17.1 Green's Theorem

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Fundamental Theorems of Vector Analysis

- Green's Theorem $\oint_{\partial\mathcal{D}} F \cdot d\mathbf{s} = \iint_{\mathcal{D}} \operatorname{curl} F \, dA$
  - $\mathcal{D}$ plane domain, $\mathbf{F} = \langle P, Q \rangle$
  - $\operatorname{curl}\langle P, Q \rangle = \frac{\partial Q}{\partial x} - \frac{\partial P}{dy}$
- Stokes' Theorem $\oint_{\partial\mathcal{S}} F \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl} F \cdot d\mathcal{S}$
  - $\mathcal{S}$ surface in space, $\mathbf{F} = \langle P, Q, R \rangle$
  - $\operatorname{curl}\langle P, Q, R \rangle = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$
- Divergence Theorem $\iint_{\partial\mathcal{W}} \mathbf{F} \cdot d\mathcal{S} = \iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV$
  - $\mathcal{W}$ region in space, $\mathbf{F} = \langle P, Q, R \rangle$
  - $\operatorname{div}\langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Green's Theorem

Theorem

$$\oint_{\partial\mathcal{D}} F_1\,dx + F_2\,dy = \iint_{\mathcal{D}} \left(\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right) dA$$

Remarks

- $\mathcal{D}$ is a domain bounded by a finite number of simple closed smooth curves.
- $\partial\mathcal{D}$ is parametrized in the mathematically positive direction, i.e., such that $\mathcal{D}$ always lies to the left of $\partial\mathcal{D}$.
- $\mathbf{F} = \langle F_1, F_2 \rangle$ is a smooth vector field.
- $\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}$ is the curl of $\mathbf{F}$. Physically, the curl is *circulation per unit of enclosed area*.

Example I

Example

Verify Green's Theorem for the line integral along the unit circle $\mathcal{C}$, oriented counterclockwise:

$$\int_{\mathcal{C}} y\,dx + xy\,dy$$

Direct Way

$$x = \cos\theta, \quad y = \sin\theta, \quad dx = -\sin\theta\,d\theta, \quad dy = \cos\theta\,d\theta$$

$$\begin{aligned}
\oint_{\mathcal{C}} y\,dx + xy\,dy &= \int_0^{2\pi} (\sin\theta)(-\sin\theta) + (\cos\theta\sin\theta)(\cos\theta)\,d\theta \\
&= \int_0^{2\pi} -\sin^2\theta + \cos^2\theta\sin\theta\,d\theta
\end{aligned}$$

Example II

Example

Verify Green's Theorem for the line integral along the unit circle $\mathcal{C}$, oriented counterclockwise:

$$\int_{\mathcal{C}} y\,dx + xy\,dy$$

Direct Way

$$\begin{aligned}\oint_{\mathcal{C}} y\,dx + xy\,dy &= \int_0^{2\pi} -\sin^2\theta + \cos^2\theta\sin\theta\,d\theta \\ &= \left[-\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\cos^3\theta}{3}\right]_0^{2\pi} = -\pi\end{aligned}$$

Example III

Example

Verify Green's Theorem for the line integral along the unit circle $\mathcal{C}$, oriented counterclockwise:

$$\int_{\mathcal{C}} y\,dx + xy\,dy$$

Green's Way

$$F_1 = y, \quad F_2 = xy, \quad \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} = y - 1$$

$$\iint_{\mathcal{D}} \left(\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} \right) dA = \iint_{\mathcal{D}} (y-1) dA = \int_0^1 \int_0^{2\pi} (r \sin\theta - 1) r \, d\theta \, dr$$

$$= \int_0^1 \left[-r^2 \cos\theta - r\theta \right]_{\theta=0}^{\theta=2\pi} dr = \int_0^1 (-2\pi r)\,dr = -\pi$$

Calculating Area

Theorem

$$\text{area}(\mathcal{D}) = \frac{1}{2}\int_{\partial\mathcal{D}} x\,dy - y\,dx$$

Proof.

$$F_1 = -y, \quad F_2 = x, \quad \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} = 1 - (-1) = 2,$$

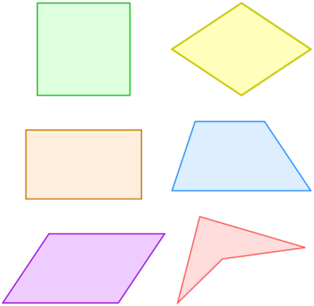
$$\frac{1}{2}\int_{\partial\mathcal{D}} x\,dy - y\,dx = \frac{1}{2}\iint_{\mathcal{D}} \left(\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right) dA = \frac{1}{2}\iint_{\mathcal{D}} 2\,dA = \text{area}(\mathcal{D}).$$

□

Area of a Quadrilateral I

Example

Find the area of the quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), using Green's Theorem.



Parametrizing one side

For $0 \leq t \leq 1$,

$$\begin{aligned} \mathbf{c}(t) &= \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1) \rangle \\ dx &= (x_2 - x_1)dt \\ dy &= (y_2 - y_1)dt \end{aligned}$$

Area of a Quadrilateral II

Example

Find the area of the quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), using Green's Theorem.

Integrating over one side

$$\mathbf{c}(t) = \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)\rangle, \quad 0 \le t \le 1$$

$$dx = (x_2 - x_1)dt, \quad dy = (y_2 - y_1)dt$$

$$\begin{aligned} x\,dy &= (x_1 + t(x_2 - x_1))(y_2 - y_1)dt \\ y\,dx &= (y_1 + t(y_2 - y_1))(x_2 - x_1)dt \\ x\,dy - y\,dx &= (x_1(y_2 - y_1) - y_1(x_2 - x_1))dt \\ &= (x_1 y_2 - x_2 y_1)dt \end{aligned}$$

Area of a Quadrilateral III

Example

Find the area of the quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), using Green's Theorem.

Integrating over one side

$$x\,dy - y\,dx = (x_1 y_2 - x_2 y_1)dt$$

$$\frac{1}{2}\int_{\mathcal{C}_1} x\,dy - y\,dx = \frac{1}{2}\int_0^1 (x_1 y_2 - x_2 y_1)dt = \frac{x_1 y_2 - x_2 y_1}{2}$$

Area of a Quadrilateral IV

Example

Find the area of the quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), using Green's Theorem.

Integrating over one side

$$\frac{1}{2}\int_{\mathcal{C}_1} x\,dy - y\,dx = \frac{x_1 y_2 - x_2 y_1}{2}$$

Integrating over all sides

$$\operatorname{area}(\mathcal{D}) = \frac{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_4 - x_4 y_3) + (x_4 y_1 - x_1 y_4)}{2}$$

The 2-D Divergence Theorem I

Definition

If $\mathcal{C}$ is a closed curve, $\mathbf{n}$ the outward-pointing normal vector, and $\mathbf{F} = \langle P, Q \rangle$, then the flux of $\mathbf{F}$ across $\mathcal{C}$ is $\oint_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{n}) ds$

Remark

If the tangent vector to the curve $\mathcal{C}$ is $\langle x'(t), y'(t) \rangle$, the outward-pointing normal vector is $\langle y'(t), -x'(t) \rangle$, so the flux is

$$\oint_{\mathcal{C}} \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \oint_{\mathcal{C}} P\, dy - Q\, dx$$

Theorem

The flux of $\mathbf{F}$ *across* $\mathcal{C}$ *is* $\iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$

The 2-D Divergence Theorem II

Theorem

The flux of $\mathbf{F}$ *across* $\mathcal{C}$ *is* $\iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$

Definition

The quantity $\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is the divergence of $\mathbf{F} = \langle P, Q \rangle$.

Theorem

$$\oint_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{n}) ds = \iint_{\mathcal{D}} \operatorname{div} F \, dA$$

The 2-D Divergence Theorem Proof

Theorem

$$Flux = \oint_{\mathcal{C}} P\,dy - Q\,dx = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

Proof.

Using Green's Theorem,

$$\begin{aligned} \oint_{\mathcal{C}} P\,dy - Q\,dx &= \oint_{\mathcal{C}} -Q\,dx + P\,dy \\ &= \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x} P - \frac{\partial}{\partial y}(-Q) \right) dA \\ &= \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \end{aligned}$$

□

2-D Divergence Example

Example

Find the flux of $\mathbf{F}(x, y) = \langle 2x + 2xy + y^2, x + y - y^2 \rangle$ across the circle $x^2 + y^2 = 4$.

Using the 2-D Divergence Theorem

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2 + 2y + 1 - 2y = 3$$

So

$$\text{Flux} = \iint_{\mathcal{D}} \operatorname{div} \mathbf{F} \, dA = \iint_{\mathcal{D}} 3 \, dA = 3 \operatorname{area}(\mathcal{D}) = 12\pi$$