16.3 Conservative Vector Fields

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Fundamental Theorem for Conservative Vector Fields

Theorem

Assume that $\mathbf{F} = \nabla V$.

- *If* $\mathcal{C}$ *is a path from* P *to* Q, *then*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = V(Q) - V(P)$$

- *If* $\mathcal{C}$ *is a closed path, then*

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

Remarks

- If $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ depends only on the endpoints of $\mathcal{C}$, $\mathbf{F}$ is path-independent.
- A path $\mathcal{C}$ is closed, if the initial point and the endpoint of $\mathcal{C}$ are the same.

Using the Fundamental Theorem

Example

Verify that $V(x, y, z) = xz + 2y$ is a potential for $\mathbf{F}(x, y, z) = \langle z, 2, x \rangle$ and evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathcal{C}$ is given by $\mathbf{c}(t) = \langle \cos t, t, \sin t \rangle$, $0 \leq t \leq \pi/2$.

Verify that V is a potential

$$\nabla V(x, y, z) = \langle V_x, V_y, V_z \rangle = \langle z, 2, x \rangle = \mathbf{F}(x, y, z)$$

Evaluate the integral

- Initial point $\mathbf{c}(0) = \langle 1, 0, 0 \rangle$
- Endpoint $\mathbf{c}(\pi/2) = \langle 0, \pi/2, 1 \rangle$
- Use the Fundamental Theorem

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = V(0, \pi/2, 1) - V(1, 0, 0) = \pi - 0 = \pi$$

Conservation of Energy

Physics conventions

- V is a potential of $\mathbf{F}$ if $\mathbf{F} = -\nabla V$.
- A particle located at P has potential energy $PE = V(P)$.
- A particle of mass m moving at speed v has kinetic energy $KE = \frac{1}{2}mv^2$.
- A particle of mass m moving along a path $\mathbf{c}(t)$ has total energy

$$E = KE + PE = \frac{1}{2}m\|\mathbf{c}'(t)\|^2 + V(\mathbf{c}(t)).$$

Theorem

The total energy of a particle moving under the influence of a conservative force field is constant in time, i.e., $\frac{dE}{dt} = 0$.

Conservation of Energy Proof

Theorem

The total energy of a particle moving under the influence of a conservative force field is constant in time, i.e., $\frac{dE}{dt} = 0$.

Proof

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left[\frac{1}{2}m\|\mathbf{c}'(t)\|^2 + V(\mathbf{c}(t))\right] \\
&= \frac{1}{2}m(2\mathbf{c}'(t)\cdot\mathbf{c}''(t)) + \nabla V(\mathbf{c}(t))\cdot\mathbf{c}'(t) \\
&= m\mathbf{c}'(t)\cdot\mathbf{c}''(t) - \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t) \\
&= [m\mathbf{c}''(t) - \mathbf{F}(\mathbf{c}(t))]\cdot\mathbf{c}'(t) = \mathbf{0}\cdot\mathbf{c}'(t) = 0,
\end{aligned}$$

using Newton's Law $\mathbf{F} = m\mathbf{a} = m\mathbf{c}''$ in the last line.

Conservation of Energy Application I

Gravitational Potential

The gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^3}\mathbf{x}$$

has potential (in the physical sense)

$$V(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -GMm(x^2+y^2+z^2)^{-1/2}$$

Check

$$V_x = -GMm\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\right)2x = \frac{GMm}{\|\mathbf{x}\|^3}x$$

$$V_y = \frac{GMm}{\|\mathbf{x}\|^3}y, \quad V_z = \frac{GMm}{\|\mathbf{x}\|^3}z, \quad \nabla V = \frac{GMm}{\|\mathbf{x}\|^3}\mathbf{x} = -\mathbf{F}$$

Conservation of Energy Application II

Gravitational Field and Potential

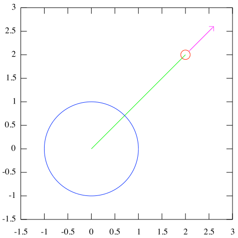
$$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^3}\mathbf{x}, \quad V(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|}$$

Example

A particle of mass m at distance r_0 from a fixed object of mass M at the origin moves straight away from the origin, with initial speed v_0.

(a) How far away from the origin will it get?

(b) What is the escape velocity, i.e., how large does v_0 have to be to get away infinitely far?



Conservation of Energy Application III

Gravitational Field and Potential

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|^3}\mathbf{x}, \quad V(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|}$$

Example

A particle of mass m at distance r_0 from a fixed object of mass M at the origin moves straight away from the origin, with initial speed v_0.

(a) How far away from the origin will it get?

Total Energy

$$E = KE + PE = \frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv(t)^2 - \frac{GMm}{r(t)}$$

Conservation of Energy Application IV

Example

A particle of mass m at distance r_0 from a fixed object of mass M at the origin moves straight away from the origin, with initial speed v_0.

(a) How far away from the origin will it get?

Total Energy

$$E = \frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv(t)^2 - \frac{GMm}{r(t)}$$

At the turnaround time $v(t_1) = 0$, so

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = -\frac{GMm}{r(t_1)}$$

Conservation of Energy Application V

Example

A particle of mass m at distance r_0 from a fixed object of mass M at the origin moves straight away from the origin, with initial speed v_0.

(a) How far away from the origin will it get?

Total Energy

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = -\frac{GMm}{r(t_1)}$$

$$\implies r(t_1) = \frac{GM}{\frac{GM}{r_0} - \frac{1}{2}v_0^2} = \frac{1}{\frac{1}{r_0} - \frac{v_0^2}{2GM}}$$

Conservation of Energy Application VI

Example

A particle of mass m at distance r_0 from a fixed object of mass M at the origin moves straight away from the origin, with initial speed v_0.

(a) How far away from the origin will it get?

(b) What is the escape velocity, i.e., how large does v_0 have to be to get away infinitely far?

Answer (a)

$$r(t_1) = \frac{1}{\frac{1}{r_0} - \frac{v_0^2}{2GM}}$$

What does a negative or zero denominator mean physically? It means that there is no turnaround, so the particle gets infinitely far away. Escape velocity is the borderline case, i.e., denominator zero.

Conservation of Energy Application VII

Example

A particle of mass m at distance r_0 from a fixed object of mass M at the origin moves straight away from the origin, with initial speed v_0.

(b) What is the escape velocity, i.e., how large does v_0 have to be to get away infinitely far?

Answer (b)

$$\frac{1}{r_0} = \frac{v_0^2}{2GM} \Leftrightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$$

Escape velocity from Earth

$$v_0 = \sqrt{\frac{2 \times 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2} \times 5.9742 \times 10^{24} kg}{6.371 \times 10^6 m}} = 11.19 km/s$$

Potentials and Path Independence

Theorem

A vector field $\mathbf{F}$ *is path-independent if and only if it is conservative.*

"Proof"

If $\mathbf{F}$ is conservative, then $\mathbf{F}$ is independent of path by the Fundamental Theorem for Conservative Vector Fields.

If $\mathbf{F}$ is independent of path, choose an arbitrary base point P_0, and define

$$V(P) = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s},$$

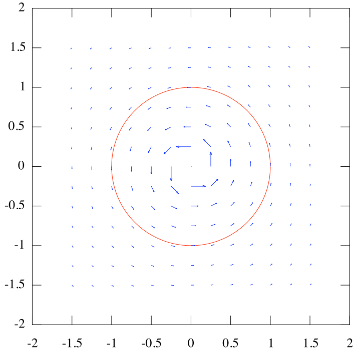
where $\mathcal{C}$ is any path from P_0 to P. Then $\nabla V = \mathbf{F}$.

Potentials and the Cross-Partial Condition I

Example

The Vortex Field

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$



Sketch of the vortex vector field with the unit circle $\mathcal{C}$. If $\mathcal{C}$ is parametrized counterclockwise, then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 2\pi.$$

So $\mathbf{F}$ is not conservative.

Potentials and the Cross-Partial Condition II

Example

The Vortex Field

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

Cross-Partial Condition

$$\begin{aligned}
\frac{\partial \mathbf{F}_1}{\partial y} &= \frac{(-1)(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial \mathbf{F}_2}{\partial x} &= \frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}$$

$$\implies \frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}$$

The cross-partial condition is satisfied, but **F** is not conservative!

Potentials and the Cross-Partial Condition III

But not all is lost!

Theorem

If a 2-D vector field $\mathbf{F} = \langle \mathbf{F}_1, \mathbf{F}_2 \rangle$ *on a simply connected domain satisfies the cross-partial condition*

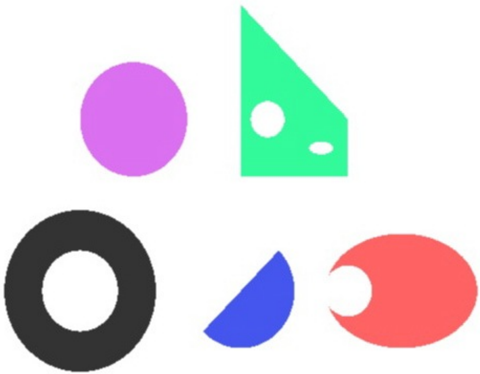
$$\frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x},$$

then it is conservative.

Simply Connected Domains

A domain is simply connected if it "has no holes". Mathematically rigorous, it is simply connected if every loop can be deformed to a point in the domain.

Simply Connected Domains



Question

Which of these domains are simply connected?

Answer

The green and black domain are not simply connected, all the others are.

Vortex field explained

The problem with the vortex field is that it is not defined at $\mathbf{0}$, so its domain is not simply connected.

Potentials and the Cross-Partial Condition in 3-D

Theorem

If a 3-D vector field $\mathbf{F} = \langle \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \rangle$ on a simply connected domain satisfies the cross-partial condition

$$\frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}, \quad \frac{\partial \mathbf{F}_1}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial x}, \quad \frac{\partial \mathbf{F}_2}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial y},$$

then it is conservative.

Simply Connected Domains

A 3-D domain is simply connected if every loop can be deformed to a point in the domain. This includes some domains with holes, like balls with a point removed and spherical shells. An example of a non-simply connected domain is a cylindrical shell.

Finding a Potential I

Example

Show that $\mathbf{F}(x, y) = \langle 2e^{2x} + \sin y, 2y + x\cos y\rangle$ is conservative and find a potential.

Checking cross-partials

$$\frac{\partial \mathbf{F}_1}{\partial y} = \cos y = \frac{\partial \mathbf{F}_2}{\partial x}$$

Checking simple connectivity

The vector field is defined in the whole plane, which is simply connected. So $\mathbf{F}$ is conservative.

Finding a Potential II

Example

Show that $\mathbf{F}(x, y) = \langle 2e^{2x} + \sin y, 2y + x\cos y\rangle$ is conservative and find a potential.

Finding antiderivatives

$$V_x = 2e^{2x} + \sin y \implies V = \int 2e^{2x} + \sin y \, dx = e^{2x} + x\sin y + g(y)$$

The "integration constant" $g(y)$ may depend on y.

$$V_y = 2y + x\cos y \text{ and } V_y = x\cos y + g'(y) \implies g'(y) = 2y$$

$$\implies g(y) = y^2 + C \implies V = e^{2x} + x\sin y + y^2 + C.$$

Finding a Potential in 3-D I

Example

Show that $\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2z, x + 2y + 4z \rangle$ is conservative and find a potential.

Checking cross-partials

$$\frac{\partial \mathbf{F}_1}{\partial y} = 2x = \frac{\partial \mathbf{F}_2}{\partial x}, \quad \frac{\partial \mathbf{F}_1}{\partial z} = 1 = \frac{\partial \mathbf{F}_3}{\partial x}, \quad \frac{\partial \mathbf{F}_2}{\partial z} = 2 = \frac{\partial \mathbf{F}_3}{\partial y}$$

Checking simple connectivity

The vector field is defined in the whole space, which is simply connected. So $\mathbf{F}$ is conservative.

Finding a Potential in 3-D II

Example

Show that $\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2z, x + 2y + 4z \rangle$ is conservative and find a potential.

Find antiderivatives

$$V_x = 2xy + z \implies V = \int 2xy + z \, dx = x^2 y + xz + g(y, z)$$

The "integration constant" $g(y, z)$ may depend on all variables but x.

$$V_y = x^2 + 2z \text{ and } V_y = x^2 + g_y(y, z) \implies g_y(y, z) = 2z.$$

$$g(y, z) = \int 2z \, dy = 2yz + h(z)$$

Now the "integration constant" $h(z)$ may only depend on z.

Finding a Potential in 3-D III

Example

Show that $\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2z, x + 2y + 4z \rangle$ is conservative and find a potential.

Find antiderivatives

$$V(x, y, z) = x^2 y + xz + g(y, z), \quad g(y, z) = 2yz + h(z)$$

$$\implies V = x^2 y + xz + 2yz + h(z)$$

$$V_z = x + 2y + 4z \text{ and } V_z = x + 2y + h'(z) \implies h'(z) = 4z$$

$$\implies h(z) = \int 4z \, dz = 2z^2 + C \implies V = x^2 y + xz + 2yz + 2z^2 + C$$