16.5 Surface Integrals of Vector Fields

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Parametrized Surfaces

Definition

An orientation on a surface $\mathcal{S}$ is a continuous choice of a unit normal vector $\mathbf{e_n}(P)$ at each point P os $\mathcal{S}$.

Example

The xy-plane has two orientations, one given by $\mathbf{e_n} = \mathbf{k}$ (pointing up), the other by $\mathbf{e_n} = -\mathbf{k}$ (pointing down).

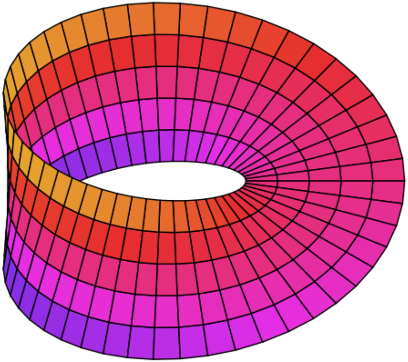
Example

The sphere $\|\mathbf{x}\| = R$ has two orientations, one given by the outward pointing vector $\mathbf{e_n}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, the other by the inward pointing normal vectors $-\mathbf{e_n}(\mathbf{x})$.

The Möbius Strip

Caution

Not all surfaces are orientable. The most popular example of a non-orientable surface is the Möbius strip depicted below.



Vector Surface Integral

Definition

The normal component of a vector field $\mathbf{F}$ at a point P on an oriented surface $\mathcal{S}$ is

$$\mathbf{F}(P) \cdot \mathbf{e_n}(P)$$

Definition

The vector surface integral of a vector field $\mathbf{F}$ over a surface $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} (\mathbf{F} \cdot \mathbf{e_n}) dS.$$

It is also called the flux of $\mathbf{F}$ across or through $\mathcal{S}$.

Applications

- Flow rate of a fluid with velocity field $\mathbf{F}$ across a surface $\mathcal{S}$.
- Magnetic and electric flux across surfaces. (Maxwell's equations)

Parametrized Vector Surface Integral

Calculating Parametrized Surface Integrals

A regular parametrization $G(u,v)$ (i.e., a parametrization with $\mathbf{n}(u,v) \neq \mathbf{0}$ for all u, v) induces an orientation by $\mathbf{e_n} = \frac{\mathbf{n}}{\|\mathbf{n}\|}$. We get

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(u,v)) \cdot \mathbf{n}(u,v)\, du\, dv.$$

Differential Version

$$d\mathbf{S} = \mathbf{n}(u,v)\, du\, dv$$

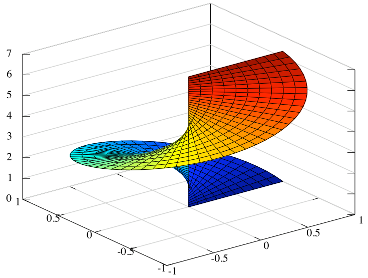
Orientation matters

Reversing the orientation of $\mathcal{S}$ changes the sign of the integral.

Vector Surface Integral Example I

Example

Find $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle 0, 0, z \rangle$ and $\mathcal{S}$ is the oriented surface parametrized by $G(u, v) = (u \cos v, u \sin v, v)$, where $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.



$$
\begin{aligned}
\mathbf{T}_u &= \langle \cos v, \sin v, 0 \rangle \\
\mathbf{T}_v &= \langle -u \sin v, u \cos v, 1 \rangle \\
\mathbf{n} &= \mathbf{T}_u \times \mathbf{T}_v = \langle \sin v, -\cos v, u \rangle
\end{aligned}
$$

Is the integral positive, negative, or 0? Positive!

Vector Surface Integral Example II

Example

Find $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle 0, 0, z \rangle$ and $\mathcal{S}$ is the oriented surface parametrized by $G(u, v) = (u \cos v, u \sin v, v)$, where $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

Evaluating the integral

$$\begin{aligned}
\mathbf{n} &= \langle \sin v, -\cos v, u \rangle \\
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \langle 0, 0, v \rangle \cdot \langle \sin v, -\cos v, u \rangle \, dv \, du \\
&= \int_0^1 \int_0^{2\pi} vu \, dv \, du = \left(\int_0^1 u \, du \right) \left(\int_0^{2\pi} v \, dv \right) \\
&= \frac{1}{2} \cdot \frac{4\pi^2}{2} = \pi^2.
\end{aligned}$$

Flow Rate Example I

Example

A fluid flows with constant velocity $\mathbf{v} = 3\mathbf{k}(m/s)$. Calculate the flow rate (in m^3/s) through the part of the elliptic paraboloid $z = x^2 + y^2$ with $z \leq 4$ and upward pointing normal vector.

Parametrize Surface

At height z the trace is a circle of radius $r = \sqrt{z}$. Using r and θ as parameters:

$$\begin{aligned}
G(r,\theta) &= (r\cos\theta, r\sin\theta, r^2), \quad 0 \leq r \leq 2,\, 0 \leq \theta \leq 2\pi \\
\mathbf{T}_r &= \langle \cos\theta, \sin\theta, 2r \rangle \\
\mathbf{T}_\theta &= \langle -r\sin\theta, r\cos\theta, 0 \rangle \\
\mathbf{n} &= \mathbf{T}_r \times \mathbf{T}_\theta = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle
\end{aligned}$$

Flow Rate Example II

Example

A fluid flows with constant velocity $\mathbf{v} = 3\mathbf{k}(m/s)$. Calculate the flow rate (in m^3/s) through the part of the elliptic paraboloid $z = x^2 + y^2$ with $z \leq 4$ and upward pointing normal vector.

Integrate

$$\begin{aligned} G(r,\theta) &= (r\cos\theta, r\sin\theta, r^2), \quad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi \\ \mathbf{n} &= \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle \end{aligned}$$

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \int_0^2\int_0^{2\pi} \langle 0,0,3\rangle \cdot \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle \, d\theta\, dr \\ &= \int_0^2\int_0^{2\pi} 3r\, d\theta\, dr = 6\pi\int_0^2 r\, dr = 12\pi. \end{aligned}$$

Another Example I

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Parametrizing $\mathcal{S}$

We have to parametrize both the base and the boundary cone, calculate the flux through both and add the results.

Parametrizing the base

Standard parametrization of a disk of radius 2:

$$G(r, \theta) = (r \cos \theta, r \sin \theta, 0), \quad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi.$$

Another Example II

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Parametrizing the base

$$\begin{aligned}
G(r, \theta) &= (r \cos\theta, r \sin\theta, 0), \quad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi, \\
\mathbf{T}_r &= \langle \cos\theta, \sin\theta, 0 \rangle \\
\mathbf{T}_\theta &= \langle -r \sin\theta, r \cos\theta, 0 \rangle \\
\mathbf{n} &= \mathbf{T}_r \times \mathbf{T}_\theta = \langle 0, 0, r \rangle
\end{aligned}$$

Problem: $r \geq 0$, so $\mathbf{n}$ points up, into the cone, not out. Solution: Change the sign of $\mathbf{n}$.

Another Example III

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Integrating over the base

$$\begin{aligned} G(r,\theta) &= (r\cos\theta, r\sin\theta, 0), \quad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi, \\ \mathbf{n} &= -\langle 0, 0, r \rangle \\ \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_0^{2\pi} \langle -r\sin\theta, r\cos\theta, 0 \rangle \cdot \langle 0, 0, -r \rangle \, d\theta \, dr = 0 \end{aligned}$$

Another Example IV

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0,0,4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Parametrizing the cone

Trace in $z = k$ is a circle of radius $(4-k)/2$ for $0 \leq k \leq 4$, so

$$\begin{aligned}
G(z,\theta) &= \left\langle \frac{4-z}{2}\cos\theta, \frac{4-z}{2}\sin\theta, z \right\rangle, \quad 0 \leq z \leq 4,\ 0 \leq \theta \leq 2\pi. \\
\mathbf{T}_z &= \left\langle -\frac{1}{2}\cos\theta, -\frac{1}{2}\sin\theta, 1 \right\rangle \\
\mathbf{T}_\theta &= \left\langle -\frac{4-z}{2}\sin\theta, \frac{4-z}{2}\cos\theta, 0 \right\rangle \\
\mathbf{n} &= \mathbf{T}_z \times \mathbf{T}_\theta = \left\langle \frac{z-4}{2}\cos\theta, \frac{z-4}{2}\sin\theta, \frac{z-4}{2} \right\rangle.
\end{aligned}$$

Another Example V

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Integrating over the cone

$$\begin{aligned} G(z, \theta) &= \left\langle \frac{4-z}{2} \cos\theta, \frac{4-z}{2} \sin\theta, z \right\rangle, \quad 0 \leq z \leq 4,\ 0 \leq \theta \leq 2\pi. \\ \mathbf{n} &= \left\langle \frac{z-4}{2} \cos\theta, \frac{z-4}{2} \sin\theta, \frac{z-4}{2} \right\rangle. \end{aligned}$$

$(z-4)/2 \leq 0$, so $\mathbf{n}$ points down, i.e., into the cone. Again we have to change the sign to fix the orientation.

Another Example VI

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Integrating over the cone, correct orientation

$$\begin{aligned} G(z,\theta) &= \left\langle \frac{4-z}{2}\cos\theta, \frac{4-z}{2}\sin\theta, z \right\rangle, \quad 0 \leq z \leq 4,\ 0 \leq \theta \leq 2\pi. \\ \mathbf{n} &= \left\langle \frac{4-z}{2}\cos\theta, \frac{4-z}{2}\sin\theta, \frac{4-z}{2} \right\rangle. \end{aligned}$$

$$\mathbf{F} \cdot \mathbf{n} = \left\langle \frac{z-4}{2}\sin\theta, \frac{4-z}{2}\cos\theta, z \right\rangle \cdot \left\langle \frac{4-z}{2}\cos\theta, \frac{4-z}{2}\sin\theta, \frac{4-z}{2} \right\rangle$$

Another Example VII

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Integrating over the cone, correct orientation

$$G(z, \theta) = \left\langle \frac{4-z}{2} \cos\theta, \frac{4-z}{2} \sin\theta, z \right\rangle, \quad 0 \leq z \leq 4, \, 0 \leq \theta \leq 2\pi.$$

$$\mathbf{F} \cdot \mathbf{n} = \left\langle \frac{z-4}{2} \sin\theta, \frac{4-z}{2} \cos\theta, z \right\rangle \cdot \left\langle \frac{4-z}{2} \cos\theta, \frac{4-z}{2} \sin\theta, \frac{4-z}{2} \right\rangle$$
$$= \frac{z(4-z)}{2} = \frac{4z - z^2}{2}$$

Another Example VIII

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Integrating over the cone, correct orientation

$$G(z, \theta) = \left\langle \frac{4-z}{2} \cos\theta, \frac{4-z}{2} \sin\theta, z \right\rangle, \quad 0 \leq z \leq 4,\ 0 \leq \theta \leq 2\pi.$$

$$\mathbf{F} \cdot \mathbf{n} = \frac{4z - z^2}{2}$$

$$\iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^4 \int_0^{2\pi} \mathbf{F} \cdot \mathbf{n} \, d\theta \, dz = \int_0^4 \int_0^{2\pi} \frac{4z - z^2}{2} \, d\theta \, dz$$

Another Example IX

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Integrating over the cone, correct orientation

$$\begin{aligned} \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^4 \int_0^{2\pi} \frac{4z - z^2}{2} \, d\theta \, dz = \pi \int_0^4 4z - z^2 \, dz \\ &= \pi \left[2z^2 - \frac{z^3}{3} \right]_{z=0}^{z=4} = \pi \left(32 - \frac{64}{3} \right) = \frac{32\pi}{3}. \end{aligned}$$

Another Example X

Example

Let $\mathcal{S}$ be the boundary of the solid cone with base $x^2 + y^2 \leq 4$ in the xy-plane and apex $(0, 0, 4)$, with outward-pointing normal vector. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z \rangle$ through $\mathcal{S}$.

Adding up the results

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} \\ &= 0 + \frac{32\pi}{3} = \frac{32\pi}{3}. \end{aligned}$$