Second Test Review Key, M 273, Fall 2011

1. Find and sketch the domain of the function $f(x, y) = \ln(x - \sqrt{y})$. Sketch a few level curves of f.

$D = \{(x, y)|y \geq 0,\, x > \sqrt{y}\}$, i.e. the region under the graph of $y = x^2$ for $x > 0$, including the positive part of the x-axis, but not the graph itself. The level lines for $z = k$ are parabolas $x = \sqrt{y} + e^k$, i.e., the boundary of the domain shifted to the right by e^k.

2. Find the partial derivatives f_x, f_y, f_z, and f_{xyz} of $f(x, y, z) = \dfrac{xy^2 - yx^2}{z}$.

$f_x = (y^2 - 2yx)/z$, $f_y = (2xy - x^2)/z$, $f_z = (-xy^2 + yx^2)/z^2$, and $f_{xyzx} = (-2y + 2x)/z^2$.

3. Find the linear approximation of $f(x, y) = x + y - \sin(x^2 - 4y^2)$ at the point $(2, 1)$, and use it to estimate $f(2.01, 0.99)$.

$f_x(x, y) = 1 - 2x\cos(x^2 - 4y^2)$, $f_y(x, y) = 1 + 8y\cos(x^2 - 4y^2)$, so $L(x, y) = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 3 - 3(x - 2) + 9(y - 1) = -3x + 9y$, and $L(2.01, 0.99) = 3 - 3 \cdot 0.01 + 9 \cdot (-0.01) = 2.88$.

4. Use the chain rule to find $\partial z/\partial r$ and $\partial z/\partial \theta$ for $z = e^{-x^2-y^2}$, $x = r\cos\theta$, $y = 2r\sin\theta$ at $r = 1, \theta = 0$.

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\
&= -2xe^{-x^2-y^2}\cos\theta - 2ye^{-x^2-y^2} \cdot 2\sin\theta \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -2xe^{-x^2-y^2}(-r\sin\theta) - 2ye^{-x^2-y^2} \cdot 2r\cos\theta
\end{aligned}$$

At $r = 1$, $\theta = 0$, we have $x = 1$, $y = 0$, $\cos\theta = 1$, and $\sin\theta = 0$, so $\dfrac{\partial z}{\partial r}(1, 0) = 8$ and $\dfrac{\partial z}{\partial r}(1, 0) = 2e^{-1}$ and $\dfrac{\partial z}{\partial \theta}(1, 0) = 0$.

5. Find the directional derivative of $f(x, y, z) = \dfrac{x^2 - y^4}{z}$ at $(4, 2, 1)$ in the direction of $\mathbf{v} = \langle 0, 3, -4\rangle$.

$$\begin{aligned}
\nabla f &= \langle 2x/z, -4y^3/z, (-x^2 + y^4)/z^2\rangle \\
\nabla f(4, 2, 1) &= \langle 8, -32, 0\rangle \\
\mathbf{u} &= \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{0^2 + 3^2 + (-4)^2}}\langle 0, 3, -4, \rangle = \left\langle 0, \frac{3}{5}, -\frac{4}{5}\right\rangle \\
D_{\mathbf{u}}(4, 2, 1) &= \nabla f(4, 2, 1) \cdot \mathbf{u} = 8 \cdot 0 + (-32) \cdot \frac{3}{5} + 0 \cdot \left(-\frac{4}{5}\right) = -\frac{96}{5}.
\end{aligned}$$

6. Find all critical points of $f(x, y) = x^3 - 3x + y^2$ and determine whether they are local maxima, minima, or saddle points.

$\nabla f = \langle 3x^2 - 3, 2y \rangle$, so the zeros of the gradient are $x = \pm 1$, $y = 0$, i.e. the critical points are $(1, 0)$ and $(-1, 0)$. Second derivatives are $f_{xx} = 6x$, $f_{yy} = 2$, and $f_{xy} = f_{yx} = 0$. For the second derivative test we also need $D = f_{xx} f_{yy} - (f_{xy})^2 = 12x$. At $(1, 0)$ we have $D = 12 > 0$ and $f_{xx} = 6 > 0$, so $(1, 0)$ is a local minimum. At $(-1, 0)$ we have $D = -12 < 0$, so $(-1, 0)$ is a saddle point.

7. Find an equation for the tangent plane to $xyz = \cos(x + 2y + 3z)$ at $(2, -1, 0)$.

Implicit differentiation, assuming $z = z(x, y)$, leads to $yz + xyz_x = -(1 + 3z_x)\sin(x + 2y + 3z)$ and $xz + xyz_y = -(2 + 3z_y)\sin(x + 2y + 3z)$. Plugging in the values $x = 2$, $y = -1$, $z = 0$, leads to $-2z_x = 0$ and $-2z_y = 0$, so the tangent plane at $(2, -1, 0)$ is $z = 0$, i.e., the xy-plane.

8. True or false? (Assume that f has continuous second partial derivatives.)

(a) If $\nabla f(x, y) = \mathbf{0}$, then f has a local extremum at (x, y).

False, could be a saddle point.

(b) If f has a local extremum at (x, y), then $\nabla f(x, y) = \mathbf{0}$.

True, theorem from the book.

(c) If f has two local maxima, then f must have a local minimum.

False, we have seen counterexamples to this.

(d) $f_{xy} = f_{yx}$.

True, Clairaut's Theorem.