1. Normal forms via Uniformization Theorem

These are alternative proofs of the local normal forms near attracting, repelling, and parabolic fixed points using the Uniformization Theorem.

1.1. **Uniformization.**

Theorem 1.1 (Uniformization Theorem)**.** *Every Riemann surface is conformally isomorphic to a quotient X/Γ, where X is either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$, and Γ is a group of Möbius transformation acting freely and properly discontinuously on X.*

Here the group acts *freely* if only the identity element has fixed points in X, and it acts *properly discontinuously* if for every compact subset $K \subset X$, there are only finitely many group elements $\gamma \in \Gamma$ with $\gamma(K) \cap K \neq \emptyset$. (This is equivalent to the property that every point $z \in X$ has a neighborhood U such that $\gamma(U) \cap U = \emptyset$ for every non-identity $\gamma \in \Gamma$.)

We say that X is the *universal cover* of the Riemann surface $S = X/\Gamma$, and we call S *elliptic* if $X = \hat{\mathbb{C}}$, *parabolic* if $X = \mathbb{C}$, and *hyperbolic* if $X = \mathbb{D}$. Since every Möbius transformation has a fixed point in $\hat{\mathbb{C}}$, the only elliptic Riemann surface is $\hat{\mathbb{C}}$ itself (up to conformal isomorphism). In the case $X = \mathbb{C}$, the only fixed-point free Möbius transformations are translations, so in this case Γ is either trivial, an infinite cyclic group generated by one translation $z \mapsto z + a$ with $a \neq 0$, or a group generated by two translations $z \mapsto z + a$ and $z \mapsto z + b$ with $a/b \notin \mathbb{R}$. The resulting parabolic surfaces are the plane $\mathbb{C}$, the cylinder $\mathbb{C}/a\mathbb{Z}$ (which is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$ for all $a \neq 0$), and the torus $\mathbb{C}/(a\mathbb{Z} + b\mathbb{Z})$. (These tori are not all conformally isomorphic, and the study of conformal invariants of tori is the beginning of Teichmüller theory.) As a consequence, most Riemann surfaces are hyperbolic, as made precise in the following theorem.

Theorem 1.2. *Let S be a Riemann surface which is not homeomorphic to the sphere, the plane, the cylinder, or the torus. Then S is hyperbolic, i.e., S is conformally isomorphic to $\mathbb{D}/\Gamma$, where Γ is a subgroup of* $\mathrm{Möb}(\mathbb{D})$ *acting freely and properly discontinuously on the unit disk $\mathbb{D}$.*

Remark. There are obviously surfaces homeomorphic to the plane which are hyperbolic (e.g., the unit disk), and there are cylinders which are hyperbolic as well, e.g., $S = \mathbb{H}/\mathbb{Z}$, where $\mathbb{H}$ is the upper halfplane (which is hyperbolic since it is conformally isomorphic to the unit disk), or $S_a = \{z \colon 0 < \operatorname{Im} z < a\}/\mathbb{Z}$, so there is no purely topological characterization of conformal type. However, these (plane and cylinder) are the only examples where there exist homeomorphic Riemann surfaces which have different conformal types (hyperbolic vs. parabolic). In particular, for surfaces of genus $g \geq 2$, the type is always hyperbolic.

1.2. **Attracting and repelling fixed points.**

Theorem 1.3 (Schröder linearization theorem)**.** *Let $f(z) = \lambda z + a_2 z^2 + \dots$ be analytic with $|\lambda| \notin \{0, 1\}$. Then there exists a unique analytic map $\phi(z) = z + b_2 z^2 + \dots$ and a constant $r > 0$ such that $\phi(f(z)) = \lambda \phi(z)$ for $|z| < r$.*

In other words, the following diagram commutes, where U and V are neighborhoods of zero, and all maps fix zero.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"] & V \arrow[d, "\phi"] \\ \mathbb{C} \arrow[r, "w \mapsto \lambda w"] & \mathbb{C} \end{tikzcd}$$

FIGURE 1. Neighborhood of zero for $f(z) = \frac{i}{2}z + z^2$. Shown are the first few images of the circle $|z| = 1/8$, with same colors corresponding to points in the same orbit. The outermost annulus is a fundamental domain for the orbit space, and the spiral arcs indicate which boundary points are glued together to get the torus S.

Proof. If ϕ is such a conjugacy for f, and if f^{-1} is the local inverse of f, then ϕ conjugates f^{-1} to $w \mapsto \lambda^{-1}w$. This shows that it is enough to show the claim of the theorem for the attracting case $0 < |\lambda| < 1$, the repelling case then follows by passing to the inverse.

Let $\mu > 0$ be chosen such that $|\lambda| < \mu < 1$. Then there exists $r > 0$ such that

$$|f(z)| \leq \mu|z| < |z| \text{ and } \left|\frac{f(z)}{\lambda z} - 1\right| < 1 \text{ for } 0 < |z| < r, \tag{1}$$

and that f is one-to-one on $\mathbb{D}_r$. Now we define a Riemann surface $S = \mathbb{D}_r^*/\sim$, where $\mathbb{D}_r^* = \{z \in \mathbb{C}\colon 0 < |z| < r\}$ is the punctured disk of radius r, and the equivalence relation is given by $z_1 \sim z_2$ iff there exists $n \geq 0$ such that $f^n(z_1) = z_2$ of $f^n(z_2) = z_1$. (This surface S is the (local) *orbit space* of f.)

By (1), every point $z \in \mathbb{D}_r^*$ has a disk neighborhood U such that the iterates $\{f^n(U)\}$ are mutually disjoint, so that distinct points in U are not equivalent. This shows that the natural projection $P : \mathbb{D}_r^* \to S$ is one-to-one on U. Any other component of $P^{-1}(P(U))$ is of the form $\mathbb{D}_r \cap f^n(U)$ with $n \in \mathbb{Z}$ (where negative n means the corresponding iterate of the local inverse), and since f^n is analytic for all such n, this projection induces a Riemann surface structure on S which makes P analytic. Note that this projection is not a covering map, though.

Fixing $\rho \in (0, r)$, every point in $\mathbb{D}_r^*$ is equivalent to exactly one point in $\mathbb{D}_\rho \setminus f(\mathbb{D}_\rho)$, so another way to visualize the Riemann surface S is to take this fundamental domain bounded by the circle $S_\rho = \partial\mathbb{D}_\rho$ and its image $f(S_\rho)$, and to glue the two boundary curves via f. Since $f(S_\rho)$ is contained in the interior of $\mathbb{D}_r$, and f is an orientation-preserving homeomorphism from S_ρ onto $f(S_\rho)$, the resulting surface is a topological torus. For an illustration of this construction, see Figure 1.

Fix $z_0 = \rho$, and consider the fundamental group $\pi_1(S, z_0)$ of S with base point z_0. Let $\alpha = S_\rho$ (parametrized in the standard way) and β be any smooth curve from z_0 to $z_1 = f(z_0)$ in $\mathbb{D}_r^*$. Then $\pi_1(S, z_0) = \langle P(\alpha), P(\beta)\rangle$, i.e., the canonical projections of these two curves generate the fundamental group of S.

By the Uniformization Theorem, S is conformally equivalent to $\mathbb{C}/\Gamma$, where Γ is a group of translations algebraically isomorphic to $\mathbb{Z}^2$, acting freely and properly discontinuously on $\mathbb{C}$. In particular, there is a universal covering map $\Theta : \mathbb{C} \to S$ with deck transformation group $\Gamma = \{T \in \text{Möb}(\mathbb{C})\colon \Theta \circ T = \Theta\}$.

It is convenient to switch to a logarithmic coordinate system at this point via the universal covering $\exp : H_r \to \mathbb{D}_r^*$, where $H_r = \{\zeta \in \mathbb{C}\colon \operatorname{Re}\zeta < \log r\}$. Writing $f(z) = \lambda z g(z)$ with $g(0) = 1$, we get from (1) that $\operatorname{Log} g(z)$ is analytic for $|z| < r$, so

$$F(\zeta) = \log f(e^\zeta) = \log\lambda + \zeta + \operatorname{Log} g(e^\zeta) \tag{2}$$

is analytic in H_r, with $F(H_r) \subseteq H_{\mu r} \subset H_r$, and $\exp\circ F = f \circ \exp$ in H_r. Here $\log\lambda$ is an arbitrary choice of the logarithm of λ, fixed for the rest of the proof.

Since H_r is simply connected, the map $P \circ \exp : H_r \to S$ can be lifted to an analytic map $\Psi : H_r \to \mathbb{C}$ satisfying $\Theta \circ \Psi = P \circ \exp$ on H_r. This construction is best illustrated by the following commutative diagram.

$$\begin{tikzcd}
H_r \arrow[r, "F"] \arrow[d, "\exp"'] & H_r \arrow[r, "\Psi"] \arrow[d, "\exp"'] & \mathbb{C} \arrow[ddl, "\Theta"] \\
\mathbb{D}_r^* \arrow[r, "f"] \arrow[d, "P"'] & \mathbb{D}_r^* \arrow[d, "P"'] & \\
S \arrow[r, "\text{id}"] & S &
\end{tikzcd}$$

Since $P(e^\zeta) = P(e^{\zeta+2\pi i}) = P(e^{F(\zeta)})$ for all $\zeta \in H_r$, we conclude that the lifts $\Psi(\zeta)$, $\Psi(\zeta+2\pi i)$, and $\Psi(F(\zeta))$ all have to be equivalent under the action of Γ. This means that there exist $a, b \in \mathbb{C}$ with

$$\Psi(\zeta + 2\pi i) = \Psi(\zeta) + a \text{ and } \Psi(F(\zeta)) = \Psi(\zeta) + b, \text{ where } a, b \in \Gamma. \tag{3}$$

Here we use the somewhat sloppy convention identifying a and b with the translations $T_a(z) = z + a$ and $T_b(z) = z + b$. A priori, a and b might depend on ζ, but the functional equations show that a and b depend continuously on ζ. Since Γ is discrete and H_r is connected, this shows that a and b are constants. Furthermore, the line segments $[\zeta, \zeta + 2\pi i]$ and $[\zeta, F(\zeta)]$ project to generators of the fundamental group of S under $P \circ \exp$, so the lifts a and b are generators of Γ.

By precomposing Θ with a linear map and replacing ψ with $-\psi$ if necessary, we may assume that $a = 2\pi i$ and $\operatorname{Re} b < 0$. This shows that $\Psi(\zeta + 2\pi i) = \Psi(\zeta) + 2\pi i$, so that the function $\zeta \mapsto \Psi(\zeta + 2\pi i) - \Psi(\zeta)$ is $2\pi i$-periodic, which means that $\Psi(\zeta) = \zeta + h(e^\zeta)$ with some analytic function $h : \mathbb{D}_r^* \to \mathbb{C}$. This implies that the function $\phi(z) = e^{\Psi(\log z)} = ze^{h(z)}$ is well-defined and analytic in $\mathbb{D}_r^*$, with

$$\phi(f(z)) = e^{\Psi(\log f(z))} = e^{\Psi(F(\log z))} = e^{\Psi(\log z)+b} = \phi(z)e^b \tag{4}$$

for all $z \in \mathbb{D}_r^*$. This implies that $\phi(f^n(z)) = \phi(z)e^{bn} \to 0$ locally uniformly as $n \to \infty$, since $\operatorname{Re} b < 0$. As a consequence, the isolated singularity at $z = 0$ is removable for ϕ and for h, with $\phi(0) = 0$, and $\phi'(0) = e^{h(0)} \neq 0$. Differentiating (4) at $z = 0$ gives $\lambda = f'(0) = e^b$. Multiplying ϕ with a constant preserves the functional equation, so the map $\phi_0(z) = \frac{\phi(z)}{\phi'(0)}$ has the desired normalization and satisfies $\phi_0(f(z)) = \lambda\phi_0(z)$.

Again, the situation is probably best understood in a commutative diagram, slightly modified and rotated from the previous one.

$$\begin{tikzcd}
\mathbb{D}_r^* \arrow[d, "f"] & H_r \arrow[l, "\exp"'] \arrow[r, "\Psi"] \arrow[d, "F"] & \mathbb{C} \arrow[r, "\exp"] \arrow[d, "T_b"] & \mathbb{C}^* \arrow[d, "M_\lambda"] \\
\mathbb{D}_r^* & H_r \arrow[l, "\exp"'] \arrow[r, "\Psi"] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^*
\end{tikzcd}$$

where $T_b(z) = z + b$ is translation by b, and $M_\lambda(w) = \lambda w$ is multiplication by $\lambda = e^b$. The crucial part in using the uniformization theorem is the existence of the map Ψ, "flattening" the orbit space S. Lifting everything, the action of F becomes a translation on the flat torus $\mathbb{C}/\Gamma$. The minor technical difficulty is that the orbit space is the quotient of $\mathbb{D}_r^*$ by a semigroup action (forward iterates of f), not a group action, so that the corresponding projection is not a covering map, and we can't quite use all the powerful theorems from the theory of covering spaces directly.

For the uniqueness, assume that ϕ_0 and ϕ_1 are two such maps. Then $\theta = \phi_1 \circ \phi_0^{-1}$ satisfies $\theta(\lambda w) = \lambda\theta(w)$ for $|w|$ small, so θ induces a conformal self-map of the torus $\mathbb{C}^*/\langle w \mapsto \lambda w\rangle$. This lifts to a conformal self-map of the cylinder $\theta : \mathbb{C}^* \to \mathbb{C}^*$. (This is just a fancy way of saying that the functional equation enables us to extend θ to the punctured plane. Since $\theta(0) = 0$, this shows that θ is a conformal self-map of the plane fixing 0, so $\theta(w) = cw$ with some constant $c \neq 0$. By the normalization $\theta'(0) = 1$, so $c = 1$, $\theta(w) = w$, and thus $\phi_0(z) = \phi_1(z)$ for all z. $\square$

1.3. **Parabolic fixed points.** The setup for parabolic points is a little more complicated, and we assume that the following is known. Let $f(z) = z + a_{m+1}z^{m+1} + \dots$ be analytic in a neighborhood of zero with $a_{m+1} \neq 0$ and let α be an attracting direction, i.e., $m\alpha + \arg a_{m+1} = \pi \pmod{2\pi}$. Then the change of coordinates $w = \psi(z) = -\frac{1}{ma_{m+1}z^m}$ conjugates f restricted to some small sector

$$A = \left\{ z \in \mathbb{C} \colon 0 < |z| < \delta,\ |\arg z - \alpha| < \frac{\pi - \varepsilon}{m} \right\}$$

to $F(w) = w + 1 + O(|w|^{-1/m})$ on some domain of the form

$$B = \left\{ w \in \mathbb{C} \colon |w| > \frac{1}{\delta},\ |\arg w| < \pi - \varepsilon \right\}.$$

With this setup, Fatou proved the following.

Theorem 1.4 (Fatou coordinates)**.** *There exists some $R > 0$ and a conformal map ϕ in the halfplane $H_R = \{w \in \mathbb{C} \colon \operatorname{Re} w > R\}$ such that $F(H_R) \subset H_R$ and $\phi(F(w)) = \phi(w) + 1$ for all $w \in H_R$. Furthermore, $\lim_{v \to \pm\infty} \operatorname{Im} \phi(u + iv) = \pm\infty$ for every $u > R$.*

Remark. As a corollary, the map f itself has a petal-shaped domain $P = \psi^{-1}(H_R)$ centered about each attracting direction and a conformal map $\phi \circ \psi$ in P conjugating f to $w \mapsto w + 1$. The exact formulation of this theorem is left up to the reader.

The last assertion about the images of vertical lines basically says that the orbit space is conformally equivalent to $\mathbb{C}/\mathbb{Z}$. With more work one can actually say quite a bit more about the asymptotics of $\phi(u+iv)$ as $v \to \pm\infty$. (Exercise!)

There are Fatou coordinates in every attracting direction, and passing to the inverse there are Fatou coordinates in every repelling direction, too. With a bit of work one can construct these in overlapping domains so that there are natural maps between parts of adjacent attracting and repelling orbit spaces. These orbit spaces are called *Ecalle cylinders*, and the transition maps are called *horn maps*, and they have proved extremely valuable in studying parabolic bifurcations and renormalizations near parabolic points. As another application, these horn maps lead to a rich family of analytic conjugacy invariants of parabolic fixed points.

Proof. Again we want to form the orbit space of F. In order to do so, note that there exists $R > 0$ such that

$$|F(w) - w - 1| < \frac{1}{2} \text{ and } |F'(w) - 1| < \frac{1}{2} \text{ for } \operatorname{Re} w > R. \tag{5}$$

The second condition in particular implies that F is one-to-one in H_R, since for all distinct $w_0, w_1 \in H_R$:

$$\operatorname{Re} \frac{F(w_1) - F(w_0)}{w_1 - w_0} = \operatorname{Re} \int_0^1 F'((1-t)w_0 + tw_1)\, dt \geq \frac{1}{2},$$

so $F(w_1) \neq F(w_2)$. (This is an instance of the more general theorem that any analytic function F satisfying $\operatorname{Re} F' > 0$ in a convex domain is one-to-one.)

We form a Riemann surface $S = H_R/\sim$ where $w_1 \sim w_2$ iff there exists $n \geq 0$ such that $F^n(w_1) = w_2$ or $F^n(w_2) = w_1$. As in the attracting case, we can find a fundamental domain as follows. Fixing $\rho > R$, every point in H_R is equivalent to exactly one point in $H_\rho \setminus F(H_\rho)$, so S is obtained topologically by taking the closure of this domain and gluing together the boundary curves (the straight vertical line of real part ρ and its "almost translated" image) via F. Topologically, this domain is a vertical strip, and the gluing map on the boundary is an orientation-preserving homeomorphism, so the resulting surface is homeomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. For an illustration of this construction, see Figure 2.

Unfortunately, topological cylinders can be either of parabolic or of hyperbolic conformal type, so we have to determine which type S has. The "standard" way to show that S has parabolic type is to use conformal modulus. (See Milnor, Problem 10-c for a guide of how to do this, even though he does not mention the word "modulus".) Here is an alternative argument.

Every conformal cylinder is conformally equivalent to $\mathbb{C}/\mathbb{Z}$, $\mathbb{H}/\mathbb{Z}$, or $S_a/\mathbb{Z}$, where $S_a = \{z \in \mathbb{C} \colon 0 < \operatorname{Re} z < a\}$ is a horizontal strip of height $a > 0$. Just as in the case of an attracting fixed point (actually, a little easier since H_R is already simply connected and we do not have to pass to its universal cover), the uniformization of the orbit space S gives us an analytic map $\phi : H_R \to X$ (in the attracting case we called this map Ψ) as a lift of the projection $P : H_R \to S$, where X is the universal cover, so it is one of $\mathbb{C}$, $\mathbb{H}$, or S_a. (The halfplane and the strip are conformally equivalent, but it is convenient to have deck transformation group $\mathbb{Z}$ in all cases.) And again as in the previous case, we get that $\phi(F(w)) = \phi(w) + 1$ for all $w \in H_R$.

Assuming that X is hyperbolic, pick any point $w_0 \in H_R$ and consider its orbit $w_n = F^n(w)$ as well as its image under ϕ given by $z_n = \phi(w_n) = z_0 + n$ by the functional equation. Note that $\operatorname{Re} w_n \to \infty$ and $|w_{n+1} - w_n| \to 1$. The halfplane H_R carries a hyperbolic metric d_{H_R} given by the length element $\rho(w)|dw| = \frac{|dw|}{\operatorname{Re} w - R}$, so $d_{H_R}(w_n, w_{n+1}) \to 0$ as $n \to \infty$. As an

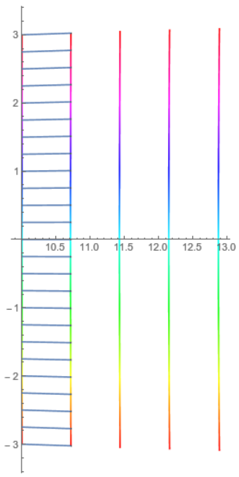

FIGURE 2. The map $f(z) = z - z^4 + z^5$ in the coordinate $w = \frac{1}{3z^3}$, as $F(w) = w + 1 + O(|w|^{-1/3})$. Shown are the first few images of the line $\operatorname{Re} w = 10$, with same colors corresponding to points in the same orbit. The leftmost vertical strip is a fundamental domain for the orbit space, and the line segments indicate how the boundary is glued together to get the cylinder S.

analytic map between hyperbolic Riemann surfaces, ϕ is weakly contracting by the Schwarz lemma, so $d_X(z_n, z_{n+1}) \to 0$ as $n \to \infty$. However, integer translations act on X as isometries, so $d_X(z_n, z_{n+1}) = d_X(z_0, z_0 + 1)$ is a positive constant independent of n. This contradiction shows that X has to be parabolic, i.e., S is conformally equivalent to $\mathbb{C}/\mathbb{Z}$.

It remains to show that ϕ is conformal, i.e., one-to-one. As a lift of a conformal map between the cylinders S and $\mathbb{C}/\mathbb{Z}$, the image of the fundamental domain $\phi(H_\rho \setminus F(H_\rho))$ is a fundamental domain for integer translation in the plane, bounded by two simple curves γ and $\gamma + 1$. Then the functional equation shows that ϕ is a conformal map from H_ρ onto the connected component of $\mathbb{C} \setminus \gamma$ which contains $\gamma + 1$. $\square$