# 17.3 Divergence Theorem

Lukas Geyer

Montana State University

M273, Fall 2011

# Fundamental Theorems of Vector Analysis

- Green's Theorem $\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \operatorname{curl} \mathbf{F} \, dA$
  - $\mathcal{D}$ plane domain, $\mathbf{F} = \langle P, Q \rangle$
  - $\operatorname{curl}\langle P, Q \rangle = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{dy}$
- Stokes' Theorem $\oint_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathcal{S}$
  - $\mathcal{S}$ surface in space, $\mathbf{F} = \langle P, Q, R \rangle$
  - $\operatorname{curl}\langle P, Q, R \rangle = \left\langle \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}, \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right\rangle$
- Divergence Theorem $\iint_{\partial\mathcal{W}} \mathbf{F} \cdot d\mathcal{S} = \iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV$
  - $\mathcal{W}$ region in space, $\mathbf{F} = \langle P, Q, R \rangle$
  - $\operatorname{div}\langle P, Q, R \rangle = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$

# Divergence Theorem

## Theorem

$$\iint_{\partial\mathcal{W}} \mathbf{F} \cdot d\mathcal{S} = \iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV$$

## Remarks

- $\mathcal{W}$ is a region in space.
- $\partial\mathcal{W}$ is oriented with outward-pointing normal vector
- $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a smooth vector field.
- $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle$

$$= \frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z}$$

# Example I

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the surface integral

The boundary of $\mathcal{W}$ consists of the upper hemisphere of radius 2 and the disk of radius 2 in the $xy$-plane. The upper hemisphere is parametrized by

$$G(\phi, \theta) = (2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi), \quad 0 \leq \phi \leq \pi/2,\, 0 \leq \theta \leq 2\pi.$$

$$\begin{aligned}
\mathbf{T}_\phi &= \langle 2 \cos \phi \cos \theta, 2 \cos \phi \sin \theta, -2 \sin \phi \rangle, \\
\mathbf{T}_\theta &= \langle -2 \sin \phi \sin \theta, 2 \sin \phi \cos \theta, 0 \rangle, \\
\mathbf{n} &= \mathbf{T}_\phi \times \mathbf{T}_\theta = \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \cos \phi \sin \phi \rangle
\end{aligned}$$

**n** points up, so the orientation is correct.

# Example II

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the surface integral

$$G(\phi, \theta) = (2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi), \quad 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq 2\pi.$$

$$\mathbf{n} = \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \cos \phi \sin \phi \rangle$$

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= \langle 2 \sin \phi \sin \theta, 2 \sin \phi \cos \theta, 1 + 2 \cos \phi \rangle \\ &\cdot \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \cos \phi \sin \phi \rangle \\ &= 16 \sin^3 \phi \cos \theta \sin \theta + 4 \cos \phi \sin \phi (1 + 2 \cos \phi) \end{aligned}$$

# Example III

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the surface integral

$$G(\phi, \theta) = (2 \sin\phi \cos\theta, 2 \sin\phi \sin\theta, 2 \cos\phi), \quad 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq 2\pi.$$

$$\mathbf{F} \cdot \mathbf{n} = 16 \sin^3\phi \cos\theta \sin\theta + 4 \cos\phi \sin\phi (1 + 2\cos\phi)$$

$$\begin{aligned} \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\pi/2} \int_0^{2\pi} 16 \sin^3\phi \cos\theta \sin\theta \, d\theta \, d\phi \\ &+ \int_0^{\pi/2} \int_0^{2\pi} 4 \cos\phi \sin\phi (1 + 2\cos\phi) \, d\theta \, d\phi \end{aligned}$$

# Example IV

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the surface integral

$$\begin{aligned}
\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\pi/2} \int_0^{2\pi} 16 \sin^3 \phi \cos \theta \sin \theta \, d\theta \, d\phi \\
&+ \int_0^{\pi/2} \int_0^{2\pi} 4 \cos \phi \sin \phi (1 + 2 \cos \phi) \, d\theta \, d\phi \\
&= 0 + 8\pi \int_0^{\pi/2} (\cos \phi + 2 \cos^2 \phi) \sin \phi \, d\phi \\
&= 8\pi \left[ -\frac{1}{2} \cos^2 \phi - \frac{2}{3} \cos^3 \phi \right]_{\phi=0}^{\phi=\pi/2} = 8\pi \left( \frac{1}{2} + \frac{2}{3} \right) = \frac{28\pi}{3}.
\end{aligned}$$

# Example V

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the surface integral

The disk of radius 2 in the *xy*-plane is parametrized by

$$G(r, \theta) = (r \cos \theta, r \sin \theta, 0), \quad 0 \leq r \leq 2, \, 0 \leq \theta \leq 2\pi.$$

$$\begin{aligned} \mathbf{T}_r &= \langle \cos \theta, \sin \theta, 0 \rangle, \\ \mathbf{T}_\theta &= \langle -r \sin \theta, r \cos \theta, 0 \rangle, \\ \mathbf{n} &= \mathbf{T}_r \times \mathbf{T}_\theta = \langle 0, 0, r \rangle \end{aligned}$$

**n** points up, so the orientation is incorrect, and we have to change the sign on **n**.

# Example VI

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1+z \rangle$.

## Computing the surface integral

$$G(r,\theta) = (r\cos\theta, r\sin\theta, 0), \quad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi.$$

$$\mathbf{n} = \langle 0, 0, -r \rangle$$

$$\begin{aligned} \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_0^{2\pi} \langle r\sin\theta, r\cos\theta, 1 \rangle \cdot \langle 0, 0, -r \rangle \, d\theta \, dr \\ &= \int_0^2 \int_0^{2\pi} (-r) \, d\theta \, dr = -2\pi \int_0^2 r \, dr = -4\pi \end{aligned}$$

# Example VII

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the surface integral

Adding up the results gives

$$\iint_{\partial \mathcal{W}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} = \frac{28\pi}{3} - 4\pi = \frac{16\pi}{3}.$$

# Example VIII

## Example

Verify the Divergence Theorem for the region given by $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$, and for the vector field $\mathbf{F} = \langle y, x, 1 + z \rangle$.

## Computing the volume integral

$$\operatorname{div} \mathbf{F} = \frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z} = 0 + 0 + 1 = 1.$$

$$\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV = \iiint_{\mathcal{W}} 1 \, dV = \operatorname{vol}(\mathcal{W}) = \frac{1}{2} \cdot \frac{4\pi \cdot 2^3}{3} = \frac{16\pi}{3}$$