Fall 2016 – M 597

Topics in Mathematics **Homework 1**

Complex Dynamics Solutions

1. Let f be a quadratic polynomial with two distinct roots $a \neq b$. Show that the associated Newton's method $N_f(z) = z - \dfrac{f(z)}{f'(z)}$ is conformally conjugate to the Newton's method for $f_0(z) = z^2 - 1$. Conclude that Newton's method with initial value $z_0 \in \mathbb{C}$ converges to a iff $|z_0 - a| < |z_0 - b|$, and that it converges to b iff $|z_0 - b| < |z_0 - a|$. (Hint: Use the results about Newton's method for f_0 from class.)

Solution: We saw that $N_{f_0}(w) = w - \frac{w^2-1}{2w} = w + \frac{1}{w}$. Now $f(z) = c(z-a)(z-b)$, so $f'(z) = c(2z - a - b)$, and $N_f(z) = z - \frac{(z-a)(z-b)}{2z-a-b}$. The conjugation T should map a and b to the roots of f_0, e.g., $T(a) = 1$ and $T(b) = -1$. The unique affine linear map doing this is $w = T(z) = \frac{2z-a-b}{a-b}$, and a little bit of straight-forward calculation shows that $T \circ N_f \circ T^{-1}(w) = N_{f_0}(w)$.

In class we saw that given any initial value $w_0 \in \mathbb{C}$, the Newton's method N_{f_0} converges to 1 iff $|w_0 - 1| < |w_0 + 1|$ (an inequality for the right halfplane), and that it converges to -1 iff $|w_0 + 1| < |w_0 - 1|$. This implies that the conjugate Newton's method N_f converges to $a = T^{-1}(1)$ iff $|T(z_0) - 1| < |T(z_0) + 1|$. Plugging in the definition of T and multiplying both sides by $|a - b|/2$, this is equivalent to $|z_0 - a| < |z_0 - b|$. Similarly, the Newton's method N_f converges to b iff the initial guess satisfies $|z_0 - b| < |z_0 - a|$.

2. Let T be a Möbius transformation which is not the identity, and assume that $z_0 \in \mathbb{C}$ is a fixed point of T with $T'(z_0) = 1$. Show that $T^n(z) \to z_0$ as $n \to \infty$, for all $z \in \hat{\mathbb{C}}$. (Hint: Derivatives of fixed points are invariant under analytic conjugation, and we classified the dynamical behavior of Möbius transformations in class.)

Solution: We classified Möbius transformations in class, and the only ones that have a fixed point with multiplier 1 are parabolic Möbius transformations. It follows that T has z_0 as the unique fixed point, and that $T^n(z) \to z_0$ for all z, as $n \to \infty$.

3. Let f be a quadratic polynomial. Show that f is conformally conjugate to a unique quadratic polynomial of the form $f_c(z) = z^2 + c$.

> **Solution:** Existence is easiest to do in two steps, first conjugating by multiplication with a constant to make the polynomial monic, then conjugating by a translation to get rid of the linear term. We use the fact that conjugation is transitive. (It is in fact an equivalence relation.) Here are the details: Let $f(z) = \alpha z^2 + \beta z + \gamma$ be a quadratic polynomial with $\alpha \neq 0$. Then $g(z) = \alpha f(\alpha^{-1}(z)) = z^2 + \beta z + \alpha\gamma$. Now $h(z) = g(z - \beta/2) + \beta/2 = (z-\beta/2)^2 + \beta(z-\beta/2) + \alpha\gamma = z^2 - \beta^2/4 + \alpha\gamma$ is of the claimed form with $c = -\beta^2/4 + \alpha\gamma$.
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> Uniqueness: If f_{c_1} and f_{c_2} are conjugate, then there exists $T(z) = az + b$ with $a \neq 0$ such that $T \circ f_{c_1} \circ T^{-1} = f_{c_2}$, so $T \circ f_{c_2} = f_{c_1} \circ T$. Plugging in the functions, we get $a(z^2 + c_1) + b = (az + b)^2 + c_2$, so $az^2 + c_1 + b = a^2 z^2 + 2abz + c_2$ for all z. Comparing coefficients, this implies that $b = 0$, $a = 1$, and $c_1 = c_2$.

4. Let $|c| < 1/4$, and let $f_c(z) = z^2 + c$ with Julia set J_c. Show that

   (a) $|f_c(z)| > |z|$ for $|z| > \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$, and

   (b) $|f_c(z)| \leq |z|$ for $|z| = \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$.

   Conclude that for every $\epsilon > 0$ there exists $\delta > 0$ such that J_c is contained in the annulus $\{z \in \mathbb{C} \colon 1 - \epsilon < |z| < 1 + \epsilon\}$ whenever $|c| < \delta$. (I.e., for small c, the Julia set J_c is close to the unit circle.)

> **Solution:** $|f_c(z)| - |z| \geq |z|^2 - |z| - |c| = \left(|z| - \frac{1}{2}\right)^2 - \frac{1}{4} - |c| > 0$ shows part (a), even without the assumption that $|c| < 1/4$.
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> For part (b), $|f_c(z)| - |z| \leq |z|^2 - |z| + |c| = \left(|z| - \frac{1}{2}\right)^2 - \frac{1}{4} + |c| = 0$.
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> For the second part of the problem we use the notation $\mathbb{D}_r = \{z \colon |z| < r\}$ and $\overline{\mathbb{D}}_r = \{z \colon |z| \leq r\}$ for the open and closed disks of radius r centered at 0.
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> Part (a) shows that $R_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ is an escape radius for f_c, so the filled-in Julia set is contained in $\overline{\mathbb{D}}_{R_c}$ by a theorem on escape radii from class. With $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$, part (b) combined with the maximum principle shows that the disk $\overline{\mathbb{D}}_{r_c}$ is forward-invariant. I.e., if $z \in \overline{\mathbb{D}}_{r_c}$, then $f_c^n(z) \in \overline{\mathbb{D}}_{r_c}$ for all $n \geq 1$, and this implies that $z \in K_c$ (since it has a bounded orbit.) Together this implies that $\overline{\mathbb{D}}_{r_c} \subseteq K_c \subseteq \overline{\mathbb{D}}_{R_c}$. As a consequence, $J_c = \partial K_c \subset \overline{\mathbb{D}}_{R_c} \setminus \mathbb{D}_{r_c} = \{z \colon r_c \leq |z| \leq R_c\}$. Since $\lim\limits_{c \to 0} r_c = \lim\limits_{c \to 0} R_c = 1$, the claim follows easily.