M431 Final Review Problems, Spring 2011

(This turned out a little longer and harder than the actual exam.)

1. Assuming the definition of a group is known, give the definition of a ring.

2. Assuming the definitions of groups and subgroups are known, give the definition of a normal subgroup.

3. What is the symmetry group of the circle $x^2 + y^2 = 1$ in the plane? Is it Abelian?

4. (Since the definition of extensions is somewhat confusing, here it is again: G is an extension of A by B if G contains a normal subgroup N such that $A \approx N$ and $G/N \approx B$.)

(a) Find all (isomorphism classes of) extensions of $\mathbb{Z}_3$ by $\mathbb{Z}_2$.

(b) Find all (isomorphism classes of) extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_3$.

5. Let G be a group, and let A and B be subgroups of G with $A \leq B \leq G$. Show that $A \lhd G$ implies $A \lhd B$, and give an example to show that the converse is not always true.

6. Find all (isomorphism classes of) homomorphic images of S_5.

7. (a) Give an example to show that if A and B are subgroups of a group G, then $AB = \{ab : a \in A,\, b \in B\}$ need not be a subgroup of G.

(b) Prove that if we additionally assume that $A \lhd G$ or $B \lhd G$, then $AB \leq G$.

(c) Prove that if we assume $A \lhd G$ and $B \lhd G$, then $AB \lhd G$.

8. Consider $\mathbb{R}^2$ with the usual (vector addition) $(a, b) + (c, d) = (a + c, b + d)$. Which of the following definitions of a product make $\mathbb{R}^2$ a ring? Which make it an integral domain? Which make it a field?

(i)	$(a, b)(c, d) = (ac, bd)$	(ii)	$(a, b)(c, d) = (ac, 0)$
(iii)	$(a, b)(c, d) = (a + c, b + d)$	(iv)	$(a, b)(c, d) = (ac - bd, ad + bc)$
(v)	$(a, b)(c, d) = (ac - 2bd, ad + bc)$	(vi)	$(a, b)(c, d) = (ad, 0)$

9. Find the greatest common divisor of $a(x) = x^3 - x - 1$ and $b(x) = x^3 + x^2 + x - 1$ as polynomials over $\mathbb{Z}_5$, and express it as $m(x)a(x) + n(x)b(x)$ with polynomials $m(x), n(x) \in \mathbb{Z}_5[x]$.

10. Let R be a ring with at least two elements. Assume that $x^2 = x$ for all $x \in R$. Show that R has characteristic 2.

11. ((c)–(e) are probably too hard for an actual exam...) Let X be a finite non-empty set, and let $P(X)$ be the ring of subsets of X with operations $A + B = (A \cup B) \setminus (A \cap B)$ and $AB = A \cap B$. (We verified in class that this is indeed a ring.)

(a) Find the characteristic of $P(X)$.

(b) Show that for every subset $Y \subseteq X$, the subsets of Y form an ideal $P(Y)$ of $P(X)$.

(c) Show that every ideal of $P(X)$ is of the form $P(Y)$ for some $Y \subseteq X$.

(d) Give an example of a subring of $P(X)$ which is not an ideal.

(e) Show that the quotient ring $P(X)/P(Y)$ is isomorphic to $P(X \setminus Y)$.