Final Exam Review Key, M182, Spring 2013

1. Let $m > 0$ be a positive constant.

(a) Find the area of the region enclosed by the graphs of $y = x^2$ and $y = mx$.

$$A = \int_0^m (mx - x^2)\,dx = \left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m = \frac{m^3}{2} - \frac{m^3}{3} = \frac{m^3}{6}.$$

(b) Set up the integrals for, but do not evaluate, the volume and the surface area of the solid obtained by rotating the region in (a) about the x-axis.

$$\begin{aligned} V &= \pi \int_0^m (m^2x^2 - x^4)\,dx \\ S &= 2\pi \left(\int_0^m x^2\sqrt{1+4x^2}\,dx + \int_0^m mx\sqrt{1+m^2}\,dx \right) \end{aligned}$$

2. How much work is done lifting a 12-m chain that has mass density 3 kg/m (initially coiled on the ground) so that its top end is 10 m above the ground?

The part of the chain between x and $x + \Delta x$ meters above the ground has mass $\Delta m = 3\Delta x$ (kg), and it is lifted $\approx x$ meters agains the force of gravity $\Delta F = (\Delta m)g = 3\Delta x \cdot 9.81$ (Newton), so the work on this part of the chain against gravity is $\Delta W = (\Delta F)x \approx 3\Delta x \cdot 9.81x$ (Joules). The total work is obtained by integrating as

$$W = 3 \cdot 9.81 \int_0^{10} x\,dx = 29.43 \left[\frac{x^2}{2}\right]_0^{10} = 29.43 \cdot 50 = 1,471.5\,\text{J}.$$

3. Evaluate the following integrals.

(a) $\displaystyle\int x^2 e^{4x}\,dx$

$$= x^2\frac{e^{4x}}{4} - \int 2x\frac{e^{4x}}{4}\,dx = \frac{x^2e^{4x}}{4} - \left(2x\frac{e^{4x}}{16} - \int 2\frac{e^{4x}}{16}\,dx\right) = \frac{x^2e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{e^{4x}}{32} + C$$

(b) $\displaystyle\int \frac{2x-1}{x^2-5x+6}\,dx$

$$= \int \left(\frac{-3}{x-2} + \frac{5}{x-3}\right) dx = -3\ln|x-2| + 5\ln|x-3| + C$$

(c) $\displaystyle\int \ln(x^2+9)\,dx$

$$\begin{aligned} &= x\ln(x^2+9) - \int x\frac{2x}{x^2+9}\,dx = x\ln(x^2+9) - \int \left(2 - \frac{18}{x^2+9}\right) dx \\ &= x\ln(x^2+9) - 2x + 6\tan^{-1}\left(\frac{x}{3}\right) + C \end{aligned}$$

(d) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}\,dx$

Substituting $x = 3\sin\theta$ gives

$$\int \frac{3\cos\theta}{3\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

4. Determine for which $p > 0$ the improper integral $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^p+1}}\,dx$ converges.

Comparing to p-integrals gives convergence for $p > 4$, divergence for $p \le 4$.

5. Find the Taylor polynomial $T_4(x)$ centered at $x = 1$, for the function $f(x) = x\ln x$.

$$T_4(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4.$$

6. Solve the differential equation $y' = 1 - y^2$ with initial value $y(0) = 0$.

Separating variables, integrating, solving for y, finding C from the initial value, gives:

$$\int \frac{dy}{1-y^2} = \int dx \implies \frac{1}{2}(\ln(1+y) - \ln(1-y)) = x + C, \text{ so } C = 0$$
$$\implies \frac{1+y}{1-y} = e^{2x} \implies y = \frac{e^{2x}-1}{e^{2x}+1}.$$

7. Determine whether the following series converge absolutely, conditionally, or not at all.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{1}{n}\right)^n$

Converges absolutely, e.g., by root test.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\sqrt{n}}$

Converges by Leibniz test, but does not converge absolutely by comparison with $\sum\frac{1}{n}$, so it converges conditionally.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{3^n}$

Converges absolutely as a geometric series with $r = \frac{-2}{3}$.

8. Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$. (No need to test the endpoints here.)

Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)!)^2(2n)!x^{n+1}}{(2n+2)!(n!)^2x^n}\right| = \frac{(n+1)^2|x|}{(2n+1)(2n+2)} \to \frac{|x|}{4} \text{ as } n \to \infty$$

This limit is less than 1 if $|x| < 4$, and greater than 1 if $|x| > 4$, so the radius of convergence is $R = 4$.

9. Find the Taylor series of $f(x) = \tan^{-1}(2x)$ centered at $c = 0$, and determine the interval on which it converges.

$$\begin{aligned} f'(x) &= \frac{2}{1+4x^2} = \frac{2}{1-(-4x^2)} = \sum_{n=0}^{\infty} 2(-4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n} \\ &= 2 - 8x^2 + 32x^4 - 128x^6 \pm \dots \end{aligned}$$

by the geometric series with $r = -4x^2$, for $|x| < 1/2$. Integrating and observing that $f(0) = 0$ gives

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{2n+1} = 2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \frac{128}{7}x^7 \pm \dots$$

Radius of convergence is again $R = 1/2$ (does not change by integrating), but the behavior at the endpoints changes:

For $x = 1/2$ we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \dots$$

which converges by the Leibniz test. For $x = -1/2$ we get the same series without alternating signs, i.e.,

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \pm \dots$$

which diverges by limit comparison with $\sum \frac{1}{n}$. Combining these statements, the interval of convergence is $(-1/2, 1/2]$.