Fall 2016 – M 597

Topics in Mathematics **Homework 1**

Complex Dynamics Due Friday, September 9, in class

1. Let f be a quadratic polynomial with two distinct roots $a \neq b$. Show that the associated Newton's method $N_f(z) = z - \dfrac{f(z)}{f'(z)}$ is conformally conjugate to the Newton's method for $f_0(z) = z^2 - 1$. Conclude that Newton's method with initial value $z_0 \in \mathbb{C}$ converges to a iff $|z_0 - a| < |z_0 - b|$, and that it converges to b iff $|z_0 - b| < |z_0 - a|$. (Hint: Use the results about Newton's method for f_0 from class.)

2. Let T be a Möbius transformation which is not the identity, and assume that $z_0 \in \mathbb{C}$ is a fixed point of T with $T'(z_0) = 1$. Show that $T^n(z) \to z_0$ as $n \to \infty$, for all $z \in \hat{\mathbb{C}}$. (Hint: Derivatives of fixed points are invariant under analytic conjugation, and we classified the dynamical behavior of Möbius transformations in class.)

3. Let f be a quadratic polynomial. Show that f is conformally conjugate to a unique quadratic polynomial of the form $f_c(z) = z^2 + c$.

4. Let $|c| < 1/4$, and let $f_c(z) = z^2 + c$ with Julia set J_c. Show that

   (a) $|f_c(z)| > |z|$ for $|z| > \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$, and

   (b) $|f_c(z)| \leq |z|$ for $|z| = \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$.

   Conclude that for every $\epsilon > 0$ there exists $\delta > 0$ such that J_c is contained in the annulus $\{z \in \mathbb{C} \colon 1 - \epsilon < |z| < 1 + \epsilon\}$ whenever $|c| < \delta$. (I.e., for small c, the Julia set J_c is close to the unit circle.)