Third Test Review Key, M273, Spring 2011

1. Find $\iiint_E y^2z^2\,dV$, where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

$$\begin{aligned}\iiint_E y^2z^2\,dV &= \iint_D \int_0^{1-y^2-z^2} y^2z^2\,dx\,dA_{y,z} = \iint_D (1-y^2z^2)y^2z^2\,dA_{y,z}\\ &= \int_0^{2\pi}\int_0^1 (1-r^2)(r\cos t)^2(r\sin t)^2 r\,dr\,dt\\ &= \int_0^{2\pi}\cos^2 t\,\sin^2 t\,dt\cdot\int_0^1 (r^5-r^7)\,dr = \frac{\pi}{4}\cdot\frac{1}{24} = \frac{\pi}{96}\approx 0.0327,\end{aligned}$$

where D is the unit disk in the yz-plane, and integration over D is done using polar coordinates $y = r\cos t$ and $z = r\sin t$.

2. Sketch the region of integration, reverse the order of integration and evaluate $\int_0^4\int_{\sqrt{y}}^2 \frac{ye^{x^2}}{x^3}\,dx\,dy$.

This is the region under the graph of $y = x^2$ from $x = 0$ to $x = 2$. Reversing the order gives

$$\begin{aligned}\int_0^4\int_{\sqrt{y}}^2 \frac{ye^{x^2}}{x^3}\,dx\,dy &= \int_0^2\int_0^{x^2}\frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^2 \frac{e^{x^2}}{x^3}\left[\frac{y^2}{2}\right]_{y=0}^{y=x^2} dx\\ &= \int_0^2 \frac{xe^{x^2}}{2}\,dx = \left[\frac{e^{x^2}}{4}\right]_0^2 = \frac{e^4-1}{4}\approx 13.4\end{aligned}$$

3. Consider a lamina that occupies the region D between the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$ in the first quadrant with mass density equal to the distance to $(0,0)$. Find the mass and center of mass of the lamina.

Using polar coordinates we get

$$m = \int_0^{\pi/2}\int_1^2 r\cdot r\,dr\,dt = \frac{\pi}{2}\left[\frac{r^3}{3}\right]_1^2 = \frac{\pi}{2}\cdot\frac{7}{3} = \frac{7\pi}{6}\approx 3.67$$

and

$$\begin{aligned}\bar{x} &= \frac{1}{m}\int_0^{\pi/2}\int_1^2 r\cos t\cdot r\cdot r\,dr\,dt = \frac{6}{7\pi}\int_0^{\pi/2}\cos t\,dt\int_1^2 r^3\,dr\\ &= \frac{6}{7\pi}\left[\sin t\right]_0^{\pi/2}\left[\frac{r^4}{4}\right]_1^2 = \frac{6}{7\pi}\cdot\frac{15}{4} = \frac{45}{14\pi}\approx 1.02\end{aligned}$$

By symmetry, $\bar{y} = \bar{x}$, so the center of mass is at $\left(\frac{45}{14\pi}, \frac{45}{14\pi}\right)\approx(1.02, 1.02)$.

4. Calculate $\int_C y\,ds$, where C is the part of the graph $y = 2x^3$ from $(0,0)$ to $(1,2)$.

With the parametrization $x = t$, $y = 2t^3$, $0 \le t \le 1$ we get $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 6t^2$, so

$$\int_C y\,ds = \int_0^1 2t^3\sqrt{1+36t^4}\,dt = \int_1^{37} \frac{\sqrt{u}}{72}\,du = \left[\frac{u^{3/2}}{108}\right]_1^{37} = \frac{37^{3/2}-1}{108} \approx 2.07.$$

(Here we used the substitution $u = 1 + 36t^4$.)

5. Find the work done by the force field $\mathbf{F}(x,y) = \langle y, -x\rangle$ on a particle that moves along the graph of $y = x^3 - x$ from $(-1,0)$ to $(1,0)$.

With the parametrization $x = t$, $y = t^3 - t$, $-1 \le t \le 1$, we get $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 3t^2 - 1$, and

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^1 \langle t^3 - t, -t\rangle \cdot \langle 1, 3t^2 - 1\rangle\,dt = \int_{-1}^1 (-2t^3)dt = 0.$$

6. Which of the following vector fields are conservative? Find a potential for one of them and use it to calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the arc of the unit circle from $(1,0)$ to $(0,1)$ in counterclockwise direction.

$$\mathbf{F}_1(x,y) = \langle x^2, x^2\rangle \qquad \mathbf{F}_2(x,y) = \langle 2xy, x^2\rangle$$
$$\mathbf{F}_3(x,y) = \langle e^y, e^x\rangle \qquad \mathbf{F}_4(x,y) = \langle e^x, e^y\rangle$$

All of these are defined in the whole plane, so they are conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. This shows that $\mathbf{F}_2$ and $\mathbf{F}_4$ are conservative, $\mathbf{F}_1$ and $\mathbf{F}_3$ are not. Potentials for the two conservative vector fields are $f_2(x,y) = x^2y$ and $f_4(x,y) = e^x + e^y$. The integrals are $\int_C \mathbf{F}_2\cdot d\mathbf{r} = f_2(0,1) - f_2(1,0) = 0$ and $\int_C \mathbf{F}_4\cdot d\mathbf{r} = f_4(0,1) - f_4(1,0) = 0$.

7. Use Green's Theorem to evaluate $\oint_C \sin(1+x^2)dx + x(1+y)\,dy$, where C is the unit circle.

Writing D for the unit disk, Green's Theorem gives

$$\oint_C \sin(1+x^2)dx + x(1+y)\,dy = \iint_D (1+y)\,dA = \pi.$$

(It is easy to explicitly integrate in polar coordinates, but here is an easy argument without calculations for the last step. The integral of 1 is the area of D, which we know to be π. The integral of y is the first moment with respect to the x-axis, and by symmetry this is 0, so the integral is the sum of π and 0.)