17.2 Stokes' Theorem

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Fundamental Theorems of Vector Analysis

- Green's Theorem $\oint_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{s} = \iint_{\mathcal{D}} \operatorname{curl}\mathbf{F}\,dA$
  - $\mathcal{D}$ plane domain, $\mathbf{F} = \langle P, Q\rangle$
  - $\operatorname{curl}\langle P, Q\rangle = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{dy}$
- Stokes' Theorem $\oint_{\partial\mathcal{S}} \mathbf{F}\cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot d\mathcal{S}$
  - $\mathcal{S}$ surface in space, $\mathbf{F} = \langle P, Q, R\rangle$
  - $\operatorname{curl}\langle P, Q, R\rangle = \left\langle \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}, \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right\rangle$
- Divergence Theorem $\iint_{\partial\mathcal{W}} \mathbf{F}\cdot d\mathcal{S} = \iiint_{\mathcal{W}} \operatorname{div}\mathbf{F}\,dV$
  - $\mathcal{W}$ region in space, $\mathbf{F} = \langle P, Q, R\rangle$
  - $\operatorname{div}\langle P, Q, R\rangle = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$

Stokes' Theorem

Theorem

$$\oint_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathcal{S}$$

Remarks

- $\mathcal{S}$ is an oriented surface in space.
- $\partial\mathcal{S}$ has the boundary orientation: If a unit normal vector is walking along $\partial\mathcal{S}$, the surface $\mathcal{S}$ is to its left.
- $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a smooth vector field.
- $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle$

$$= \left\langle \frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x}, \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} \right\rangle$$

Example I

Example

Verify Stokes' Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the line integral

The boundary $\partial\mathcal{S}$ is the circle $x^2 + y^2 = 4$ in the $z = 4$ plane. Standard parametrization is

$$\mathbf{c}(\theta) = (2\cos\theta, 2\sin\theta, 4), \quad \mathbf{c}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle.$$

The orientation is correct, too: An upward-pointing normal vector moving around $\partial\mathcal{S}$ counterclockwise sees $\mathcal{S}$ to its left.

Example II

Example

Verify Stokes' Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the line integral

$$\mathbf{c}(\theta) = (2\cos\theta, 2\sin\theta, 4), \quad \mathbf{c}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle.$$

$$\begin{aligned} \oint_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \langle -4\sin\theta, 6\cos\theta, 4 \rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0 \rangle d\theta \\ &= \int_0^{2\pi} 8\sin^2\theta + 12\cos^2\theta \, d\theta = 8\pi + 12\pi = 20\pi. \end{aligned}$$

Example III

Example

Verify Stokes' Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the surface integral

We can use x and y as parameters over the disk $x^2 + y^2 \leq 4$. Then

$$G(x, y) = (x, y, x^2 + y^2),$$

$$\mathbf{T}_x = \langle 1, 0, 2x \rangle, \quad \mathbf{T}_y = \langle 0, 1, 2y \rangle \quad \mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \langle -2x, -2y, 1 \rangle.$$

Orientation is correct because the third component of $\mathbf{n}$ is positive, so $\mathbf{n}$ is pointing up.

Example IV

Example

Verify Stokes' Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the surface integral

$$G(x,y) = (x, y, x^2 + y^2), \quad x^2 + y^2 \leq 4, \quad \mathbf{n} = \langle -2x, -2y, 1 \rangle.$$

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \left\langle \frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x}, \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} \right\rangle \\ &= \langle 0 - 0, 0 - 0, 3 - (-2) \rangle = \langle 0, 0, 5 \rangle \end{aligned}$$

Example V

Example

Verify Stokes' Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the surface integral

$$G(x,y) = (x, y, x^2 + y^2), \quad x^2 + y^2 \leq 4, \quad \mathbf{n} = \langle -2x, -2y, 1 \rangle.$$

$$\operatorname{curl} \mathbf{F} = \langle 0, 0, 5 \rangle$$

$$\begin{aligned} \int_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathcal{D}} \langle 0, 0, 5 \rangle \cdot \langle -2x, -2y, 1 \rangle dA \\ &= \iint_{\mathcal{D}} 5\, dA = 5 \operatorname{area}(\mathcal{D}) = 5 \cdot 4\pi = 20\pi. \end{aligned}$$