Second Test Review, M273, Fall 2011

1. Assume that $f(x, y)$ is a differentiable function. Are the following statements true or false? Justify your answers.

(a) If $\nabla f(a, b) = \mathbf{0}$, then $D_{\mathbf{u}} f(a, b) = 0$ for all unit vectors $\mathbf{u}$.

(b) If $D_{\mathbf{u}} f(a, b) = 0$ for some unit vector $\mathbf{u}$, then $\nabla f(a, b) = \mathbf{0}$.

(c) If $f(x, y) = 0$ for all points on the circle $x^2 + y^2 = 1$, then there exists a point (a, b) in the disk $a^2 + b^2 < 1$ with $\nabla f(a, b) = \mathbf{0}$.

(d) If $x = x(t)$ and $y = y(t)$ are differentiable functions, then $\dfrac{df}{dt} = \nabla f \cdot \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle$.

2. (a) Find the domain and the range of $f(x, y) = 1 - \sqrt{x^2 - y^2}$.

(b) Sketch the domain and a few level curves of $f(x, y) = 1 - \sqrt{x^2 - y^2}$.

3. Find all second partial derivatives of $f(x, y) = \sqrt{x(1 + y)}$.

4. (a) Find the linear approximation of the function $f(x, y) = x^y$ at $(1, 3)$.

(b) Use the result from (a) to estimate $0.9^{3.1}$ without a calculator.

5. The elevation above sea level at a point (x, y) is given by $f(x, y) = \dfrac{10 + x - 2y}{10 + x^2 + y^2}$.

(a) Find the gradient of f.

(b) What is the slope at $(2, 1)$ when you walk in a straight line towards the point $(-2, -2)$?

6. Find and classify all critical points of $f(x, y) = x^3 + xy^2 - x$.

7. Find an equation of the tangent plane at $P = (0, 3, -1)$ to the surface with equation $ze^x + e^{z+1} = xy + y - 3$.

8. Let $f(x, y, z)$ be a function with gradient $\nabla f(x, y, z) = \left\langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2} \frac{z}{x^2+y^2+z^2} \right\rangle$. Use the chain rule to calculate $\partial f/\partial \rho$, $\partial f/\partial \theta$, and $\partial f/\partial \phi$, where (ρ, θ, ϕ) are spherical coordinates, i.e., $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$.