin t\rangle, \quad 0 \le t \le 2\pi.$$

Compute ds

$$\begin{aligned} \mathbf{c}'(t) &= \langle -2\sin t, 2\cos t\rangle \\ \|\mathbf{c}'(t)\| &= \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2 \\ ds &= \|\mathbf{c}'(t)\| dt = 2dt. \end{aligned}$$

Scalar Line Integral Applied Example III

Example

Find the mass of a wire in the shape of a circle of radius 2 centered at (3,4) with linear mass density $\rho(x, y) = y^2$.

Parametrize $\mathcal{C}$, compute ds

$$\mathbf{c}(t) = \langle 3 + 2\cos t, 4 + 2\sin t \rangle, \quad 0 \le t \le 2\pi, \quad ds = 2dt.$$

Write out integral and evaluate

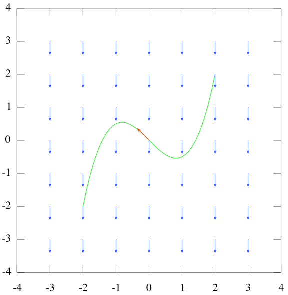
$$\begin{aligned} m &= \int_{\mathcal{C}} y^2\, ds = \int_0^{2\pi} (4 + 2\sin t)^2 \cdot 2\, dt \\ &= \int_0^{2\pi} 32 + 32\sin t + 8\sin^2 t\, dt \\ &= [32t - 32\cos t + 4(t - \sin t \cos t)]_0^{2\pi} \\ &= 64\pi + 8\pi = 72\pi \end{aligned}$$

Vector Line Integrals

Definition

The line integral of a vector field **F** along an oriented curve $\mathcal{C}$ is the scalar line integral of the tangential component of **F**.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T}) ds.$$



Vector field $\mathbf{F}(x, y) = \langle 0, -1 \rangle$ with an oriented curve $\mathcal{C}$ (in green) and one unit tangent vector (in red).

Question

Is $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ positive, negative, or zero? Positive

Applications of Vector Line Integrals

- Work done by a force field $\mathbf{F}$ on a particle moving along $\mathcal{C}$

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}.$$

- Work done against a force field $\mathbf{F}$, moving along $\mathcal{C}$

$$W = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}.$$

Computing a Vector Line Integral

Theorem

If $\mathbf{c}(t)$ *is a regular parametrization of an oriented curve* $\mathcal{C}$ *for* $a \leq t \leq b$*, then*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

Differential version

$$d\mathbf{s} = \mathbf{c}'(t)dt$$

$\mathbf{c}(t)$ is regular if it is smooth and $\mathbf{c}'(t) \neq \mathbf{0}$.

Scalar vs. Vector Line Integrals

- Orientation matters for vector line integrals: If $-\mathcal{C}$ is the curve $\mathcal{C}$ in negative orientation, then

$$\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$

- Orientation does not matter for scalar line integrals:

$$\int_{-\mathcal{C}} \mathbf{F}\, ds = \int_{\mathcal{C}} \mathbf{F} ds$$

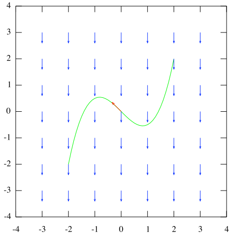
- $$\left| \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \right| \leq \int_{\mathcal{C}} \|\mathbf{F}\| ds$$

  Why? The tangential component $\mathbf{F} \cdot \mathbf{T}$ always satisfies $|\mathbf{F} \cdot \mathbf{T}| \leq \|\mathbf{F}\|$.

Vector Line Integral Example I

Example

Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = \langle 0, -1 \rangle$, and $\mathcal{C}$ is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.



Parametrize $\mathcal{C}$

$$\mathbf{c}(t) = \langle t, \frac{1}{2}t^3 - t \rangle, \quad -2 \leq t \leq 2.$$

Problem

The orientation is wrong. What to do about this? The easiest solution is to calculate with the wrong orientation and reverse the sign of the result.

Vector Line Integral Example II

Example

Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = \langle 0, -1 \rangle$, and $\mathcal{C}$ is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

Parametrize $-\mathcal{C}$

$$\mathbf{c}(t) = \langle t, \frac{1}{2}t^3 - t \rangle, \quad -2 \le t \le 2.$$

Write out integral and evaluate

$$\begin{aligned}
\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_{-2}^{2} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{-2}^{2} \langle 0, -1 \rangle \cdot \langle 1, \frac{3}{2}t^2 - 1 \rangle\, dt \\
&= \int_{-2}^{2} -\frac{3}{2}t^2 + 1\, dt = \left[-\frac{1}{2}t^3 + t \right]_{-2}^{2} \\
&= -4 + 2 - 4 + 2 = -4
\end{aligned}$$

Vector Line Integral Example III

Example

Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = \langle 0, -1 \rangle$, and $\mathcal{C}$ is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

Write out integral and evaluate

$$\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -4$$

Fix the orientation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -(-4) = 4.$$

Alternative Notation for Vector Line Integrals

2-D Version

For $\mathbf{F}(x, y) = \langle \mathbf{F}_1(x, y), \mathbf{F}_2(x, y) \rangle$ we write $d\mathbf{s} = \langle dx, dy \rangle$ and

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} F_1\, dx + F_2\, dy$$

3-D Version

For $\mathbf{F}(x, y, z) = \langle \mathbf{F}_1(x, y, z), \mathbf{F}_2(x, y, z), \mathbf{F}_3(x, y, z) \rangle$ we write $d\mathbf{s} = \langle dx, dy, dz \rangle$ and

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz$$

Alternative Notation II

3-D Parametrized Version

For $\mathbf{F}(x, y, z) = \langle \mathbf{F}_1(x, y, z), \mathbf{F}_2(x, y, z), \mathbf{F}_3(x, y, z) \rangle$ and a regular parametrization $\mathbf{c}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$ of $\mathcal{C}$,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz \\ &= \int_a^b F_1(\mathbf{c}(t))x'(t) + F_2(\mathbf{c}(t))y'(t) + F_3(\mathbf{c}(t))z'(t)\, dt \end{aligned}$$

Differential version

$$dx = x'(t)\, dt, \quad dy = y'(t)\, dt, \quad dz = z'(t)\, dt$$

3-D Line Integral Example I

Example

Calculate $\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

Parametrize $\mathcal{C}$

$$\mathbf{c}(t) = \langle 1,1,1\rangle + t(\langle 3,2,0\rangle - \langle 1,1,1\rangle) = \langle 1+2t, 1+t, 1-t\rangle, \quad 0 \le t \le 1.$$

Write out differentials

$$\begin{aligned} dx &= x'(t)\,dt = 2\,dt \\ dy &= y'(t)\,dt = dt \\ dz &= z'(t)\,dt = -dt \end{aligned}$$

3-D Line Integral Example II

Example

Calculate $\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

Parametrize $\mathcal{C}$, write out differentials

$$\mathbf{c}(t) = \langle 1+2t, 1+t, 1-t \rangle, \quad 0 \le t \le 1, \quad dx = 2\,dt,\ dy = dt,\ dz = -dt$$

Write out integral and evaluate

$$\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz$$
$$= \int_0^1 (1+t)(1-t)\cdot 2 + (1+2t)(1-t) + (1+2t)(1+t)(-1)\,dt$$

3-D Line Integral Example III

Example

Calculate $\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

Write out integral and evaluate

$$\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz$$

$$= \int_0^1 (1+t)(1-t)\cdot 2 + (1+2t)(1-t) + (1+2t)(1+t)(-1)\,dt$$

$$= \int_0^1 2 - 2t^2 + 1 + t - 2t^2 - 1 - 3t - 2t^2\,dt = \int_0^1 2 - 2t - 6t^2\,dt$$

$$= \left[2t - t^2 - 2t^3\right]_0^1 = 2 - 1 - 2 = -1.$$