Third Test Review Key, M273Q-03, Spring 2011

1. Calculate $\iint_D (1+x^2)\,dA$, where D is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

$$\iint_D (1+x^2)\,dA = \int_0^1\int_0^y (1+x^2)\,dx\,dy = \int_0^1 \left[x+\frac{x^3}{3}\right]_{x=0}^{x=y} dy = \int_0^1 \left(y+\frac{y^3}{3}\right) dy$$
$$= \left[\frac{y^2}{2}+\frac{y^4}{12}\right]_{y=0}^{y=1} = \frac{1}{2}+\frac{1}{12} = \frac{7}{12} = 0.5833$$

2. Change the order of integration and evaluate

$$\int_0^9\int_0^{\sqrt{y}} \frac{x\,dx\,dy}{(x^2+y)^{1/2}} = \int_0^3\int_{x^2}^9 \frac{x\,dy\,dx}{(x^2+y)^{1/2}} = \int_0^3 \left[2x(x^2+y)^{1/2}\right]_{y=x^2}^{y=9} dx$$
$$= \int_0^3 2x(x^2+9)^{1/2} - 2x(2x^2)^{1/2}\,dx = \int_0^3 2x(x^2+9)^{1/2} - 2^{3/2}x^2\,dx = \left[\frac{2}{3}(x^2+9)^{3/2} - \frac{2^{3/2}}{3}x^3\right]_{x=0}^{x=3}$$
$$= \frac{2}{3}(9+9)^{3/2} - \frac{2^{3/2}}{3}3^3 - \frac{2}{3}(0+9)^{3/2} = 18\sqrt{2}-18 = 7.456.$$

3. Find the centroid of the region $\mathcal{W}$ bounded in spherical coordinates by $\phi=\phi_0$ and the sphere $\rho=R$.

Assuming that $0<\phi_0\le\pi$, and that the region is the one including the north pole, the bounds for integration in spherical coordinates are $0\le\rho\le R$, $0\le\phi\le\phi_0$, and $0\le\theta\le 2\pi$. By symmetry around the z-axis, $\bar{x}=\bar{y}=0$. To find $\bar{z}$ we calculate

$$V = \int_0^R\int_0^{2\pi}\int_0^{\phi_0} \rho^2\sin\phi\,d\phi\,d\theta\,d\rho = \int_0^R \rho^2\,d\rho\int_0^{2\pi}d\theta\int_0^{\phi_0}\sin\phi\,d\phi = \frac{R^3}{3}2\pi(1-\cos\phi_0)$$
$$M_{xy} = \int_0^R\int_0^{2\pi}\int_0^{\phi_0} (\rho\cos\phi)(\rho^2\sin\phi)\,d\phi\,d\theta\,d\rho = \int_0^R\int_0^{2\pi}\int_0^{\phi_0}\rho^3\cos\phi\sin\phi\,d\phi\,d\theta\,d\rho$$
$$= \int_0^R\rho^3\,d\rho\int_0^{2\pi}d\theta\int_0^{\phi_0}\cos\phi\sin\phi\,d\phi = \frac{R^4}{4}2\pi\frac{\sin^2\phi_0}{2} = \frac{\pi R^4\sin^2\phi_0}{4}$$
$$\bar{z} = \frac{M_{xy}}{V} = \frac{\pi R^4\sin^2\phi_0}{4}\cdot\frac{3}{2\pi R^3(1-\cos\phi_0)} = \frac{3R\sin^2\phi_0}{8(1-\cos\phi_0)} = \frac{3R(1-\cos^2\phi_0)}{8(1-\cos\phi_0)} = \frac{3R}{8}(1+\cos\phi_0)$$

4. (a) Parametrize the circle $\mathcal{C}$ of radius 2 with center $(4,5)$ in counterclockwise orientation.

$$\mathbf{c}(t) = \langle 4+2\cos t, 5+2\sin t\rangle,\quad 0\le t\le 2\pi$$

(b) Find $\oint_{\mathcal{C}}(x+y)\,ds$.

$$\oint_{\mathcal{C}}(x+y)\,ds = \int_0^{2\pi}(9+2\cos t+2\sin t)2\,dt = 36\pi = 113.10$$

(c) What could a possible physical interpretation of the integral in (b) be? Give one example. (There are many correct answers here.)

Mass of a circular wire with linear density $\rho=x+y$, one-sided surface area of a circular garden fence with height $h=x+y$.

5. One of the following vector fields is conservative. Find a potential for it, and use the potential to calculate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where the curve $\mathcal{C}$ is given by $\mathbf{r}(t) = \langle t^{3/2}, \cos(\pi t^2)\rangle$, $0 \le t \le 1$.

$$\begin{aligned} \mathbf{F}_1(x,y) &= \langle ye^{xy} + y, xe^{xy} - x\rangle \\ \mathbf{F}_2(x,y) &= \langle ye^{xy} - x, xe^{xy} + y\rangle \end{aligned}$$

Cross-partial condition is satisfied by $\mathbf{F}_2$, but not by $\mathbf{F}_1$. Since both of these are defined in the whole plane, the cross-partial condition is equivalent to being conservative, so $\mathbf{F}_2$ is conservative, $\mathbf{F}_1$ is not. A potential for $\mathbf{F}_2$ is given by

$$f_2(x,y) = e^{xy} - \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

The curve $\mathcal{C}$ has initial point $\mathbf{r}(0) = \langle 0, 1\rangle$ and endpoint $\mathbf{r}(1) = \langle 1, -1\rangle$, so by the Fundamental Theorem for Line Integrals,

$$\int_{\mathcal{C}} \mathbf{F}_2 \cdot d\mathbf{r} = f_2(1,-1) - f_2(0,1) = e^{-1} - \frac{1}{2} + \frac{1}{2} - e^0 + 0 - \frac{1}{2} = \frac{1}{2} + \frac{1}{e} = 0.8679.$$

6. Calculate

$$\iint_{\mathcal{S}} (x^2+y^2)e^{-z}\, dS,$$

where $\mathcal{S}$ is the cylinder with equation $x^2 + y^2 = 9$ for $0 \le z \le 10$.

A parametrization of the cylinder is given by

$$G(\theta, z) = (3\cos\theta, 3\sin\theta, z), \quad 0 \le \theta \le 2\pi,\ 0 \le z \le 10$$

Then

$$\begin{aligned} \mathbf{T}_\theta &= \langle -3\sin\theta, 3\cos\theta, 0\rangle, \\ \mathbf{T}_z &= \langle 0, 0, 1\rangle, \\ \mathbf{n} &= \mathbf{T}_\theta \times \mathbf{T}_z = \langle 3\cos\theta, 3\sin\theta, 0\rangle, \\ dS &= \|\mathbf{n}\|\, d\theta\, dz = 3\, d\theta\, dz, \end{aligned}$$

so

$$\iint_{\mathcal{S}} (x^2+y^2)e^{-z}\, dS = \int_0^{10}\int_0^{2\pi} (9e^{-z})3\, d\theta\, dz = 27(1-e^{-10})2\pi = 54\pi(1-e^{-10}) = 169.64$$