Final Review Problems Key, M 273, Fall 2011

1. Show that the equation $\cot\phi = 2\cos\theta + \sin\theta$ in spherical coordinates describes a plane through the origin, and find a normal vector to this plane.

Multiply by $\rho\sin\phi$ to get $\rho\cos\phi = 2\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta$, i.e., $z = 2x + y$. A normal vector is $\langle 2, 1, -1\rangle$.

2. Find the length of the path $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t - 1\rangle$ for $1 \le t \le 3$.

$$L = \int_1^3 \|\mathbf{r}'(t)\|\, dt = \int_1^3 \sqrt{13}\, dt = 2\sqrt{13}$$

3. A force $\mathbf{F} = \langle 12t + 4, 8 - 24t\rangle$ (Newtons) acts on a 2-kg mass. Find the position of the mass at $t = 2$ (seconds) if it is located at $(4, 6)$ at $t = 0$ and has initial velocity $\langle 2, 3\rangle$ (m/s).

$\mathbf{F} = m\mathbf{a}$, so $\mathbf{a}(t) = \langle 6t + 2, 4 - 12t, \rangle$, $\mathbf{v}(t) = \langle 2 + 2t + 3t^2, 3 + 4t - 6t^2\rangle$, $\mathbf{r}(t) = \langle 4 + 2t + t^2 + t^3, 6 + 3t + 2t^2 - 2t^3\rangle$, $\mathbf{r}(2) = \langle 20, 4\rangle$.

4. Find an equation of the tangent plane at $P = (0, 3, -1)$ to the surface with equation $ze^x + e^{z+1} = xy + y - 3$.

The equation is a level surface of $\mathbf{F}(x, y, z) = ze^x + e^{z+1} - xy - y$, whose gradient is $\nabla F(x, y, z) = \langle ze^x - y, -x - 1, e^x + e^{z+1}\rangle$, so the normal vector is $\mathbf{n} = \nabla F(0, 3, -1) = \langle -4, -1, 2\rangle$, and the equation of the plane is $-4x - y + 2z = -4 \cdot 0 - 1 \cdot 3 + 2 \cdot (-1) = -5$.

5. Find the minimum and maximum values of $f(x, y, z) = x - z$ on the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Lagrange multipliers give equations $1 = 2\lambda x + 2\mu x$, $0 = 2\lambda y$, and $-1 = 2\mu z$. The second equation implies $\lambda = 0$ or $y = 0$. The case $y = 0$ leads to $x = \pm 1$ and $z = 0$, so $f(x, y, z) = \pm 1$. The case $\lambda = 0$ leads to $2\mu x = 1 = -2\mu z$. Dividing by 2μ gives $x = -z$. Using $x^2 + z^2 = 1$ we have the two solutions $x = 1/\sqrt{2}$ and $z = -1/\sqrt{2}$, or $x = -1/\sqrt{2}$ and $z = 1/\sqrt{2}$. In either case $y = \pm 1/\sqrt{2}$, and the two possible values for $f(x, y, z)$ are $\pm 2/\sqrt{2} = \pm\sqrt{2}$. So the maximum is $\sqrt{2}$, the minimum is $-\sqrt{2}$.

6. Find the double integral of $f(x, y) = x^3 y$ over the region between the curves $y = x^2$ and $y = x(1 - x)$.

$$\begin{aligned}
\iint_{\mathcal{D}} x^3 y\, dA &= \int_0^{1/2} \int_{x^2}^{x(1-x)} x^3 y\, dy\, dx = \int_0^{1/2} x^3 \left[\frac{y^2}{2}\right]_{y=x^2}^{y=x-x^2} dx \\
&= \int_0^{1/2} x^3 \frac{(x - x^2)^2 - x^4}{2}\, dx = \int_0^{1/2} \frac{x^5 - 2x^6}{2}\, dx \\
&= \frac{1}{2}\left(\frac{1}{6} \cdot \frac{1}{2^6} - \frac{2}{7} \cdot \frac{1}{2^7}\right) = \frac{1}{5376}
\end{aligned}$$

7. Use cylindrical coordinates to find the mass of the solid bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$, assuming a mass density of $f(x, y, z) = (x^2 + y^2)^{1/2}$.

$$\begin{aligned} m &= \iiint_{\mathcal{W}} f(x, y, z)\, dV = \iiint_{\mathcal{W}} r\, dV = \int_0^2 \int_0^{2\pi} \int_{r^2}^{8-r^2} r^2\, dz\, d\theta\, dr \\ &= 2\pi \int_0^2 (8r^2 - 2r^4)\, dr = 2\pi \left(\frac{8}{3} \cdot 8 - \frac{2}{5} \cdot 32 \right) = \frac{256\pi}{15}. \end{aligned}$$

8. Calculate the work required to move an object from $P = (1, 1, 1)$ to $Q = (3, -4, -2)$ against the force field $\mathbf{F}(x, y, z) = -12r^{-4}\langle x, y, z\rangle$ (distance in meters, force in Newtons), where $r = \sqrt{x^2 + y^2 + z^2}$. Hint: Find a potential function for $\mathbf{F}$.

Potential is $V(x, y, z) = 6(x^2 + y^2 + z^2)^{-1} = 6r^{-2}$, so the work required (where $\mathcal{C}$ is any path from P to Q) is $W = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -(V(Q) - V(P)) = V(P) - V(Q) = V(1, 1, 1) - V(3, -4, -2) = \frac{6}{3} - \frac{6}{29} = \frac{52}{29}$.

9. Find the flow rate of a fluid with velocity field $\mathbf{v} = \langle 2x, y, xy\rangle$ m/s across the part of the cylinder $x^2 + y^2 = 9$ where $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 4$ (distance in meters).

Parametrization $G(\theta, z) = (3\cos\theta, 3\sin\theta, z)$, with $0 \leq \theta \leq \pi/2$ and $0 \leq z \leq 4$. Then $\mathbf{T}_\theta = \langle -3\sin\theta, 3\cos\theta, 0\rangle$, $\mathbf{T}_z = \langle 0, 0, 1\rangle$, $\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = \langle 3\cos\theta, 3\sin\theta, 0\rangle$ (outward-pointing), and

$$\begin{aligned} \text{Flow rate} &= \int_0^{\pi/2} \int_0^4 \langle 6\cos\theta, 3\sin\theta, 9\cos\theta\sin\theta\rangle \cdot \langle 3\cos\theta, 3\sin\theta, 0\rangle\, dz\, d\theta \\ &= \int_0^{\pi/2} \int_0^4 (18\cos^2\theta + 9\sin^2\theta)\, dz\, d\theta = 4\left(18 \cdot \frac{\pi}{4} + 9 \cdot \frac{\pi}{4}\right) = 27\pi \end{aligned}$$

10. Use Green's Theorem to evaluate $\oint_{\mathcal{C}} xy\, dy - y^2\, dx$, where $\mathcal{C}$ is the unit circle in counterclockwise orientation.

$$\begin{aligned} \oint_{\mathcal{C}} xy\, dy - y^2\, dx &= \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x} xy - \frac{\partial}{\partial y}(-y^2) \right) dA \\ &= \iint_{\mathcal{D}} 3y\, dA = 0 \end{aligned}$$

by symmetry. (The integral is the first moment M_x of a unit disk with constant mass density $\rho = 3$.)