Final Review Key, M 273, Spring 2011

1. True or false? Correct the false statements.

(a) If two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \cdot \mathbf{b} = 0$.

(b) The curve with vector equation $\mathbf{r}(t) = \langle 2t^3, -t^3, 3t^3 \rangle$ is a line.

(c) If $\mathbf{u}$ is a unit vector perpendicular to $\nabla f(\mathbf{x}_0)$, then the directional derivative $D_{\mathbf{u}} f(\mathbf{x}_0)$ is zero.

(d) $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{x^2+y^2}\, dy\, dx = \int_0^2 \int_0^{\pi/2} \frac{1}{r}\, d\theta\, dr.$

(e) If C is a circle of radius R, then $\displaystyle\oint_C x\, dy = R^2$.

(a) False. To correct, replace "parallel" with "perpendicular" or the dot product with the cross product (and the scalar 0 with the 0 vector).

(b) True, it is the line through $\mathbf{0}$ with direction vector $\langle 2, -1, 3 \rangle$.

(c) True, the directional derivative is the dot product of the direction and the gradient.

(d) True.

(e) False, by Green's theorem the integral is $\iint_D 1\, dx\, dy$, where D is the disk bounded by the circle, so the result is the area of the circle, i.e., πR^2.

2. Find the area of the triangle with vertices $(-1, 0, 1)$, $(0, 0, 3)$, and $(-2, 1, 1)$

Calling the three vertices P, Q and R, resp., we get

$$A = \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2} = \frac{|\langle 1, 0, 2 \rangle \times \langle -1, 1, 0 \rangle|}{2} = \frac{|\langle -2, -2, 1 \rangle|}{2} = \frac{3}{2} = 1.5$$

3. Identify and sketch the surface $x^2 = y^2 + 4z^2 - 4z$.

Completing the square we get $x^2 = y^2 + 4(z - 1/2)^2 - 1$. This is a hyperboloid of one sheet centered at $(0, 0, 1/2)$, with an axis running parallel to the x-axis.

4. An athlete throws a ball at an angle of $30°$ to the horizontal at an initial speed of 30 m/s. It leaves his hand 2 meters above the ground.

(a) Find the velocity $\mathbf{v}(t)$ and the position $\mathbf{r}(t)$ at time t.

(b) How high does the ball go?

(Assume that the only relevant force is gravity with an acceleration of $\approx 10\,\mathrm{m/s}^2$.)

(a) Using the x-axis for the horizontal, throwing in the direction of the positive x-axis, we get $\mathbf{v}(0) = 30\langle \cos 30°, \sin 30° \rangle = \langle 15\sqrt{3}, 15 \rangle$, $\mathbf{r}(0) = \langle 0, 2 \rangle$, and $\mathbf{a}(t) = \langle 0, -10 \rangle$. This implies

$$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a}(s)\,ds = \langle 15\sqrt{3}, 15\rangle + \langle 0, -10t\rangle = \langle 15\sqrt{3}, 15 - 10t\rangle$$

and

$$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(s)\,ds = \langle 0, 2\rangle + \langle 15\sqrt{3}t, 15t - 5t^2\rangle = \langle 15\sqrt{3}t, 2 + 15t - 5t^2\rangle$$

(b) At the highest point the vertical component of the velocity is 0, so it occurs when $15 - 10t_h = 0$, i.e., at $t_h = 15/10 = 3/2$. The vertical component of the position at this time is $2 + 15t_h - 5t_h^2 = 2 + 45/2 - 45/4 = \frac{8+90-45}{4} = \frac{53}{4} = 13.25$. So the highest point is 13.25 meters above the ground, occurring 1.5 seconds after the throw.

5. Find the tangent plane to the surface $z = \dfrac{6}{1 + x^2 + y^2}$ at the point $(1, 2, 1)$.

The surface is the level surface $F(x, y, z) = (1 + x^2 + y^2)z = 6$. We know that gradients are perpendicular to level surfaces. Then $\nabla F(x, y, z) = \langle 2xz, 2yz, 1 + x^2 + y^2\rangle$, and $\nabla F(1, 2, 1) = \langle 2, 4, 6\rangle$ is a normal vector to the tangent plane, giving the equivalent equations (one of these is good enough)

$$\begin{aligned}\langle x - 1, y - 2, z - 3\rangle \cdot \langle 2, 4, 6\rangle &= 0,\\ \langle x, y, z\rangle \cdot \langle 2, 4, 6\rangle = \langle 1, 2, 1\rangle \cdot \langle 2, 4, 6\rangle &= 16,\\ 2x + 4y + 6z &= 16.\end{aligned}$$

6. Find and classify the critical points of $f(x, y) = x^3 - y^3 + 3xy$.

First and second derivatives are

$$\begin{aligned} f_x(x, y) &= 3x^2 + 3y\\ f_y(x, y) &= -3y^2 + 3x\\ f_{xx}(x, y) &= 6x\\ f_{yy}(x, y) &= -6y\\ f_{xy}(x, y) &= 3\end{aligned}$$

The critical points are solutions of $3x^2 + 3y = 0$ and $-3y^2 + 3x = 0$. From the second equation $x = y^2$, plugging into the first equation and dividing by 3 gives $3y^4 + 3y = 0$. This equation has two solutions, $y = 0$ and $y = -1$. From $x = y^2$ we get the two critical points $(0, 0)$ and $(1, -1)$.

For the second derivative test, $D = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = -36xy - 9$. At $(0, 0)$ we get $D = -9 < 0$, so there is a saddle at $(0, 0)$. At $(1, -1)$ we get $D = 36 - 9 > 0$ and $f_{xx}(1, -1) = 6 > 0$, so there is a local minimum at $(1, -1)$.

7. Find the maximum and minimum values of $f(x, y, z) = 8x - 4z$ on the ellipsoid $x^2 + 10y^2 + z^2 = 5$.

Lagrange multipliers with $g(x, y, z) = x^2 + 10y^2 + z^2$ gives

$$\begin{aligned} 8 &= 2\lambda x \\ 0 &= 20\lambda y \\ -4 &= 2\lambda z \end{aligned}$$

From the first and/or third equation we see that $\lambda \neq 0$, so from the second equation we get $y = 0$. Multiplying the third equation by 2 and adding it to the first equation gives $0 = 2\lambda x + 4\lambda z = 2\lambda(x + 2z)$. Again, since $\lambda \neq 0$ we get $x + 2z = 0$, so $x = -2z$. Plugging everything into $x^2 + 10y^2 + z^2 = 5$ gives $5 = (-2z)^2 + 0^2 + z^2 = 5z^2$. This has two solutions, $z = \pm 1$, so the whole system has the two solutions $(-2, 0, 1)$ and $(2, 0, 1)$ for (x, y, z). (We do not really care what λ is.) Plugging both into f we get $f(-2, 0, 1) = 8 \cdot (-2) - 4 \cdot 1 = -20$ and $f(2, 0, -1) = 8 \cdot 2 - 4 \cdot (-1) = 20$, so the maximum is 20 and the minimum is -20.

8. Consider a triangular lamina D with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$, with mass density $\rho(x, y) = 1 + x + y$. Find its total mass.

$$\begin{aligned} m &= \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{1-x} (1 + x + y)\, dy\, dx = \int_0^1 \left[y + xy + \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 (1 - x) + x(1 - x) + \frac{(1 - x)^2}{2}\, dy = \int_0^1 \frac{3 - 2x - x^2}{2}\, dx = \left[\frac{3x - x^2 - x^3/3}{2} \right]_0^1 \\ &= \frac{3 - 1 - 1/3}{2} = \frac{5}{6}. \end{aligned}$$

9. Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-z^2-y^2}} z\, dx\, dy\, dz$ by changing to spherical coordinates.

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-z^2-y^2}} z\, dx\, dy\, dz &= \int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} (\rho \cos\phi)(\rho^2 \sin\phi)\, d\theta\, d\phi d\rho \\ &= \frac{\pi}{2} \int_0^2 \int_0^{\pi/2} \rho^3 \cos\phi\, \sin\phi\, d\phi\, d\rho = \frac{\pi}{2} \int_0^2 \rho^3 \left[\frac{\sin^2\phi}{2} \right]_{\phi=0}^{\phi=\pi/2} d\rho \\ &= \frac{\pi}{4} \int_0^2 \rho^3\, d\rho = \frac{\pi}{4} \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\rho=2} = \pi. \end{aligned}$$

10. Let the curve C be given by $\mathbf{r}(t) = \langle \cos 2t, t, \sin 2t \rangle$, $0 \leq t \leq \pi$. Sketch the curve and find $\displaystyle\int_C y\, ds$.

C is one winding of a helix of radius 1 along the y-axis, advancing by π in the y-direction.

$$\int_C y\, ds = \int_0^\pi t\sqrt{(-2\sin 2t)^2 + 1^2 + (2\cos 2t)^2}\, dt = \int_0^\pi t\sqrt{5}\, dt = \frac{\pi^2\sqrt{5}}{2} \approx 11.035.$$

11. Find a potential for $\mathbf{F}(x, y) = \langle 2 + y\cos(xy), 3 + x\cos(xy)\rangle$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the upper half of the unit circle $x^2 + y^2 = 1$, in counterclockwise direction.

Solving for the potential $f(x, y)$, we get

$$\begin{aligned} f(x, y) &= \int 2 + y\cos(xy)\, dx = 2x + \sin(xy) + C(y) \\ f_y(x, y) &= x\cos(xy) + C'(y) = 3 + x\cos(xy) \\ C(y) &= \int 3\, dy = 3y + C, \end{aligned}$$

so one potential is $f(x, y) = 2x + 3y + \sin(xy)$, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-1, 0) - f(1, 0) = -2 - 2 = -4.$$