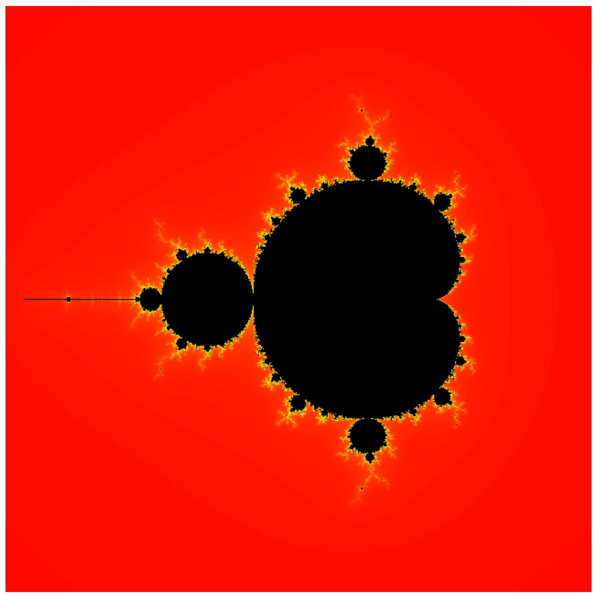
M 597 LECTURE NOTES
TOPICS IN MATHEMATICS
COMPLEX DYNAMICS

LUKAS GEYER



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1. Introduction

In complex dynamics we study the behavior of iterated analytic maps. Typical set-up is an analytic self-map $f : X \to X$, where X is some Riemann surface, typically the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. (An analytic map to $\hat{\mathbb{C}}$ is just a meromorphic map.) We will denote the n-th iterate of a function f by $f^n = f \circ f \circ \ldots \circ f$.

There are many good references for the required Complex Analysis background, e.g., the classic book by Ahlfors [Ahl78], or the more recent one by Gamelin [Gam01]. The basic theory of Complex Dynamics is well covered in the books by Beardon [Bea91], Carleson and Gamelin [CG93], Milnor [Mil06], and Steinmetz [Ste93]. The early history of the theory is covered in the books of Alexander [Ale94], and the one by Alexander, Iavernaro, and Rosa [AIR12].

Since at least the 1980s, computer pictures popularized, motivated, and helped the research into complex dynamics. Most of the pictures in these notes were produced using Wolf Jung's excellent *Mandel* software, freely available with source code at `http://www.mndynamics.com/indexp.html`. I encourage everyone to explore dynamics using this or other software.

2. Newton's method

One of the earliest examples of complex dynamics were the papers on Newton's method by Schröder [Sch70] and Cayley [Cay79] in the 1870s. Newton's method for an analytic function f is the root-finding algorithm given by the recursive formula $z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$, starting from some initial guess z_0. Cayley and Schröder both proved the following theorem.

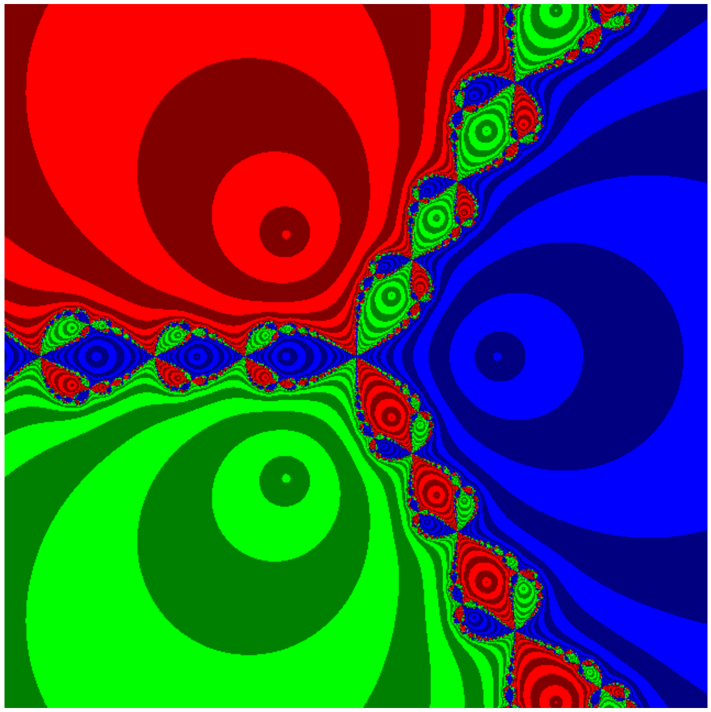


FIGURE 1. Newton's method for $f(z) = z^3 - 1$ with different colors for the basins of the three roots

Theorem 2.1. *Let f be a quadratic polynomial with distinct roots $a \neq b$. Then Newton's method converges to a iff $|z_0 - a| < |z_0 - b|$. It converges to b iff $|z_0 - b| < |z_0 - a|$, and it does not converge to either root if $|z - a| = |z - b|$.*

Proof. Assuming first that $f_0(z) = z^2 - 1 = (z-1)(z+1)$ with $a = 1$ and $b = -1$, Newton's method is $z_{n+1} = z_n - \frac{z_n^2 - 1}{2z_n} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right)$. Defining a new sequence (w_n) by $w_n = \frac{z_n - 1}{z_n + 1}$, one gets the recursive formula $w_{n+1} = w_n^2$, so by induction $w_n = w_0^{2^n}$. This shows that $w_n \to 0$ iff $|w_0| < 1$, and $w_n \to \infty$ iff $|w_0| > 1$. Translating this back to the sequence (z_n), we get that $z_n \to 1$ iff $|z_0 - 1| < |z_0 + 1|$, and that $z_n \to -1$ iff $|z_0 + 1| < |z_0 - 1|$.

Now if $f(\zeta) = c(\zeta - a)(\zeta - b)$ with distinct roots $a, b \in \mathbb{C}$, then we can first transform the Newton's method sequence (ζ_n) to $z_n = \frac{2\zeta_n - a - b}{a - b}$ and verify easily that $z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right)$, i.e., that (z_n) is the Newton's method for f_0 given above. The claims then easily follow from the first part. Details are left to the reader. □

The proof given here is not one of the original proofs, which were slightly more involved and can be found in [Ale94]. The transformation of one recursive sequence into another by means of an invertible transformation is an example of *conjugation*, a very powerful tool in dynamics (and other branches of mathematics) which will be introduced more formally later. Cayley and Schröder both failed to deal with Newton's method for polynomials of higher degree, for which the dynamics are considerably more complicated, with infinitely many basins with fractal boundaries, even for simple polynomials like $f(z) = z^3 - 1$, as illustrated in Figure 1.

3. Möbius transformations

Möbius transformations, or *fractional linear transformations*, are maps of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. They are exactly the invertible conformal self-maps of the Riemann sphere.

Lemma 3.1. *Every non-identity Möbius transformation f has one or two fixed points in $\hat{\mathbb{C}}$.*

Proof. If f fixes ∞, then it can be written as $f(z) = az + b$ with $a \neq 0$. If $a \neq 1$, then the fixed point equation $az + b = z$ has a unique solution. Otherwise we know $f(z) = z + b$ with $b \neq 0$ (since f is not the identity), and the equation $z + b = z$ has no solution.

If f does not fix ∞, then $f(z) = \frac{az+b}{cz+d}$ with $c \neq 0$, so the fixed point equation $\frac{az+b}{cz+d} = z$ is equivalent to $az + b = cz^2 + dz$. This is a quadratic equation which either has one (double) root, or no roots at all.

In both cases, the total number of fixed points is 1 or 2. □

In order to classify the dynamics of Möbius transformations, it is useful to introduce the concept of conjugation.

Definition 3.2. Two maps f and g are *(conformally) conjugate* iff there exists a conformal map ϕ such that $g = \phi \circ f \circ \phi^{-1}$. If we want to emphasize the map ϕ (which might not be unique), we say that f and g are conjugate *via* ϕ. We also write $f \sim g$ or $f \overset{\phi}{\sim} g$.

There are many different variations of conjugation, local, global, analytic, quasiconformal, topological, etc. For now we will only work with global conformal conjugacy on the Riemann sphere, in which case ϕ will be a Möbius transformation.

Lemma 3.3. *Conjugation is an equivalence relation. In particular, if $f \overset{\phi}{\sim} g$ and $g \overset{\psi}{\sim} h$, then $g \overset{\phi^{-1}}{\sim} f$ and $f \overset{\psi\circ\phi}{\sim} g$. Also, $f \overset{\mathrm{id}}{\sim} f$.*

The proof of this lemma is straightforward and left to the reader. Classification of Möbius transformations will depend crucially on the behavior at their fixed points, so the following definition is useful.

Definition 3.4. If $z_0 \in \mathbb{C}$ is a fixed point of the analytic map f, then its *multiplier* is defined as $\lambda = f'(z_0)$. The fixed point is

- *super-attracting* iff $\lambda = 0$,
- *attracting* iff $0 < |\lambda| < 1$,
- *repelling* iff $|\lambda| > 1$,
- *rationally indifferent* iff λ is a root of unity
- *irrationally indifferent* iff $|\lambda| = 1$, but λ is not a root of unity.

Lemma 3.5. *Multipliers are invariant under conjugation, i.e., if f has a fixed point z_0 with multiplier λ, and if $g = \phi \circ f \circ \phi^{-1}$ with ϕ conformal, then g has a fixed point at $w_0 = \phi(z_0)$ with multiplier λ.*

Proof. $g(w_0) = \phi(f(\phi^{-1}(w_0))) = \phi(f(z_0)) = \phi(z_0) = w_0$, so w_0 is fixed by g. Then $g'(w_0) = \phi'(z_0)f'(z_0)(\phi^{-1})'(w_0)$ by the chain rule, and $(\phi^{-1})'(w_0) = 1/\phi'(z_0)$ by the inverse function rule. Together this implies $g'(w_0) = f'(z_0) = \lambda$. □

Since multipliers of fixed points in $\mathbb{C}$ are invariant under conjugation, we can define the multiplier of a fixed point at ∞ by conjugation. E.g., if $f(\infty) = \infty$, then the multiplier of this fixed point is defined to be the multiplier of 0 for $g(w) = 1/f(1/w)$. (This is using $\phi(z) = 1/z$, but any other ϕ with $\phi(\infty) \neq \infty$ would work as conjugation, too.)

Now we can proceed to classify dynamics of non-identity Möbius transformations. If f has only one fixed point at z_0, then we can conjugate f by $T(z) = \frac{1}{z-z_0}$ to get a Möbius transformation $g = T \circ f \circ T^{-1}$ with $g(\infty) = \infty$ and no other fixed points. This shows that $g(z) = z + b$ with $b \neq 0$. We can further conjugate g to h by $S(z) = b^{-1}z$, so $h(z) = b^{-1}g(bz) = z+1$. This shows that any Möbius transformation with one fixed point is conjugate to $h(z) = z + 1$, with $h^n(z) = z + n \to \infty$ for all $z \in \hat{\mathbb{C}}$. Back to the original function f, we get that $f^n(z) \to z_0$ for $n \to \infty$, for all $z \in \hat{\mathbb{C}}$. Furthermore, every orbit $(f^n(z))$ lies on a circle or lines through z_0. Möbius transformations like this are callled *parabolic*.

If f has two fixed points at z_0 and z_1, we can again conjugate by a Möbius transformation T with $T(z_0) = \infty$ and $T(z_1) = 0$, to get a Möbius transformation $g = T \circ f \circ T^{-1}$ with fixed points at 0 and ∞, so that $g(z) = az$ and $g^n(z) = a^n z$ with some $a \notin \{0, 1\}$.

If $0 < |a| < 1$, then $g^n(z) = a^n \to 0$ as $n \to \infty$, for all $z \in \mathbb{C}$. If $|a| > 1$, then $g^n(z) \to \infty$ for all $z \in \hat{\mathbb{C}} \setminus \{0\}$. (These two situations are actually conjugate to each other by $S(z) = 1/z$.) Möbius transformation like these are called *hyperbolic* or *loxodromic*. They are characterized by having one repelling fixed point with multiplier a, where $|a| > 1$, and one attracting fixed point with multiplier $1/a$. Under iteration, the whole sphere except for the repelling fixed point converges to the attracting fixed point.

If $a = e^{2\pi i p/q}$ is a root of unity with $p, q \in \mathbb{Z}$, then $g^q(z) = z$, so all orbits are periodic of period q, and Möbius transformations like this are called *elliptic of finite order*.

If $|a| = 1$, but a is not a root of unity, then $g^n(z)$ is dense on the circle of radius $r = |z|$, and these Möbius transformations are called *elliptic of infinite order*.

4. A first look at polynomials and the Mandelbrot set

Before we start with the general theory, let us first look at dynamics of quadratic polynomials in a slightly informal and exploratory way. First of all, we would like to simplify our life by conjugating quadratic polynomials to a simple form, just like we did for Möbius transformations. If $f(z) = \alpha z^2 + \beta z + \gamma$ is a quadratic polynomial with $\alpha \neq 0$, then we can first conjugate with $T(z) = \alpha z$, so that $g(z) = T \circ f \circ T^{-1}(z) = \alpha\left(\alpha\left(\frac{z}{\alpha}\right)^2 + \beta\frac{z}{\alpha} + \gamma\right) = z^2 + \beta z + \alpha\gamma$, and then with $S(z) = z + \frac{\beta}{2}$, so that $h(z) = S \circ g \circ S^{-1}(z) = g\left(z - \frac{\beta}{2}\right) + \frac{\beta}{2} = \left(z - \frac{\beta}{2}\right)^2 + \beta\left(z - \frac{\beta}{2}\right) + \alpha\gamma + \frac{\beta}{2} = z^2 + \alpha\gamma + \frac{\beta}{2} - \frac{\beta^2}{4}$. If we define $c = \alpha\gamma + \frac{\beta}{2} - \frac{\beta^2}{4}$, then $h(z) = z^2 + c$. This shows that every polynomial is conjugate to a polynomial of the form $f_c(z) = z^2 + c$. It is also straight-forward to check that c is unique.

In the section on Newton's method we already looked at the dynamics of $f_0(z) = z^2$ which turned out to be very simple (in general terms.)

Computer experiments or some basic mathematical curiosity with quadratic and higher-degree polynomials indicate that there always is a region of the plane where orbits converge to ∞, and that all other orbits stay within some bounded region of the plane. This suggest to define these two sets for a polynomial f of degree at least 2, which (for reasons explained later) are called the *basin of infinity* $A_f(\infty) = \{z \in \mathbb{C} \colon f^n(z) \to \infty\}$, and the *filled-in Julia set* $K_f = \{z \in \mathbb{C} \colon \{f^n(z)\}_n \text{ is bounded.}\}$. Their common boundary is called the *Julia set* $J_f = \partial K_f = \partial A_f(\infty)$. When talking about the quadratic family $f_c(z) = z^2 + c$, we will also use the notation $A_c(\infty)$, K_c, and J_c.

Theorem 4.1. *Let f be a polynomial of degree at least 2. Then $A_f(\infty)$ is open and connected, K_f and J_f are compact, and all three sets are* completely invariant *under f, i.e., $f(K_f) = K_f = f^{-1}(K_f)$ and similar for the other two sets.*

Before proving this theorem, let us first prove the following helpful result.

Lemma 4.2. *Let $f(z) = a_d z^d + \ldots + a_0$ be a polynomial with $a_d \neq 0$, and $d \geq 2$. Then there exists $R > 0$ such that for all z with $|z| > R$ we have $|f(z)| > |z|$. For any such radius R, we have that $f^n(z) \to \infty$ for all $|z| > R$.*

Remark. Any such radius R is called an *escape radius* for f.

Proof of Lemma 4.2. We know that $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = \lim_{z\to\infty} |z|^{d-1}|a_d + a_{d-1}z^{-1} + \ldots + a_0 z^{1-d}| = \infty$ (since $d - 1 \geq 1$ and $a_d \neq 0$), so there exists $R > 0$ such that $|f(z)| > |z|$ for $|z| > R$.

For the second part of the claim, let z_0 be arbitrary with $|z_0| > R$, and let $z_n = f^n(z_0)$. Let $R_0 = |z_0|$. By induction, $|z_n| \geq R_0$ for all n, and since $g(z) = \frac{|f(z)|}{|z|}$ is a continuous function from $K = \{z \in \hat{\mathbb{C}} \colon |z| \geq R_0\}$ to $(1, \infty]$, with $g(\infty) = \infty$, it attains a minimum $\lambda > 1$. This implies that $|z_n| \geq \lambda^n |z_0|$, and so $z_n \to \infty$. □

Now we can go on to prove the theorem.

Proof of Theorem 4.1. Let $U = \{z \in \mathbb{C} \colon |z| > R\}$, where R is an escape radius for f. Then $A_f(\infty) = \{z \in \mathbb{C} \colon f^n(z) \in U \text{ for some } n \geq 0\} = \bigcup_{n=0}^{\infty} f^{-n}(U)$. Since U is open and f is continuous, this is a union of open sets, so it is open. This implies immediately that K_f is closed, and J_f is closed as the boundary of a set. Now $A_f(\infty)$ contains U, so K_f and J_f are both contained in $\overline{\mathbb{D}}_R$, showing that they both are compact.

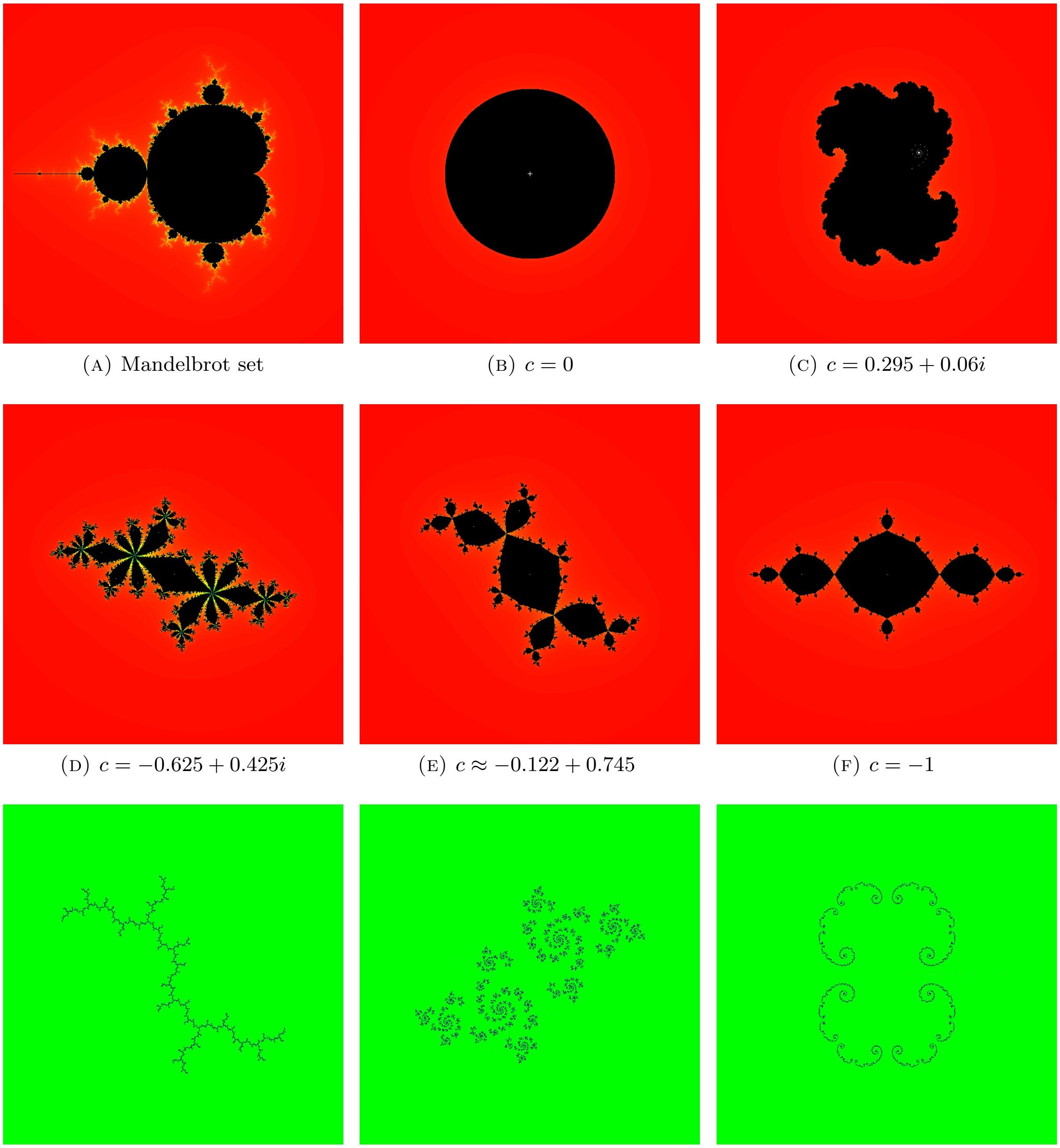
In order to show connectedness of $A_f(\infty)$, assume that this is false. Then $A_f(\infty)$ has a bounded component G. We also know that $\partial G \subset K_f$, so $|f^n(z)| \leq R$ for all $z \in \partial G$ and $n \geq 1$. By the maximum principle, $|f^n(z)| \leq R$ for all $z \in G$ and $n \geq 1$. This implies that $G \subseteq K_f$, contradicting the assumption.

If we have $w = f(z)$, then $f^n(w) = f^{n+1}(z)$, so $f^n(w) \to \infty$ iff $f^n(z) \to \infty$, and so either both z and w are in $A_f(\infty)$, or neither of them is. We also know that every point $w \in \mathbb{C}$ has at least one preimage $z \in \mathbb{C}$ with $f(z) = w$. Together this shows complete invariance of $A_f(\infty)$ and K_f. By continuity and openness of f, the complete invariance of the common boundary J_f follows, too. □

Experimenting with computer pictures for the quadratic family, it looks like J_c is frequently quite complicated and fractal, that for many parameters all orbits in K_c eventually converge to some periodic cycle, that there is a quite complicated bifurcation locus, and that for large $|c|$, the filled-in Julia set almost disappears into a faint "dust."

As we will see later, the filled-in Julia set K_c and the Julia set J_c are connected if and only if the orbit of the point $z = 0$ is bounded, otherwise $K_c = J_c$ is a Cantor set, i.e., a non-empty compact totally disconnected sets without isolated points. The *Mandelbrot set* is defined as $M = \{c \in \mathbb{C} \colon J_c \text{ is connected}\}$, and by the previous remark it is the set of parameters $c \in \mathbb{C}$ for which $\{f_c^n(0)\}_{n=1}^{\infty}$ is a bounded sequence. For pictures of the Mandelbrot set and various examples of Julia sets, see Figure 2.

For $c \notin M$, computer experiments suggest that almost all orbits diverge to ∞, and for c in the interior of M, the orbits of points in the interior of the filled-in Julia set seem to always converge to some periodic cycle. This (experimentally very well verified) statement is actually the largest open conjecture in the field.



(A) Mandelbrot set (B) $c = 0$ (C) $c = 0.295 + 0.06i$

(D) $c = -0.625 + 0.425i$ (E) $c \approx -0.122 + 0.745$ (F) $c = -1$

(G) $c = i$ (H) $c = -0.29 - 0.695i$ (I) $c = 0.3$

FIGURE 2. Mandelbrot set M and various quadratic (filled-in) Julia sets K_c, all drawn in the square window $[-2, 2] \times [-2, 2]$. Figure 2e is known as "Douady's rabbit" and 2f as the "basilica". Figure 2g is from the boundary of M, and the last two figures are Cantor Julia sets from the complement of M. Since they do not have interior, they are drawn with different colors to actually see them.

Conjecture 4.3 (Hyperbolicity Conjecture)**.** *Every component of the interior of M is* hyperbolic*, i.e., for every c in the interior of M, and for every z in the interior of the filled-in Julia set K_c, the orbit $\{f_c^n(z)\}$ converges to an attracting periodic orbit.*

It turns out that it is easy to prove that there can be at most one such attracting periodic orbit for a given c, and if it exists, it does indeed attract every point in the interior of K_c. So the conjecture is equivalent to the existence of an attracting periodic orbit for parameters c in the interior of M. Douady and Hubbard proved that this conjecture would follow from the following open conjecture, which so far has been the main line of attack at the hyperbolicity conjecture.

Conjecture 4.4 (Local Connectivity Conjecture)**.** *M is* locally connected*, i.e., every point $c \in M$ has a basis of connected neighborhoods.*

This conjecture is also known as MLC (**M**andelbrot set is **L**ocally **C**onnected), and it has been verified at many points of ∂M (local connectivity is trivial at interior points), but current techniques seem unlikely to yield the full result.

In order to start a more systematic study of dynamics of the quadratic family and polynomials in general, a good point to start is the study of fixed points and periodic points.

Definition 4.5. A *fixed point* of an analytic map f is a point z_0 such that $f(z_0) = z_0$. The *multiplier* of f at z_0 (sometimes also referred to as the multiplier of z_0) is the derivative $\lambda = f'(z_0)$. The fixed point is

- *super-attracting* if $\lambda = 0$,
- *attracting* if $0 < |\lambda| < 1$,
- *rationally indifferent* if $|\lambda| = 1$, and λ is a root of unity,
- *irrationally indifferent* if $|\lambda| = 1$, and λ is not a root of unity,
- *repelling* if $|\lambda| > 1$.

A *periodic point* of period $q \geq 1$ is a fixed point of the q-th iterate f^q which is not fixed by any iterate f^n with $1 \leq k < q$, i.e., a point z_0 such that $z_k = f^k(z_0)$ are distinct points for $k = 0, 1, \ldots, q-1$, and $z_q = z_0$. The associated *periodic orbit* is the set $\{z_0, z_1, \ldots, z_{q-1}\}$, and its *multiplier* is $\lambda = (f^q)'(z_0) = f'(z_0)f'(z_1) \cdots f'(z_{q-1})$. Periodic points and orbits are also classified as attracting, repelling, etc., in the same way as fixed points, based on the multiplier λ.

Lemma 4.6. *Fixed points and multipliers are invariant under analytic conjugation. I.e., if f has a fixed point at z_0 with multiplier λ, and if $g = \phi \circ f \circ \phi^{-1}$ with ϕ analytic and invertible in some neighborhood of z_0, then g has a fixed point at $w_0 = \phi(z_0)$ with the same multiplier λ.*

Proof. $g(w_0) = \phi(f(\phi^{-1}(z_0))) = \phi(f(z_0)) = \phi(z_0) = w_0$, so w_0 is fixed for g. Furthermore, $g'(w_0) = \phi'(z_0)f'(z_0)(\phi^{-1})'(w_0)$ by the chain rule, and $(\phi^{-1})'(w_0) = \frac{1}{\phi'(z_0)}$ by the inverse function rule. Combining these we get $g'(w_0) = f'(z_0) = \lambda$. $\square$

Remark. The same invariance result holds for periodic points z_0, their orbits, and their multipliers, under the assumption that ϕ is analytic and invertible in some neighborhood of the orbit of z_0.

The invariance of multipliers under conjugation suggests the following extension of this concept to fixed points at ∞.

Definition 4.7. Let f be analytic with $f(\infty) = \infty$. Then the *multiplier* of f at ∞ is defined as the multiplier of $g(w) = 1/f(1/w)$ at 0.

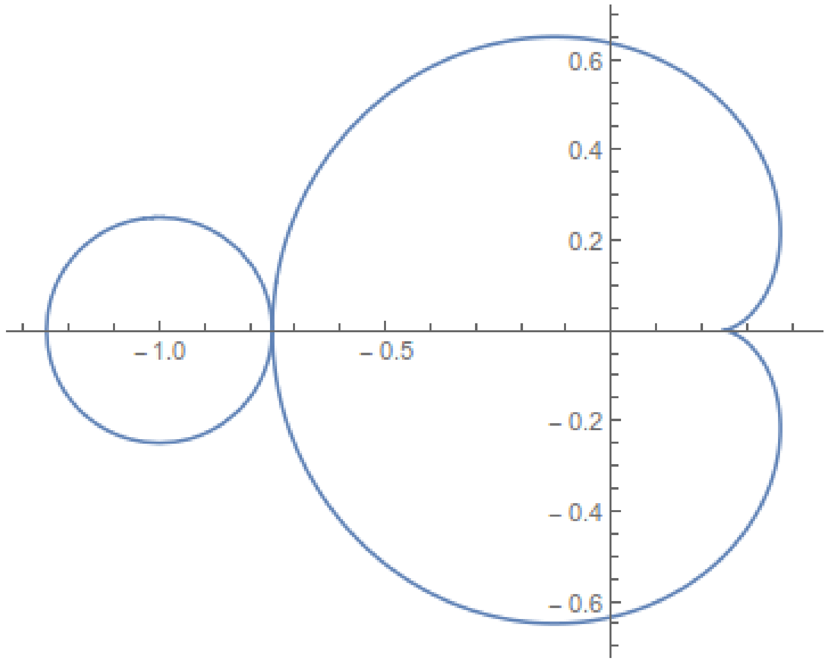


FIGURE 3. Graph in the c-plane showing the main cardioid containing the parameters corresponding to attracting fixed points and the circle containing those with attracting 2-cycles.

Here we used conjugation with the explicit map $\phi(z) = 1/z$, but we would get the same multiplier for any analytic invertible map ϕ mapping ∞ to $\mathbb{C}$. Similarly, if f has a periodic orbit containing ∞, we can define its multiplier as the multiplier of an analytic conjugate by a function ϕ which maps all points of the periodic orbit into the plane.

Example 4.8. If $f(z) = a_d z^d + \ldots + a_0$ is a polynomial of degree $d \geq 2$ with $a_d \neq 0$, then $1/f(1/w) = \frac{1}{a_d w^{-d} + \ldots + a_0} = \frac{w^d}{a_d + a_{d-1} w + \ldots + a_0 w^d} = \frac{1}{a_d} w^d + O(w^{d+1})$, so the multiplier of f at ∞ is $g'(0) = 0$, and ∞ is a super-attracting fixed point for f.

As an elementary exercise, we are going to find the parameters c such that f_c has an attracting or super-attracting fixed point, as well as those for which it has an attracting or super-attracting period two orbit.

A fixed point z_0 of the polynomial $f_c(z) = z^2 + c$ satisfies the equation $z_0^2 + c = z_0 0$, and its multiplier is $\lambda = f_c'(z_0) = 2z_0$. So we have to find all values of c such that there exists z_0 with

$$z_0^2 + c = z_0 \quad \text{and} \quad |2z_0| < 1.$$

The first equation is explicitly solved for c as $c = z_0 - z_0^2$, and so the answer is the image of the disk of radius $1/2$ centered at 0 under the map $z \mapsto z - z^2 = \frac{1}{4} - \left(z - \frac{1}{2}\right)^2$. This is easily seen to be the interior of a cardioid which is the largest prominent "bulb" of the Mandelbrot set.

The second iterate of f_c is $f_c^2(z) = (z^2 + c)^2 + c = z^4 + 2cz^2 + c^2 + c$, so the periodic points of period 2 are the roots of the polynomial $f_c^2(z) - z = z^4 + 2cz^2 - z + c^2 + c$. However, fixed points are also solutions of this equation, so $f_c(z) - z = z^2 - z + c$ actually divides this polynomial. Long division gives $z^4 + 2cz^2 - z + c^2 + c = (z^2 - z + c)(z^2 + z + c + 1)$, so if f_c has an actual periodic cycle $\{z_2, z_3\}$ of period 2 with $z_2 \neq z_3$, then $z^2+z+c+1 = (z-z_2)(z-z_3)$, so in particular $z_2 z_3 = c+1$. Its multiplier is $(f_c^2)'(z_2) = f_c'(z_2) f_c'(z_3) = 4z_2z_3 = 4(c+1)$. This shows that the set of parameters c for which there is an attracting orbit of period 2 is given by $|4(c+1)| < 1$, or equivalently $|c+1| < \frac{1}{4}$, i.e., the circle of radius $\frac{1}{4}$ centered at $c = -1$.

Figure 3 shows the boundaries of the domains in the c-plane which correspond to these attracting fixed points and 2-cycles.

For higher periods, the equations are not explicitly solvable anymore, but the same method still works in principle and it shows that all these "hyperbolic components", i.e., domains in the c-plane corresponding to some attracting periodic orbit with a given period, are bounded by algebraic curves. By calculating the degrees, one can also count the number of these components. For further explorations of the algebraic and arithmetic aspects of complex dynamics, see the book of Silverman [Sil07].

5. Some two-dimensional topology

This section presents a few important results from two-dimensional topology and covering theory which we will need later.

5.1. **Covering spaces and deck transformation groups.** TO BE ADDED LATER.

5.2. **Proper maps and Riemann-Hurwitz formula.** For a good treatment of this topic from a slightly different angle, see the book of Steinmetz [Ste93].

Definition 5.1. Let $U, V \subseteq \hat{\mathbb{C}}$ be open sets. A non-constant meromorphic map $f : U \to V$ is *proper* if $f^{-1}(K)$ is compact for every compact set $K \subseteq V$.

Remark. In the case where f extends continuously to a map between the closures $f : \overline{U} \to \overline{V}$, this definition is equivalent to $f(\partial U) = \partial V$.

A very important property of proper maps is the fact that they have a finite (topological) degree.

Theorem 5.2. *Let $f : U \to V$ be proper. Then there exists an integer $d \geq 1$ such that every point $w \in V$ has d preimages under f, counted with multiplicity. Furthermore, there exist discrete sets $E \subset U$ and $F \subset V$ (i.e., sets without accumulation points in U or V, respectively) of critical points and critical values such that every point $w \in V \setminus F$ has d distinct simple preimages under f.*

Remark. Note that E and F can in general be infinite sets (though it is challenging to construct such examples). However, in our context, f will usually be the restriction of a rational map, which has only finitely many critical points and critical values.

Definition 5.3. If $U \subseteq \hat{\mathbb{C}}$ is a domain, we define the *Euler characteristic* of U as $\chi(U) = 2 - c(U)$, where $c(U)$ is the number of connected components of the complement $\hat{\mathbb{C}} \setminus U$. In the case where the complement has infinitely many connected components, we set $\chi(U) = -\infty$. For a general non-empty open set U we define $\chi(U) = \sum_k \chi(U_k)$ where U_k are the connected components of U, as long as this sum is well-defined.

Remark. This definition agrees with the usual definition of Euler characteristic by triangulations, suitably adapted to the case of an open possibly infinitely connected set U. For the case of a domain $U \subsetneq \hat{\mathbb{C}}$, we call $c(U)$ the *connectivity* of U. In particular, domains with $c(U) = 1$ are *simply connected*, with $c(U) = 2$ are *doubly connected*, etc. Note that $\hat{\mathbb{C}}$ is also simply connected, even though $c(\hat{\mathbb{C}}) = 0$.

A very important and useful tool is the following classical result.

Theorem 5.4 (Riemann-Hurwitz formula). *Let $U, V \subseteq \hat{\mathbb{C}}$ be open non-empty sets, and $f : U \to V$ be a proper meromorphic map of degree d. Assume that f has n critical points, counted with multiplicity. Then*

$$\chi(U) = d\chi(V) - n$$

One simple application of this is the count of critical points of a rational map of degree d. In that case, $U = V = \hat{\mathbb{C}}$, $\chi(U) = \chi(V) = 2$, and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is obviously proper (as a continuous map with a compact domain) of degree d, so the Riemann-Hurwitz formula gives $2 = 2d - n$, or $n = 2d - 2$ for the number of critical points, counted with multiplicity.

6. A complex analysis interlude

6.1. **Basic complex analysis.** The real start of a systematic study of the dynamics of iterated analytic maps was conducted by Pierre Fatou and Gaston Julia in the 1910s, closely following Paul Montel's development of the theory of normal families of analytic functions (and long before the advent of computer pictures of fractal Julia sets.) We will review the necessary background from complex analysis quickly, mostly without proofs. An excellent reference for this theory is the book of [Gam01].

In general we will work on the Riemann sphere $\hat{\mathbb{C}}$, equipped with either the chordal metric $d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ (which is the distance in $\mathbb{R}^3$ of the preimages of z and w under stereographic projection), or the spherical metric $\sigma(z, w)$ given by the length element $ds = \frac{2|dz|}{1+|z|^2}$ (which is the distance of the same points along shortest paths on the sphere.) These two metrics satisfy the inequalities $d(z, w) \leq \sigma(z, w) \leq \frac{\pi}{2} d(z, w)$, and for most of the theory it does not matter which one is used.

A *domain* $D \subseteq \hat{\mathbb{C}}$ is an open non-empty connected set, and a function $f : D \to \mathbb{C}$ is *analytic* if it is complex differentiable in D. Here we define complex differentiability (and analyticity) of f at ∞ as complex differentiability (and analyticity) of the map $g(w) = f(1/w)$ at $w = 0$. A function is *meromorphic* if it is analytic outside of a set of poles. (I.e., all singularities are isolated, and none of them are essential.)

Meromorphic maps from the sphere to the sphere are exactly the rational function $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials, and q is not the zero polynomial. The group of conformal self-maps of $\hat{\mathbb{C}}$ is exactly the group of Möbius transformations $\text{Möb}(\hat{\mathbb{C}})$ which consists of maps of the form $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$. (By multiplying numerator and denominator by a constant we can actually always achieve $ad - bc = 1$.) The subgroup of conformal isometries is given by the Möbius transformations of the form $f(z) = \frac{az-b}{\bar{b}z+\bar{a}}$ where $|a|^2 + |b|^2 = 1$. (Note that we only work with orientation-preserving maps here, and that this group of isometries is canonically identified with the matrix group $SO(3)$.)

A sequence of meromorphic maps $f_n : D \to \hat{\mathbb{C}}$ on a domain $D \subset \hat{\mathbb{C}}$ converges *locally uniformly* (or *normally*) to some function $f : D \to \hat{\mathbb{C}}$ if $f_n \to f$ uniformly on compact subsets of D (with respect to the spherical or chordal metric.) Equivalently, every $z \in D$ has a neighborhood U on which $f_n \to f$ uniformly. The limit function f is then again meromorphic or the constant function ∞.

A fundamental result from real analysis, adapted to this setting is the following

Theorem 6.1 (Arzelà-Ascoli). *Let $D \subseteq \hat{\mathbb{C}}$ be a domain, and let $\mathcal{F}$ be a family of continuous functions $f : D \to \hat{\mathbb{C}}$. Then the following two statements are equivalent.*

(1) *The family* $\mathcal{F}$ *is* equicontinuous *at every point* $z \in D$*, i.e., for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $f \in \mathcal{F}$*, and for all* $w \in \hat{\mathbb{C}}$ *with* $d(z,w) < \delta$ *we have* $d(f(z), f(w)) < \varepsilon$*.*
(2) *Every sequence* $\{f_n\}$ *in* $\mathcal{F}$ *has a locally uniformly convergent subsequence* $\{f_{n_k}\}$*.*

Definition 6.2. A family of meromorphic functions on a domain $D \subset \hat{\mathbb{C}}$ satisfying the two equivalent properties in Theorem 6.1 is called a *normal family.*

There are whole volumes of normality criteria by now, but most fundamental are two famous theorems by Montel.

Theorem 6.3 (Montel's Little Theorem)**.** *Let* $\mathcal{F}$ *be a* locally uniformly bounded *family of analytic functions in a domain* $D \subset \hat{\mathbb{C}}$*, i.e., assume that for every* $z \in D$ *there exists a neighborhood* U *of* z *and a constant* $M \in \mathbb{R}$ *such that* $|f(w)| \leq M$ *for all* $f \in \mathcal{F}$ *and* $w \in U$*. Then* $\mathcal{F}$ *is a normal family.*

The proof of this theorem is very straight-forward, using Cauchy's estimate to show that the derivatives of functions in $\mathcal{F}$ are also locally uniformly bounded, and using this to derive a locally uniform Lipschitz condition which implies equicontinuity.

The following much stronger theorem is the main tool Fatou and Julia used to develop the theory of complex dynamics.

Theorem 6.4 (Montel's Big Theorem)**.** *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* $D \subset \hat{\mathbb{C}}$*, and assume that there exist three distinct points* $a, b, c \in \hat{\mathbb{C}}$ *such that all functions in* D *omit* a, b, c*. Then* $\mathcal{F}$ *is a normal family.*

In other words, the family of all meromorphic functions into the triply punctured sphere $\hat{\mathbb{C}} \setminus \{a, b, c\}$ is normal. (Obviously, a subfamily of a normal family is again normal.) Gamelin's book gives a proof of this theorem using Zalcman's lemma. We will sketch the more classical proof using an explicit universal covering of the triply punctured sphere by the unit disk. In order to do this, we need a few more results from complex analyis (which we will also use in other contexts anyway.)

We will frequently use the following result about extending meromorphic maps by reflections. The reflection in the real line $\mathbb{R}$ is given by complex conjugation $\tau_{\mathbb{R}}(z) = \overline{z}$. If L is any other line or circle, we can define the reflection in L as follows. Let D be one of the components of $\hat{\mathbb{C}} \setminus L$, and let T be a Möbius transformation with $T(\mathbb{H}) = D$. Then it is an easy exercise to show that the map $\tau_L(z) = T\left(\overline{T^{-1}(z)}\right)$ only depends on L, and not on the particular choice of D or T. We define τ_L as the *reflection in* L. Geometrically, it is the ordinary reflection in the case where L is a line, and it is the inversion in L if L is a circle.

Theorem 6.5 (Schwarz Reflection Principle)**.** *Let* L_1 *and* L_2 *be circles or lines, and let* τ_1 *and* τ_2 *be the associated reflections. Let* $D \subset \hat{\mathbb{C}}$ *be a domain symmetric with respect to* τ_1*, and let* D^+ *be one component of* $D \setminus L_1$*. Assume that* $f : D^+ \to \hat{\mathbb{C}}$ *is meromorphic, and that* $f(z) \to L_2$ *as* $z \to L_1$*. Then* f *extends to an analytic function on* D *satisfying* $f(\tau_1(z)) = \tau_2(f(z))$*.*

This is often stated only for the case where $L_1 = L_2 = \mathbb{R}$ and $\tau_1(z) = \tau_2(z) = \overline{z}$, but this more general statement easily follows by composition with Möbius transformations. The proof is considerably easier if one assumes that f extends continuously to $L_1 \cap D$, and often we actually know this in situations where we have to apply it.

Recall that a domain G is *simply connected* if every loop in G is null-homotopic, i.e., if it is continuously contractible to a point in G. One of the most famous theorems in complex analysis (and typically the goal of a one-semester graduate course on the topic) is the following.

Theorem 6.6 (Riemann Mapping Theorem)**.** *Let $G \subsetneq \mathbb{C}$ be simply connected. Then there exists a conformal map f from G onto the unit disk $\mathbb{D}$. Any two such maps differ by a Möbius transformation of $\mathbb{D}$, i.e., if g is another such map, then $g \circ f^{-1} \in \text{Möb}(\mathbb{D})$.*

Here $\text{Möb}(\mathbb{D})$ is the group of conformal self-maps of the unit disk, which (by the reflection principle) is a subgroup of the full group of Möbius transformations $\text{Möb}(\hat{\mathbb{C}})$. Explicitly, it is given by maps of the form $f(z) = \lambda \frac{z-a}{1-\bar{a}z}$ where $a, \lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $|a| < 1$.

6.2. **Riemann surfaces and the Uniformization Theorem.** By Liouville's theorem there is no conformal map from the plane $\mathbb{C}$ onto $\mathbb{D}$. An easy consequence of the Riemann Mapping Theorem is that up to conformal equivalence there are only three possible types of simply connected subdomains of the sphere, namely the sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, and the disk $\mathbb{D}$. Amazingly, this is still true for general Riemann surfaces, as proved by Koebe and Poincaré in 1907.

Theorem 6.7 (Uniformization Theorem)**.** *Every simply connected Riemann surface is conformally isomorphic to either the sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, or the disk $\mathbb{D}$.*

Here a *Riemann surface* is a connected two-dimensional manifold where the chart transitions are complex-analytic maps. (The analyticity of the chart transitions makes it possible to define the concept of analytic and conformal maps on a Riemann surface.)

ADD SOME RESULTS ABOUT COVERING SPACES.

Combining this with the topological theory of covering surfaces, we obtain the following general result about not necessarily simply connected surfaces.

Theorem 6.8. *Every Riemann surface is conformally isomorphic to a quotient X/Γ, where X is either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$, and Γ is a group of Möbius transformation acting freely and properly discontinuously on X.*

Here the group acts *freely* if only the identity element has fixed points in X, and it acts *properly discontinuously* if for every compact subset $K \subset X$, there are only finitely many group elements $\gamma \in \Gamma$ with $\gamma(K) \cap K \neq \emptyset$. (This is equivalent to the property that every point $z \in X$ has a neighborhood U such that $\gamma(U) \cap U = \emptyset$ for every non-identity $\gamma \in \Gamma$.)

We say that X is the *universal cover* of the Riemann surface $S = X/\Gamma$, and we call S *elliptic* if $X = \hat{\mathbb{C}}$, *parabolic* if $X = \mathbb{C}$, and *hyperbolic* if $X = \mathbb{D}$. Since every Möbius transformation has a fixed point in $\hat{\mathbb{C}}$, the only elliptic Riemann surface is $\hat{\mathbb{C}}$ itself (up to conformal isomorphism). In the case $X = \mathbb{C}$, the only fixed-point free Möbius transformations are translations, so in this case Γ is either trivial, an infinite cyclic group generated by one translation $z \mapsto z + a$ with $a \neq 0$, or a group generated by two translations $z \mapsto z + a$ and $z \mapsto z + b$ with $a/b \notin \mathbb{R}$. The resulting parabolic surfaces are the plane $\mathbb{C}$, the cylinder $\mathbb{C}/a\mathbb{Z}$ (which is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$ for all $a \neq 0$), and the torus $\mathbb{C}/(a\mathbb{Z} + b\mathbb{Z})$. (These tori are not all conformally isomorphic, and the study of conformal invariants of tori is the beginning of Teichmüller theory.) As a consequence, most Riemann surfaces are hyperbolic, as made precise in the following theorem.

Theorem 6.9. *Let S be a Riemann surface which is not homeomorphic to the sphere, the plane, the cylinder, or the torus. Then S is hyperbolic, i.e., S is conformally isomorphic to*

$\mathbb{D}/\Gamma$, where Γ is a subgroup of Möb($\mathbb{D}$) *acting freely and properly discontinuously on the unit disk $\mathbb{D}$.*

Remark. There are obviously surfaces homeomorphic to the plane which are hyperbolic (e.g., the unit disk), and there are cylinders which are hyperbolic as well, e.g., $S = \mathbb{H}/\mathbb{Z}$, where $\mathbb{H}$ is the upper halfplane (which is hyperbolic since it is conformally isomorphic to the unit disk), so there is no purely topological characterization of conformal type. However, for surfaces of genus $g \geq 2$, the type is always hyperbolic.

Returning to the Riemann Mapping Theorem, a natural question is whether the conformal mapping extends to the boundary. In this question, it is a little more natural to work with the inverse of the map given in the theorem, i.e., the conformal map from the unit disk onto G. The following two results are not in Gamelin's book, but they are classical and well-known by now. For a reference, see the book of Pommerenke [Pom92].

Theorem 6.10 (Continuity Theorem)**.** *Let f be a conformal map from $\mathbb{D}$ onto $G \subset \hat{\mathbb{C}}$. Then f extends to a continuous map from $\overline{\mathbb{D}}$ onto $\overline{G}$ if and only if ∂G is locally connected.*

And the question when f extends to a homeomorphism of the closures has also been answered completely.

Theorem 6.11 (Carathéodory Theorem)**.** *Let f be a conformal map from $\mathbb{D}$ onto $G \subset \hat{\mathbb{C}}$. Then f extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{G}$ if and only if ∂G is a* Jordan curve*, i.e., iff it is homeomorphic to the unit circle S^1.*

As a corollary, we can normalize a conformal map onto a Jordan domain (i.e., a domain bounded by a Jordan curve) by prescribing three points on the boundary.

Theorem 6.12. *Let $G \subsetneq \mathbb{C}$ be a Jordan domain, and let $z_1, z_2, z_3 \in \partial\mathbb{D}$ and $w_1, w_2, w_3 \in \partial G$ points in counterclockwise order on the respective boundaries. Then there exists a unique conformal map $f : \mathbb{D} \to G$ with $f(z_k) = w_k$ for $k = 1, 2, 3$.*

Here we identify f with its extension to $\overline{\mathbb{D}}$ given by Carathéodory's theorem. By the Riemann mapping theorem there exists a conformal map $g : G \to \mathbb{D}$. By Carathéodory it extends to a homeomorphism of $\overline{G}$ to $\overline{\mathbb{D}}$. Let $z'_k = g(w_k)$. Then z'_1, z'_2, z'_3 are distinct points in counterclockwise order on ∂G, so there exists a Möbius transformation T with $T(z'_k) = z_k$. Since T maps three points on the unit circle to three points on the unit circle, it fixes the unit circle (as a set.) Since T preserves the cyclic order of the three points, it maps the interior of the unit circle to itself, so it fixes the unit disk (again, as a set.) This shows that $h = T \circ g$ is a conformal map from G onto $\mathbb{D}$ with $h(w_k) = z_k$ for $k = 1, 2, 3$. Then $f = h^{-1}$ is the desired conformal map. If $\tilde{f}$ is another such map, then $T = \tilde{f}^{-1} \circ f$ is a conformal map of the unit disk onto itself fixing z_1, z_2, and z_3. This shows that T is a Möbius transformation fixing three points, so it is the identity.

Now we are ready to give a sketch of a proof of Montel's Big Theorem. Let $z_k = e^{2\pi ik/3}$ be the third roots of unity, and let G_0 be the subdomain of the unit disk bounded by circular arcs from z_0 to z_1, from z_1 to z_2, and from z_2 to z_0 which hit the unit circle at right angles. Let $\lambda : G_0 \to \mathbb{H}$ be the conformal map from G_0 onto the upper halfplane with $f(z_0) = 0$, $f(z_1) = 1$, and $f(z_2) = \infty$. (By Carathéodory we know that this map extends to a homeomorphism of the closures, and Theorem 6.12 tells us that we can normalize by prescribing the value of three boundary points.) Now we extend λ by reflection across the three circular arcs to a map from a domain G_1 to $\mathbb{C}$. The values 0, 1, and ∞ are only attained at the vertices z_1, z_2, and z_3 on the boundary, so this extended map actually maps

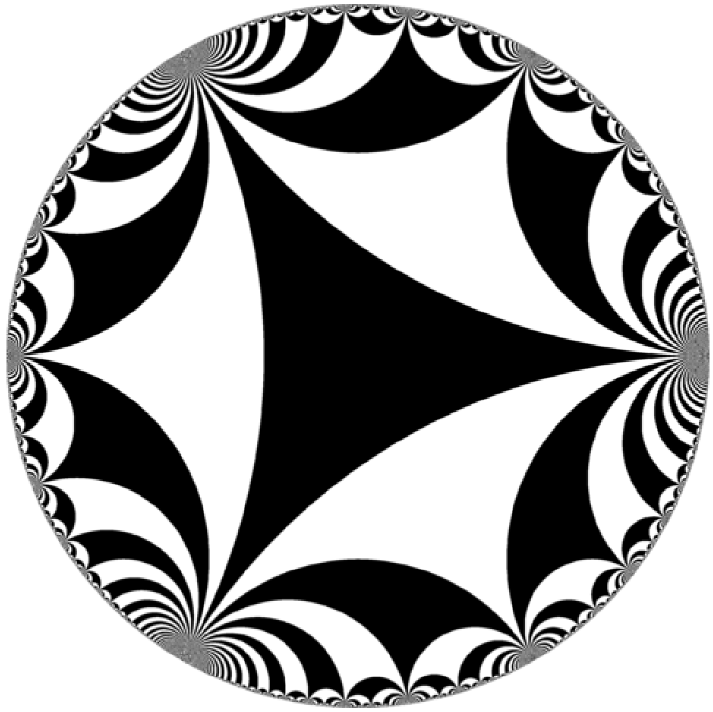


FIGURE 4. Visualization of the construction of the λ-function. The central black triangle G_0 maps to the upper halfplane, the vertices to 0, 1, and ∞. The sides map to the intervals $(0,1)$, $(1,\infty)$, and $(-\infty,0)$ on the real line. The continued extension by reflections maps all the black triangles to the upper halfplane, all the white triangles to the lower halfplane. The vertices on $\partial\mathbb{D}$ map to 0, 1, ∞ for all approximations after finitely many reflections. The preimages of 0, 1, and ∞ are all dense in $\partial\mathbb{D}$ in the limit, so this function does not extend continuously to any boundary point.

into the triply punctured sphere $\mathbb{C} \setminus \{0,1\}$. The reflections across all three boundary arcs leave the unit circle and the unit disk invariant since it hits the arcs at a right angle, so the new domain G_1 is still contained in $\mathbb{D}$, and it is bounded by circular arcs meeting the unit circle at a right angle. The extended map still maps ∂G_1 to the (extended) real line, and we can inductively repeat the argument to get an increasing sequence of domains $\{G_n\}$ to which we can extend λ analytically. It is not too hard to check that $\bigcup_{n=0}^{\infty} G_n = \mathbb{D}$, and that the extension $\lambda : \mathbb{D} \to \mathbb{C} \setminus \{0,1\}$ is surjective and satisfies $\lambda'(z) \neq 0$ for all $z \in \mathbb{D}$. The reflections in the circular arcs generate a group $\tilde{\Gamma}$, and the subgroup Γ of index 2 of any composition of an even number of reflections is a subgroup of $\text{Möb}(\mathbb{D})$ such that $f \circ \gamma = f$ for every $\gamma \in \Gamma$. Taking this all together (plus a few details) shows that $f : \mathbb{D} \to \mathbb{C} \setminus \{0,1\}$ is a universal covering with deck transformation group Γ. For a geometric illustration of the construction in the proof see Figure 4.

Now if $\mathcal{F}$ is a family of meromorphic maps in some disk D, omitting 0, 1, and ∞, then we can lift each map $f \in \mathcal{F}$ to a map $F : D \to \mathbb{D}$, i.e., we can find an analytic map $F : D \to \mathbb{D}$ such that $\lambda \circ F = f$. (This lift is not quite unique, but in this context we do not need uniqueness.) Now the family $\mathcal{G}$ of all these lifts is a uniformly bounded family of analytic function, so it is normal by Montel's Little Theorem. If $\{F_n\}$ is a sequence of such lifts, then there exists a locally uniformly convergent subsequence $F_{n_k} \to F$, and so $f_{n_k} = \lambda \circ F_{n_k} \to \lambda \circ F$ is also locally uniformly convergent, showing that $\mathcal{F}$ is normal.

This argument works for disks and, more generally, for simply connected domains, otherwise the existence of the lift is not guaranteed. However, if $\mathcal{F}$ is such a family in an arbitrary domain $G \subseteq \hat{\mathbb{C}}$, then the previous argument shows that it is normal in every disk $D \subset G$, which implies that it is normal in G.

If the family omits three arbitrary distinct values a, b, c, we use a Möbius transformation T with $T(a) = 0$, $T(b) = 1$, and $T(c) = \infty$, and consider the family of all compositions $T \circ f$ for $f \in \mathcal{F}$. By the previous argument, this is a normal family since it omits 0, 1, and ∞, so every sequence $\{T \circ f_n\}$ with $f_n \in \mathcal{F}$ has a locally uniformly convergent subsequence $T \circ f_{n_k} \to g$. Then $f_{n_k} \to T^{-1} \circ g$ locally uniformly, and it follows that $\mathcal{F}$ is normal.

The following corollary to Montel's theorem is frequently useful, replacing the omitted values a, b, c, with three omitted meromorphic functions.

Corollary 6.13 (Montel's Theorem for Omitted Functions)**.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathbb{D} \subseteq \hat{\mathbb{C}}$, and let $g_1, g_2, g_3 : D \to \hat{\mathbb{C}}$ be meromorphic functions with $g_j(z) \neq g_k(z)$ for all $z \in D$ and $j \neq k$. Assume further that all functions in $\mathcal{F}$ omit g_1, g_2, and g_3, i.e., for every $f \in \mathcal{F}$, and for all $z \in D$, $j \in \{1, 2, 3\}$, we have $f(z) \neq g_j(z)$. Then $\mathcal{F}$ is normal.*

Proof. Let $z_0 \in D$, and let U be an open disk centered at z_0 such that $\overline{U} \subset D$, and that the images $\overline{V_j} = g_j(\overline{U})$ are mutually disjoint. By replacing $\mathcal{F}$ with the family $T \circ \mathcal{F} = \{T \circ f \colon f \in \mathcal{F}\}$ (which is normal if and only if the original family $\mathcal{F}$ is normal) with a suitably chosen Möbius transformation T we may assume that $\overline{V_j} \subset \mathbb{C}$ for $j = 1, 2, 3$, so that the functions g_j are actually analytic in some neighborhood of $\overline{U}$. We define

$$T_z(w) = \frac{w - g_1(z)}{w - g_3(z)} \cdot \frac{g_2(z) - g_3(z)}{g_2(z) - g_1(z)},$$

so that T_z is the Möbius transformation with $T_z(g_1(z)) = 0$, $T_z(g_2(z)) = 1$, and $T_z(g_3(z)) = \infty$. We can write $T_z(w) = \frac{a_z w + b_z}{c_z w + d_z}$ and $T_z^{-1}(w) = \frac{d_z w - b_z}{-c_z w + a_z}$, with coefficients a_z, b_z, c_z, d_z depending analytically on z. Now let $\hat{\mathcal{F}}$ be the family of functions $\hat{f}(z) = T_z(f(z))$ for $z \in U$, where $f \in \mathcal{F}$. Then $\hat{\mathcal{F}}$ is a family of meromorphic functions omitting 0, 1, and ∞, so it is normal by Montel's Big Theorem. If $\{f_n\}$ is any sequence of functions from $\mathcal{F}$, then the corresponding sequence $\{\hat{f}_n\}$ has a subsequence $\{\hat{f}_{n_k}\}$ which converges locally uniformly in U. This implies that the sequence $f_{n_k}(z) = T_z^{-1}(\hat{f}_{n_k}(z))$ converges locally uniformly in U, too. This shows that $\mathcal{F}$ is normal in U. □

7. Basic theory of complex dynamics

Fatou's starting point for the theory of complex dynamics was the partitioning of the plane into a set with relatively "tame" dynamics and its complement with "wild" dynamical behavior. (Julia started with the periodic points and developed basically the same theory from a slightly different framework. Modern presentations almost always follow Fatou's line these days.) The "tame" set is the maximal open set on which the iterates form a normal family. As a consequence of Montel's theorem, for any analytic function $f : S \to S$ on a hyperbolic Riemann surface S the sequence of iterates $\{f^n\}$ is normal on S, since it can be lifted to self-maps of the unit disk $\mathbb{D}$. (It also turns out that most Riemann surfaces do not have many analytic self-maps to start with.) This shows that the richest theory of complex dynamics is the study of iterated maps on parabolic or elliptic Riemann surfaces. The only analytic self-maps of complex tori $\mathbb{C}/\Gamma$ are covered by affine maps of the form $f(z) = az + b$ which have very simple dynamics. The remaining cases are (up to conformal equivalence) the Riemann sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, and the cylinder $\mathbb{C}/\mathbb{Z}$, which is conformally equivalent to the punctured plane $\mathbb{C} \setminus \{0\}$ (via $z \mapsto e^{2\pi i z}$). Dynamics on $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$ are very interesting and the subject of active research. However, the theory in general is much harder than the study of dynamics on the sphere, due to the non-compactness of the plane and the cylinder.

7.1. **Rational maps, degree, critical points.** The analytic self-maps of the Riemann sphere (viewed as a Riemann surface) are exactly the rational maps (and the constant ∞). Every rational map can be written as $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials without a common root. The *(algebraic) degree* of f is $d = \max(\deg p, \deg q)$.

Theorem 7.1. *Let f be a rational function of degree $d \geq 1$. Then for every $w \in \hat{\mathbb{C}}$, the equation $f(z) = w$ has exactly d solutions, counted with multiplicity. For all $w \in \hat{\mathbb{C}}$ with finitely many exceptions, the equation $f(z) = w$ has exactly d distinct solutions.*

Proof. First assume that $w = \infty$. If $d = \deg p > \deg q$, then f has a pole of multiplicity $d - \deg q$ at ∞, and the poles in $\mathbb{C}$ are the solutions of $q(z) = 0$, of which there are $\deg q$, counted with multiplicity. This shows that the total number of poles, counted with multiplicity, is $d - \deg q + \deg q = d$.

Otherwise $\deg p \leq \deg q = d$, which implies that f does not have a pole at ∞, and since the poles in the plane are the roots of q, we get that the total number of poles in this case is $\deg q = d$.

If $w \neq \infty$, consider $g(z) = \frac{1}{f(z)-w} = \frac{q(z)}{p(z)-wq(z)}$. Any common root of $q(z)$ and $p(z) - wq(z)$ would be a common root of $p(z)$ and $q(z)$, so by assumption this representation also does not have common roots of the numerator and denominator. Furthermore, both the degrees of the numerator and denominator are $\leq d$, and at least one of them has to be d, since otherwise p and q would be polynomials of degree $\leq d - 1$. By the first part, g has d poles counted with multiplicity. This shows that the equation $f(z) = w$ has d solutions, counted with multiplicity, as claimed.

The points in the plane where f is not locally injective are contained in the set of poles of f and the zeros of the derivative f'. These are both finite sets, and the only points w where $f(z) = w$ has solutions of multiplicity larger than 1 are contained in the image of this finite set, so this is again a finite subset of the sphere, showing the second claim. □

It is well-known from complex analysis that a non-constant analytic map locally behaves like $z \mapsto z^k$. More precisely, if f is analytic and non-constant in a neighborhood of z_0, with $f(z_0) = w_0$, then there exist local analytic diffeomorphisms ϕ and ψ with $\phi(z_0) = 0$, $\psi(w_0) = 0$, and

$$f(z) = \psi^{-1}(\phi(z_0)^k) \tag{7.1}$$

for z near z_0. This still holds if one or both of z_0 and w_0 are ∞. The number k is the *local degree* of f at z_0. In the case where both z_0 and w_0 are finite, we also have that $k = \inf\{j \geq 1 \colon f^{(k)}(z_0) \neq 0\}$, i.e., the degree is the order of the first non-vanishing derivative at z_0. From (7.1) we can also see that the degree can be characterized as follows: There exist arbitrarily small $\varepsilon, \delta > 0$ such that the equation $f(z) = w$ for any given w with $0 < |w - w_0| < \varepsilon$ has exactly k solutions $z_1, \ldots, z_k$ satisfying $0 < |z_j - z_0| < \delta$ for $j = 1, \ldots, k$. Furthermore, each one of these is a simple solution of the equation, i.e., $f'(z_j) \neq 0$ for $j = 1, \ldots, k$. (In the case where z_0 or w_0 is ∞, the same statement holds with the spherical instead of the Euclidean metric.) The *critical points* of a map f are the points where the local degree is at least 2. If the local degree at z_0 is $k \geq 2$, then z_0 is a critical point *of multiplicity* $k-1$. From (7.1) we can also see that every critical point is isolated. The set of *critical values* is the image of the set of critical points under the map f. A point which is not a critical value is called a *regular value.* If w_0 is a regular value, and $z_0 \in f^{-1}(\{w_0\})$ is any preimage of w_0, then there exists a local analytic inverse branch g of f^{-1} with $g(w_0) = z_0$.

Theorem 7.2. *A rational map f of degree $d \geq 1$ has $2d-2$ critical points, counted with multiplicity.*

Proof. If f is rational, and if T and S are Möbius transformations, then $g = T \circ f \circ S$ is rational of the same degree, and the local degree of g at z_0 is the same as the local degree of f at $S(z_0)$. This shows that the number of critical points is the same for f and g, counted with or without multiplicity. So we may assume that ∞ is not a critical point, and that $f(\infty) = \infty$, so that $f(z) = \frac{p(z)}{q(z)}$ with relatively prime polynomials satisfying $\deg q = m < d = \deg p$. Conjugating by $z \mapsto 1/z$, it is straightforward to see that the local degree of f at ∞ is $d-m$, so the assumption that ∞ is not critical implies that $m = d-1$. Multiplying f with a non-zero constant does not change the number of critical points or the degree, so we may assume that p and q are monic, i.e., $p(z) = z^d + O(z^{d-1})$ and $q(z) = z^{d-1} + O(z^{d-2})$

$$f'(z) = \frac{p'(z)q(z) - p(z)q'(z)}{q(z)^2} = \frac{z^{2d-2} + O(z^{2d-3})}{q(z)^2}$$

Since p and q were assumed to be relatively prime, the same is true for the numerator and denominator of f', and by the fundamental theorem of algebra, the number of zeros of f' in the plane, counted with multiplicity, is $2d-2$. Since ∞ is not a critical point, this is the number of critical points of f, counted with multiplicity. □

7.2. **Fatou and Julia sets.** Next is the rigorous fundamental definition of the partition of the sphere into "tame" and "wild" sets.

From here on, unless otherwise specified, we will always assume that f is a rational map of degree $d \geq 2$.

Definition 7.3. The *Fatou set* $\mathcal{F}(f)$ of f is defined as the set of points $z \in \hat{\mathbb{C}}$ for which there exists a neighborhood U such that the sequence of iterates $\{f^n\}$ is normal in U. The *Julia set* is defined as its complement $\mathcal{J}(f) = \hat{\mathbb{C}} \setminus \mathcal{F}(f)$.

Remark. By definition the Fatou set is open and the Julia set is compact.

Example 7.4. Let $f(z) = z^2$. Then $f^n(z) = z^{2^n}$ converges locally uniformly to ∞ in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and it converges locally uniformly to 0 in the unit disk $\mathbb{D}$, so both of these domains are contained in the Fatou set. If $|z| = 1$, and if $D_r(z)$ is an open disk of radius $r > 0$ about z, then both $D_r(z) \cap \mathbb{D}$ and $D_r(z) \setminus \overline{\mathbb{D}}$ are non-empty and open, and f^n converges locally uniformly to 0 on the first, and to ∞ on the second set. However, the possible locally uniform limit functions are meromorphic (possibly constant ∞), showing that no such locally uniform convergent subsequence exists on $D_r(z)$. This shows that $z \in \mathcal{J}(f)$. Combining this with the first part this implies that $\mathcal{J}(f) = \partial\mathbb{D}$ is the unit circle.

In this example, the Julia set according to our new definition coincides with the Julia set according to the original definition for polynomials as the boundary of the filled-in Julia set from section 4. This is in fact true in general.

Theorem 7.5. *For polynomials the two definitions agree, i.e., if f is a polynomial with filled-in Julia set K_f, then $\mathcal{J}(f) = \partial K_f$.*

Proof. Since $\deg f \geq 2$ (according to our standing assumption), there exists $R > 0$ such that $|f(z)| \geq 2|z|$ for $|z| \geq R$. Then by induction $|f^n(z)| \geq 2^n|z| \geq 2^n R$ for $|z| \geq R$, so $f^n \to \infty$ uniformly for $|z| \geq R$. If $z \in A_f(\infty)$, then there exists n_0 such that $|f^{n_0}(z)| > R$. By continuity there exists a neighborhood U of z such that $|f^{n_0}(w)| > R$ for all $w \in U$. Then $|f^{n_0+k}(w)| > 2^k R$ for all $k \geq 0$ and all $w \in U$, which implies that $f^n \to \infty$ uniformly on U. This shows that $z \in \mathcal{F}(f)$, so $A_f(\infty) \subseteq \mathcal{F}(f)$.

If z is an interior point of K_f, then there exists an open neighborhood U of z such that $U \subseteq K_f$, so that $|f^n(w)| \leq R$ for all $w \in U$ and all $n \geq 0$. By Montel's theorem this implies that the family of iterates is normal on U (since it is uniformly bounded), and so $z \in \mathcal{F}(f)$. This shows that $\text{int}(K_f) \subseteq \mathcal{F}(f)$ (where $\int(K_f)$ denotes the interior of K_f.)

If $z \in \partial K_f$, and $r > 0$, then any disk $D_r(z)$ intersects $A_f(\infty)$ in an open set, so $f^n \to \infty$ locally uniformly on an open non-empty subset of $D_r(z_0)$. This shows that the only possible limit function (of subsequences of iterates) is the constant ∞. However, $|f^n(z)| \leq R$ for all n, so any subsequential limit will be finite at z. This shows that no such limit can exist, and that the sequence of iterates is not normal in any neighborhood of z. By definition this means that $z \in \mathcal{J}(f)$.

Combining all three of these, we see that $\mathcal{J}(f) = \partial K_f$, and that $\mathcal{F}(f) = A_f(\infty) \cup \text{int}(K_f)$. □

Theorem 7.6 (Conjugation)**.** *If $g = T \circ f \circ T^{-1}$ with a Möbius transformation T, then $\mathcal{F}(g) = T(\mathcal{F}(f))$ and $\mathcal{J}(g) = T(\mathcal{J}(f))$.*

Proof. Induction gives $g^n = T \circ f^n \circ T^{-1}$. Since T is a conformal map from the sphere to itself, the sequence f^{n_k} converges uniformly in an open set U to some limit f_0 if and only if the sequence g^{n_k} converges uniformly in the open set $T(U)$ to $g_0 = T \circ f_0 \circ T^{-1}$. Further details are left to the reader. □

Theorem 7.7 (Invariance)**.** *Both $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant, i.e., $f(\mathcal{F}(f)) = \mathcal{F}(f) = f^{-1}(\mathcal{F}(f))$ and $f(\mathcal{J}(f)) = \mathcal{J}(f) = f^{-1}(\mathcal{J}(f))$.*

Proof. It is enough to show that $z \in \mathcal{F}(f)$ if and only if $f(z) \in \mathcal{F}(f)$.

Let $z_0 \in \mathcal{F}(f)$, and let $w_0 = f(z)$. Let U be a disk about z_0 which is still contained in $\mathcal{F}(f)$, and let $V = f(U)$. Then V is an open connected neighborhood of w_0. We claim that the sequence of iterates is normal on V. Let $\{f^{n_k}\}$ be any subsequence of the sequence of iterates of f. We know that the subsequence $\{f^{n_k+1}\}$ has a subsequence which converges locally uniformly on U. This means that there exists $k_j \to \infty$ such that $f^{n_{k_j}+1} \to g$ locally uniformly on U, with some limit function g. This implies that $f^{n_{k_j}}(f(z)) \to g(z)$ locally uniformly for $z \in U$. If $f|_U$ is invertible, then this implies $f^{n_{k_j}}(w) \to g(f^{-1}(w))$ locally uniformly on V. Otherwise, there is a multivalued local inverse function f^{-1} on V whose branches are analytic outside of a finite set of branched values. Since g is a limit of iterates of f, it satisfies $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$. So the composition $g \circ f^{-1}$ is well-defined in V, it is analytic outside of the finite branch set, and it is bounded, since it maps into the disk U. (Here we are assuming that $z_0 \in \mathbb{C}$, the case $z_0 = \infty$ can be dealt with by conjugation.) By Riemann's removability theorem, this implies that g is analytic in V, and so $f^{n_{k_j}} \to g \circ f^{-1}$ locally uniformly in either case. This shows that $V \subseteq \mathcal{F}(f)$. (This argument is admittedly a little messy, and the more elegant arguments I came up with require more theory. If someone can figure out a simpler argument, please let me know.)

The other direction is easier since we do not have to deal with inverses. The details are left to the reader. ☐

Theorem 7.8 (Iteration). *$\mathcal{F}(f^n) = \mathcal{F}(f)$ and $\mathcal{J}(f^n) = \mathcal{J}(f)$ for all $n \geq 1$.*

Proof. The sequence of iterates of f^n is $\{f^{nk}\}_{k=1}^{\infty}$, which is a subsequence of the sequence of all iterates $\{f^k\}_{k=1}^{\infty}$. Subsequences of normal families are normal, so if $z \in \mathcal{F}(f)$, then $z \in \mathcal{F}(f^n)$.

For the other direction, observe that normality of $\{f^{nk}\}_k$ in some open set U implies normality of $\{f^{nk+j}\}_k$ for $j = 0, 1, 2, \dots, n-1$ in the same set U, since $f^{nk+j} = f^j \circ f^{nk}$. It is easy to see that the finite union of normal families is again normal, so the union of these subsequences is normal in U. However, this is the family of all iterates. This argument shows that $\mathcal{F}(f^n) \subseteq \mathcal{F}(f)$. ☐

7.3. **Fixed points and periodic points.** Next we are going to look at how the various types of periodic points fit into the global picture of Julia and Fatou sets. Recall from Definition 4.5 that fixed points z_0 of an analytic map f are classified as super-attracting, attracting, repelling, rationally or irrationally indifferent based on their multiplier $\lambda = f'(z_0)$. The following theorem is basically a simple real analysis exercise, but very important nonetheless.

Theorem 7.9. *If $z_0 \in \mathbb{C}$ is an attracting or super-attracting fixed point for an analytic map f, then there exists $r > 0$ such that $f^n(z) \to z_0$ uniformly for $|z| \leq r$.*

Proof. By definition, $|f'(z_0)| < 1$, so there exists c such that $|f'(z_0)| < c < 1$, and $r > 0$ with $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leq c$ for $|z - z_0| \leq r$. Since $f(z_0) = z_0$, this implies $|f(z) - z_0| \leq c|z - z_0|$ for $|z - z_0| \leq r$, and by induction $|f^n(z) - z_0| \leq c^n|z - z_0| \leq c^n r \to 0$ for $|z - z_0| \leq r$, which shows the claimed uniform convergence. ☐

Remark. We will mostly apply this theorem to the case of rational functions, but the proof works under the much weaker assumption that f is analytic in some neighborhood of z_0. Obviously, in the case of an attracting or super-attracting fixed point at $z_0 = \infty$ we can conjugate by $T(z) = 1/z$, and obtain uniform convergence to ∞ for $|z - z_0| \geq R$ for some $R > 0$. In any case, this shows that the name "attracting" is justified, since the fixed point attracts nearby points under iteration of the map.

Example 7.10. If f is some analytic function with a zero at some $z_0 \in \mathbb{C}$, and if $N_f(z) = z - \frac{f(z)}{f'(z)}$ is the associated Newton's method, then $N_f(z_0) = z_0$, and simple power series arithmetic shows that $N_f'(z_0) = \frac{n-1}{n}$ if the multiplicity of z_0 as a zero of f is n. In particular, z_0 is a super-attracting fixed points of N_f is z_0 is a simple root of f, and it is an attracting fixed point of N_f if it is a multiple root of f. Theorem 7.9 shows that the Newton's method converges to z_0 in some neighborhood of z_0.

It is straightforward to generalize the previous result to periodic cycles as follows.

Theorem 7.11. *Let $Z = \{z_0, z_1, \dots, z_{q-1}\}$ be an attracting or super-attracting periodic cycle of the analytic map f. Then there exists $r > 0$ such that $\operatorname{dist}(f^n(z), Z) \to 0$ uniformly for all z with $\operatorname{dist}(z, Z) \leq r$.*

Proof. Since f^q has attracting or super-attracting fixed points at $z_0, z_1, \dots, z_{q-1}$, Theorem 7.9 gives $r_0, r_1, \dots, r_{q-1} > 0$ such that $f^{nq}(z) \to z_k$ uniformly whenever $|z - z_k| \leq r_k$. Taking $r = \min\{r_0, r_1, \dots, r_{q-1}\}$, the claim follows. Details are left to the reader. ☐

Again, by conjugation it easily follows that the (suitably modified) claim is still true for the case where the periodic cycle contains ∞.

Definition 7.12. Let z_0 be an attracting or super-attracting fixed point of the rational map f. Then the *basin of attraction* of z_0 under f is $A_f(z_0) = \{z \in \hat{\mathbb{C}} \colon f^n(z) \to z_0 \text{ as } n \to \infty\}$. The *immediate basin of attraction* $A_f^*(z_0)$ is the connected component of $A_f(z_0)$ containing z_0.

More generally, the *basin of attraction* of an attracting or super-attracting periodic cycle $Z = \{z_0, z_1, \dots, z_{q-1}\}$ is $A_f(Z) = \{z \in \hat{\mathbb{C}} \colon \operatorname{dist}(f^n(z), Z) \to 0 \text{ as } n \to \infty\}$, and the *immediate basin of attraction* $A_f^*(Z)$ is the union of the components of $A_f(Z)$ containing points of Z.

By the previous theorems, the basin of attraction of an attracting or super-attracting fixed point or periodic cycle always contains a neighborhood of the fixed point or the periodic cycle. Furthermore, it is easy to verify that $A_f(Z) = \bigcup_{k=0}^{q-1} A_{f^q}(z_k)$, i.e., the basin of attraction of a periodic cycle $Z = \{z_0, \dots, z_{q-1}\}$ with period q is the union of the basins of attraction of the fixed points z_k of f^q for $k = 0, 1, \dots, q-1$. (The same statements are true about the immediate basins.) This observation can often be used to reduce statements about periodic basins to statements about fixed basins by passing to an iterate of the map (which does not change its Julia set.)

Theorem 7.13 (Attractive basins and repelling cycles)**.** *Basins of attractions are contained in the Fatou set. Repelling periodic points are contained in the Julia set.*

Proof. As remarked before the statement of the theorem, by passing to an iterate it is enough to prove both statement for the case of a fixed point z_0. By conjugation we may assume that the fixed point is at $z_0 = 0$. Let us first assume that f has an attracting or super-attracting fixed point at 0 with basin of attraction $A = A_f(0)$. Theorem 7.9 gives $r > 0$ such that $f^n(z) \to 0$ uniformly for $|z| \le r$. If $z_1 \in A$, then there exists n_0 such that $|f^{n_0}(z_1)| < r$, and this implies that there exists $\delta > 0$ such that $|f^{n_0}(z)| < r$ whenever $d(z, z_1) < \delta$. This shows that $f^{n_0+k}(z) \to 0$ uniformly for $d(z, z_1) < \delta$ as $k \to \infty$, and this shows $z_1 \in \mathcal{F}(f)$.

If 0 is a repelling fixed point of f with multiplier λ, then by the chain rule $(f^n)'(0) = \lambda^n \to \infty$. If some subsequence of iterates $\{f^{n_k}\}$ would converge uniformly to a limit function g in some neighborhood of 0, then we know from basic complex analysis, that g is analytic at 0, with $g'(0) = \lim_{k\to\infty} (f^{n_k})'(0) = \infty$, which is impossible. This contradiction shows that $0 \in \mathcal{J}(f)$, as claimed. ☐

For rationally indifferent cycles (also known as *parabolic cycles*), a similar argument as for repelling cycles still works.

Theorem 7.14 (Parabolic cycles)**.** *Rationally indifferent cycles are contained in the Julia set.*

Proof. We can again pass to an iterate and conjugate to assume that we have a rationally indifferent fixed point at 0, i.e., $f(0) = 0$ with $\lambda = f'(0)$ being a q-th root of unity, where $q \ge 1$. From here we can pass to yet another iterate, $F = f^q$, so that $F(0) = 0$ and $F'(0) = \lambda^q = 1$. Now we have to invoke the standing assumption that we started with a rational map of degree $d \ge 2$. The degree increases with iteration, so in particular we know that $\deg F \ge 2$, and that F can not be the identity map. This means that F has a power series expansion $F(z) = z + a_m z^m + \dots$ with $m \ge 2$ and $a_m \ne 0$. Simple power series arithmetic and induction give $F^n(z) = z + n a_m z^m + \dots$. Since $n a_m \to \infty$ as $n \to \infty$, by the same argument as in the case of repelling fixed points, any subsequential uniform limit function of the iterates of F in a neighborhood of 0 would have to have an infinite m-th power series coefficient which is impossible. This shows that $0 \in \mathcal{J}(F) = \mathcal{J}(f)$. □

Remark. Note that Theorem 7.13 would still be true for Möbius transformations, whereas Theorem 7.14 crucially uses the fact that we are dealing with maps of degree ≥ 2. In fact, while the dynamics of Möbius transformations with rationally indifferent fixed points is very simple (since some iterate is the identity map), the dynamics for the case of degree ≥ 2 is qualitatively different, with attracting and repelling directions, and is locally modeled on the dynamics of parabolic Möbius transformations (as we will see later.)

7.4. **More results on Fatou and Julia sets.**

Theorem 7.15 (The Julia set is non-empty)**.** *$\mathcal{J}(f) \neq \emptyset$.*

Remark. Note that this theorem also crucially depends on f having degree ≥ 2, and that this is wrong for (elliptic) Möbius transformations.

Proof. The standard proof by contradiction uses the fact that the uniform limit f of a sequence of rational functions f_n is again rational, and that $\deg f = \deg f_n$ for large enough n. Applied to the situation of complex dynamics, this can never happen because the degrees of the iterates diverge to ∞.

Here we will give a different proof using the holomorphic fixed point index which is a useful tool in many contexts.

We define the *holomorphic index* $\iota(f, z_0)$ of a fixed point $z_0 \in \mathbb{C}$ of a rational map f which is not the identity as the residue of $\frac{1}{z-f(z)}$ at z_0, i.e.,

$$\iota(f, z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{dz}{z - f(z)} \tag{7.2}$$

where $r > 0$ is chosen so that f has no other fixed points z with $|z - z_0| \leq r$. If $\lambda = f'(z_0) \neq 1$, then the integrand has a simple pole and $\iota(f, z_0) = \frac{1}{1-\lambda}$. (In particular, this shows that the holomorphic index is invariant under conjugation in this case. This is in fact also true for the case $\lambda = 1$, but a little harder to prove, and we do not need it at this point.)

Before we apply this to our function f, let us first pass to a conjugate function so that $f(\infty) = 0$, so $f(z) = \frac{p(z)}{q(z)}$ with $\deg p < \deg q = d$ and $d \geq 2$. If we assume that $\mathcal{J}(f) = \emptyset$, then by Theorems 7.13 and 7.14 the map f can have no repelling or rationally indifferent fixed points. In particular, it can have no fixed point with multiplier 1, and all fixed points are simple roots of the equation $f(z) - z = 0$. This equation is equivalent to $p(z) - zq(z) = 0$, which is an equation of degree $d + 1$, so it has $d + 1$ roots. Since none of them is a multiple root, there are $d + 1$ distinct fixed points $z_0, \ldots, z_d \in \mathbb{C}$ of f, with corresponding multipliers $\lambda_0, \ldots, \lambda_d$ satisfying $|\lambda_k| \leq 1$ and $\lambda_k \neq 1$ for all k. The holomorphic indices are given by $\iota_k = \iota(f, z_k) = \frac{1}{1-\lambda_k}$, and the assumption that $f(\infty) = 0$ implies that there exists $R_0 > 0$ such that $|f(z)| \leq R_0/2$ and $|z - f(z)| \geq R/2$ for $|z| = R \geq R_0$. This implies in particular that $|z_k| \leq R_0$ for all k, and the residue theorem gives

$$\begin{aligned} 2\pi \left| \sum_{k=0}^{d} \iota_k - 1 \right| &= \left| \int_{|z|=R} \frac{dz}{z - f(z)} - \int_{|z|=R} \frac{dz}{z} \right| = \left| \int_{|z|=R} \frac{f(z)}{(z - f(z))z} \, dz \right| \\ &\leq 2\pi R \frac{R_0/2}{(R/2)R} = \frac{2\pi R_0}{R} \end{aligned}$$

for all $R \geq R_0$. Now $\frac{2\pi R_0}{R} \to 0$ as $R \to \infty$, so

$$\sum_{k=0}^{d} \iota_k = 1, \tag{7.3}$$

which is commonly known as the *holomorphic fixed point formula.* We had assumed that $|\lambda_k| \leq 1$, and $\lambda_k \neq 1$ for all k. Now the map $\iota(\lambda) = \frac{1}{1-\lambda}$ maps the closed unit disk to the halfplane $\operatorname{Re}\iota \geq \frac{1}{2}$, so the holomorphic fixed point indices satisfy $\operatorname{Re}\iota_k \geq \frac{1}{2}$. Since $d \geq 2$, we get that $\operatorname{Re}\sum_{k=0}^{d} \iota_k \geq \frac{d}{2} > 1$, contradicting the holomorphic fixed point formula (7.3). □

This proof actually shows that the Julia set always contains at least one fixed point.

Definition 7.16. The *grand orbit* $[z]$ of a point $z \in \hat{\mathbb{C}}$ under a map f is defined as the set of all points w such that there exists $m, n \geq 0$ with $f^m(z) = f^n(w)$. The *exceptional set* $\mathcal{E}(f)$ is the set of all points whose grand orbit under f is finite.

Theorem 7.17 (Finite grand orbits)**.** *The exceptional set $\mathcal{E}(f)$ has at most two elements. Any point in $\mathcal{E}(f)$ is a super-attracting fixed point or periodic point of period* 2*. Furthermore, if $\mathcal{E}(f)$ has one element, then f is conjugate to a polynomial g (for which $\mathcal{E}(g) = \{\infty\}$), and if $\mathcal{E}(f)$ has two elements, then f is conjugate to $g(z) = z^{\pm d}$, where $d = \deg f$ (for which $\mathcal{E}(g) = \{0, \infty\}$.)*

Proof. Let z_0 be a point with finite grand orbit $[z_0] = \{z_0, z_1, \ldots z_{q-1}\}$. By conjugation we may assume that $[z_0] \subseteq \mathbb{C}$. Then both f and f^{-1} map $[z_0]$ into itself. Since f is surjective on the sphere, this implies that f maps $[z_0]$ onto itself. Surjective self-maps of a finite set are automatically bijective, so f permutes the elements of $[z_0]$. This implies that every point $z_k \in [z_0]$ has exactly one preimage $f^{-1}(\{z_k\}) = \{z_j\}$ for some j. By our standing assumption, f has degree $d \geq 2$, so f has local degree d at z_j, i.e., $f'(z_j) = f''(z_j) = \ldots = f^{(d-1)}(z_j) = 0$. This means that each $z_j \in [z_0]$ is a zero of multiplicity $d-1$ of the derivative f'. Writing $f(z) = \frac{p(z)}{q(z)}$ with $\deg p, \deg q \leq d$, we get that $f'(z) = \frac{p'(z)q(z)-p(z)q'(z)}{q(z)^2}$, so the zeros of f' are the solutions of $p'(z)q(z) - p(z)q'(z) = 0$. Writing $p(z) = a_d z^d + O(z^{d-1})$ and $q(z) = b_d z^d + O(z^{d-1})$, we get that $p'(z) = d a_d z^{d-1} + O(z^{d-2})$ and $q'(z) = d b_d z^{d-1} + O(z^{d-2})$. Multiplying out, we see that the leading term in the numerator is $(d a_d b_d - a_d d b_d) z^{2d-1} = 0$, so the numerator of f' has degree $\leq 2d-2$. Since every point with finite grand orbit is a zero of f' of multiplicity $d-1$, there can be at most 2.

Note that in this part of the proof we crucially used the surjectivity of (non-constant) rational functions, as well as the fact that $d - 1 > 0$.

In order to show the claim about the explicit form of f, let us first assume that $\mathcal{E}(f) = \{z_0\}$. Let T be a Möbius transformation with $T(z_0) = \infty$, and let $g = T \circ f \circ T^{-1}$ Then $\mathcal{E}(g) = T(\mathcal{E}(f)) = \{\infty\}$, so $g^{-1}(\{\infty\}) = \{\infty\}$. This means that g is a rational map whose poles are all at ∞, so g is a polynomial.

If $\mathcal{E}(f) = \{z_0, z_1\}$ with $z_0 \neq z_1$, then let T be a Möbius transformation with $T(z_0) = \infty$ and $T(z_1) = 0$. We again define $g = T \circ f \circ T^{-1}$, and observe that g is a rational map with $\mathcal{E}(g) = \{0, \infty\}$. This means that either both 0 and ∞ are fixed by g, or they form a periodic 2-cycle. In the first case $g^{-1}(\{\infty\}) = \{\infty\}$, so g is a polynomial, and $g^{-1}(\{0\}) = \{0\}$, so g has all zeros at zero. This implies that $g(z) = az^d$ with some $a \neq 0$. Now $g_\lambda(z) = \lambda g(\lambda^{-1} z) = \frac{a}{\lambda^{d-1}} z^d$ for $\lambda \neq 0$. If we pick λ to be one of the $(d-1)$-st roots of a, then $g_\lambda(z) = z^d$, as claimed.

In the second case we get that g is a rational map with all its poles at 0, and all its zeros at ∞, so $g(z) = az^{-d}$ with some $a \neq 0$. We again conjugate by the linear map $z \mapsto \lambda z$ to get $g_\lambda(z) = \lambda g(\lambda^{-1}z) = \lambda^{d+1}az^{-d}$. If we pick λ to be one of the $(d+1)$-st roots of a^{-1}, then $g_\lambda(z) = z^{-d}$, as claimed. $\square$

Theorem 7.18 (Topological transitivity)**.** *Let $z \in \mathcal{J}(f)$, and let U be an open neighborhood of z. Then $V = \bigcup_{n=1}^\infty f^n(U)$ contains $\mathcal{J}(f)$. In fact, $\mathcal{J}(f) \subseteq \hat{\mathbb{C}} \setminus \mathcal{E}(f) \subseteq V$.*

In fact, this theorem immediately implies the following stronger result.

Corollary 7.19. *Let U be an open set with $U \cap \mathcal{J}(f) \neq \emptyset$. Then there exists $N \in \mathbb{N}$ such that $\mathcal{J}(f) \subseteq \bigcup_{n=1}^N f^n(U)$.*

Proof of Corollary 7.19. By Theorem 7.18, $\mathcal{J}(f) \subseteq \bigcup_{n=1}^\infty f^n(U)$. Since f is an open map, this gives an open cover of the compact set $\mathcal{J}(f)$ by the forward iterates of U. By compactness, there is a finite set of indices I such that $\mathcal{J}(f) \subseteq \bigcup_{n\in I} f^n(U)$. The claim now follows with $N = \max I$. $\square$

Proof of Theorem 7.18. By definition, $f(V) = \bigcup_{n=2}^\infty f^n(U) \subseteq V$, so f and all of its iterates map the open set V into itself. If $E = \hat{\mathbb{C}} \setminus V$ had more than two elements, then the family of iterates $\{f^n\}$ would be normal in V by Montel's theorem. This would imply $V \subseteq \mathcal{F}(f)$, contradicting the fact that V contains $f(z) \in \mathcal{J}(f)$. Since $f(V) \subseteq V$, we get that $f^{-1}(E) \subseteq E$, which by finiteness of E implies that f permutes the elements of E, and that every point in E has a finite grand orbit. This shows $E \subseteq \mathcal{E}(f)$, so $\hat{\mathbb{C}} \setminus \mathcal{E}(f) \subseteq V$. The inclusion $\mathcal{J}(f) \subseteq \hat{\mathbb{C}} \setminus \mathcal{E}(f)$ follows from Theorem 7.17. $\square$

The topological transitivity property has a few strong and very useful corollaries.

Corollary 7.20 (Interior)**.** *If $\operatorname{int} \mathcal{J}(f) \neq \emptyset$, then $\mathcal{J}(f) = \hat{\mathbb{C}}$.*

Proof. If $U \subseteq \mathcal{J}(f)$ is open and non-empty, then Theorem 7.18 shows that $V = \bigcup_{n=1}^\infty f^n(U)$ is dense in $\hat{\mathbb{C}}$. By invariance of the Julia set, $V \subseteq \mathcal{J}(f)$, so $\mathcal{J}(f)$ is dense in $\hat{\mathbb{C}}$. Since $\mathcal{J}(f)$ is closed, this implies that $\mathcal{J}(f) = \hat{\mathbb{C}}$. $\square$

Corollary 7.21 (Basin boundaries)**.** *If $A \subseteq \hat{\mathbb{C}}$ is the basin of attraction of some attracting or super-attracting cycle of f, then $\partial A = \mathcal{J}(f)$.*

Proof. If U is an open neighborhood of some point $z \in \mathcal{J}(f)$, then Theorem 7.18 implies that there exists $n \in \mathbb{N}$ such that $f^n(U) \cap A \neq \emptyset$, so $U \cap f^{-n}(A) \neq \emptyset$. Basins of attraction are completely invariant, so $f^{-n}(A) = A$, which implies that $U \cap A \neq \emptyset$. This shows that $z \in \overline{A}$, and since A is contained in the Fatou set, $z \in \partial A$. The point $z \in \mathcal{J}(f)$ was arbitrary, so $\mathcal{J}(f) \subseteq \partial A$.

If $z_0 \in \partial A$, then any locally uniform limit $f^{n_k} \to g$ of a subsequence of iterates of f in some open neighborhood U of z_0 would be a constant z_1 on $A \cap U$, namely one of the periodic points in the attracting cycles associated to A, and it would have to satisfy $g(z_0) = w_0 \in \mathcal{J}(f)$. This shows that g would necessarily be discontinuous at z_0, contradicting the fact that all such limit functions are meromorphic. This contradiction shows that $z_0 \in \mathcal{J}(f)$, so $\partial A \subseteq \mathcal{J}(f)$. $\square$

Remark. As a corollary, we again get the fact that $\mathcal{J}(f) = \partial A_f(\infty)$ for polynomials of degree ≥ 2, previously proved in Theorem 7.5.

The following corollary is especially important if one wants to produce pictures of Julia sets.

Corollary 7.22 (Iterated preimages are dense)**.** *If $z_0 \in \mathcal{J}(f)$, then the topological closure of the backward orbit $Z = \bigcup_{n=1}^\infty f^{-n}(\{z_0\})$ equals the Julia set $\mathcal{J}(f)$.*

Proof. If $z_1 \in \mathcal{J}(f)$ is arbitrary, and U is any open neighborhood of z_1, then Theorem 7.18 shows that there exist $n \geq 1$ with $z_0 \in f^n(U)$. This implies that $f^{-n}(\{z_0\}) \cap U \neq \emptyset$. Since U was arbitrary, this shows that z_1 is in the closure of Z, and since $z_1 \in \mathcal{J}(f)$ was arbitrary, it implies that $\mathcal{J}(f) \subseteq \overline{Z}$. For the other inclusion, invariance of $\mathcal{J}(f)$ gives $Z \subseteq \mathcal{J}(f)$, and since the Julia set is closed, $\overline{Z} \subseteq \mathcal{J}(f)$. □

This Corollary suggests an algorithm to produce pictures of Julia sets. Start by finding a point $z_0 \in \mathcal{J}(f)$, e.g., a parabolic or repelling fixed point, and then calculate all preimages and draw them on a computer screen. This is easiest for quadratic maps, since quadratic equations are easy to solve, and since the number of preimages in $f^{-n}(\{z_0\})$ is d^n (assuming that there are no critical points in the preimages of z_0, otherwise it is a little smaller.) Still, the number of preimages grows exponentially, and the results vary greatly, depending on the particular map f. I.e., since we can only calculate preimages f^n for $n \leq N$, the quality of the resulting pictures depends on the speed of convergence of these preimages to $\mathcal{J}(f)$ (e.g., measured in the Hausdorff metric.) It turns out that this algorithm works pretty well for "hyperbolic" rational maps, but it produces very poor pictures in the presence of indifferent periodic points (and in some other nasty circumstances, too.)

Corollary 7.23 (No isolated points)**.** *$\mathcal{J}(f)$ has no isolated points.*

Proof. Since $\mathcal{J}(f)$ is completely invariant, not empty, and contains no points with finite grand orbit, it is automatically an infinite set. Since the sphere is compact, $\mathcal{J}(f)$ has an accumulation point z_0, and since $\mathcal{J}(f)$ is closed, $z_0 \in \mathcal{J}(f)$ is a non-isolated point. We know that the backward orbit of z_0 is dense in $\mathcal{J}(f)$, and that every point in the backward orbit of z_0 is also a non-isolated point of $\mathcal{J}(f)$, so $\mathcal{J}(f)$ contains a dense subset of non-isolated points, implying the claim. □

A closed set without isolated points is called *perfect*, so this shows that Julia sets are always perfect.

Corollary 7.24. *$\mathcal{J}(f)$ is always uncountable.*

Proof. This is in fact true for all perfect subsets of the plane (or any Euclidean space), and the proof is an application of the Baire Category Theorem. Details are left to a library or Internet search. □

8. Examples

Now that we have developed some of the basic theory, let us take a break and look at a few examples of Julia sets.

Example 8.1. Let $f(z) = z^2 - 2$. There are several different ways to find the Julia set of f, here is one: Since f is a polynomial, and since $f([-2, 2]) = [-2, 2]$, we know that $[-2, 2] \subseteq K_f$. Now it is also easy to see that the equation $z^2 - 2 = w$ has both solutions in $[-2, 2]$, for all $w \in [-2, 2]$, so $f^{-1}([-2, 2]) = [-2, 2]$. Since $[-2, 2]$ is a compact completely invariant set with more than two elements, it has to contain the Julia set, so $\mathcal{J}(f) \subseteq [-2, 2]$. Since $\mathcal{J}(f) = \partial K_f$, and $K_f \supseteq [-2, 2]$, we get that $\mathcal{J}(f) = K_f = [-2, 2]$. Note that this in particular shows that $c = -2$ is an element of the Mandelbrot set.

To put this example in a different context, $f(z) = z^2 - 2$ is conjugate to the second Chebyshev polynomial $T_2(z) = \cos(2 \arccos z) = 2z^2 - 1$, with $\mathcal{J}(T_2) = [-1, 1]$. It turns out that the Julia set of any Chebyshev polynomial $T_n(z) = \cos(n \arccos z)$ is $\mathcal{J}(T_n) = [-1, 1]$, and it is not hard to prove either.

Even before developing the general theory, Fatou constructed examples of quadratic rational maps with Cantor Julia sets in 1906. For details of Fatou's proof, see [Ale94, Ch. 7]. Here a *Cantor set* is a perfect compact totally disconnected set, or equivalently a set which is homeomorphic to the standard middle-third Cantor set. At this point we are not going to prove Fatou's result in full generality (this will need a little more tools from complex analysis), but the following slightly weaker result.

Theorem 8.2. *There exists $\rho > 0$ such that for all $|c| > \rho$, the Julia set J_c of $f_c(z) = z^2 + c$ is a Cantor set.*

Remark. Note that this automatically implies that $K_c = J_c$, so the filled-in Julia set is the same Cantor set. It also shows that the Mandelbrot set M is contained in the closed disk of radius ρ centered at the origin. The proof will in fact show that the Mandelbrot set is contained in the closed disk $|c| \leq 2$, and while it is also true that J_c is a Cantor set for $|c| > 2$, we will need a slightly larger ρ at this point.

Proof. It is easy to check using the triangle inequality that $R_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ is an escape radius for f_c, i.e., that $|f_c(z)| > |z|$ for $|z| > R_c$. Let $D_0 = \overline{\mathbb{D}}_{R_c}$ be the closed disk of radius R_c centered at 0, and let $D_n = f_c^{-n}(D_0)$ be its n-th preimage under f_c. Since R_c is an escape radius, we get that $D_0 \supseteq D_1 \supseteq D_2 \supseteq \ldots$ is a nested sequence of compact sets with $f(D_n) = D_{n-1}$. This implies that $K_c = \bigcap_{n=0}^{\infty} D_n$.

In order to show that K_c is a Cantor set, we want to represent it as the invariant set for the invariant function system generated by the two inverse branches $g_{c,0}$ and $g_{c,1}$ of f_c. In order to make this work, we analytic inverse branches to start with, and we need to show that they are actually contractions, and we need some compact set E that is mapped by these branches to disjoint sets E_0 and E_1 which are contained in the interior of E. In our context, E will be a slightly larger disk than D_0.

The inverse branches of f_c are $g_{c,j}(z) = \pm\sqrt{z-c}$, where we use the $\pm$ sign to denote the two different branches of the square root in the complex plane. In order for these to be well-defined and analytic in a neighborhood of D_0, we need that $c \notin D_0$ (so that the expression under the square root does not vanish in D_0), so $|c| > R_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$, which is equivalent to $|c| > 2$. Note that this already shows that the Julia set J_c is disconnected for $|c| > 2$, implying that the Mandelbrot set is contained in the disk $|c| \leq 2$. Since we saw in the previous example that $-2 \in M$, this estimate is sharp.

The derivatives of the inverse branches in D_0 can be estimated as

$$|g_{c,j}'(z)|^2 = \frac{1}{4|z-c|} \leq \frac{1}{4(|c| - R_c)} = \frac{1}{4\left(|c| - \frac{1}{2} - \sqrt{\frac{1}{4} + |c|}\right)} =: \lambda_{|c|}^2$$

for $|c| > 2$, where $\lambda_r = \frac{1}{2}\left(r - \frac{1}{2} - \sqrt{\frac{1}{4} + r}\right)^{-1/2}$ is strictly decreasing and positive on $(2, \infty)$, with $\lambda_\rho = 1$ for $\rho = \frac{5+2\sqrt{5}}{4} \approx 2.368$. In particular, for $|c| > \rho$ and $z \in D_0$ we have that $|g_{c,j}'(z)| \leq \lambda_{|c|} < 1$. Now fix such a parameter c, and choose some λ such that $\lambda_{|c|} < \lambda < 1$. From here on, since we are working with a fixed c, we drop the subscript c from the escape radius, and the function and its inverse branches.

There exist a radius $R' > R$ such that the inverse branches g_j are still defined in the closed disk $E = \overline{\mathbb{D}}_{R'}$ and satisfy $|g_j'(z)| \leq \lambda$ for $|z| \leq R'$. We also know that $|f(z)| > |z|$ for $|z| > R$, so by compactness of the circle $|z| = R'$ we get that there exists $R'' > R'$ such that $|f(z)| \geq R''$ for $|z| = R'$. This implies that the inverse images $E_j = g_j(E)$ of E are actually

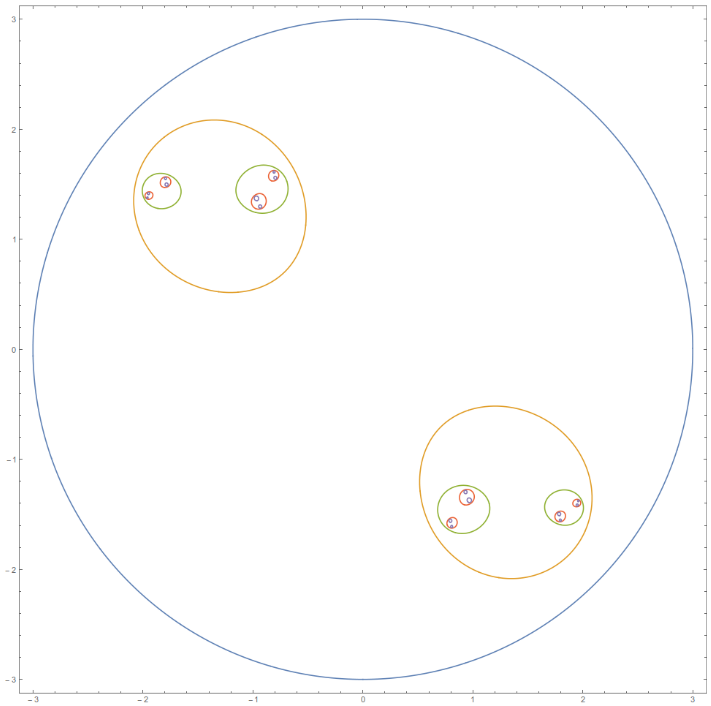


FIGURE 5. Initial stages of the construction in the proof for $f(z) = z^2 + 4i$, showing (the boundaries of) the nested preimages of the disk $\overline{\mathbb{D}}_{R'}$ with $R' = 3$. (In this case, the escape radius is $R \approx 2.56$.) The Julia set J_{4i} is the intersection of this infinite nested sequence of compact preimages. The large disk is E, the two smaller subdomains are E_0 and E_1, the smaller domains nested inside of these are $E_{0,0}$, $E_{0,1}$, $E_{1,0}$, and $E_{1,1}$, etc.

contained in the interior of E. Since the inverses g_j are still analytic and map to disjoint sets in a neighborhood of E, we get that $E_0 \cap E_1 = \emptyset$.

Summarizing, for a fixed quadratic polynomial $f(z) = z^2 + c$ with $|c| > \rho \approx 2.368$ we constructed a closed disk E such that the two inverse branches g_1 and g_2 are analytic in a neighborhood of E, and such that the images $E_j = g_j(E)$ are disjoint and contained in the interior of E. We also found $\lambda < 1$ such that $|g_j'(z)| \leq \lambda$ for $z \in E$ and both inverse branches.

Since $\mathbb{C} \setminus E$ is contained in the basin of ∞, the filled in Julia set is $K_f = \bigcap_{n=0}^{\infty} f^{-n}(E)$. Now for any subset $S \subseteq E$, we have that $f^{-1}(S) = g_1(E) \cup g_2(E)$. With the notation $E_{j_1,j_2,\dots,j_n} = g_{j_1} \circ g_{j_2} \circ \dots \circ g_{j_n}(E)$, simple induction gives $f^{-n}(E) = \bigcup E_{j_1,j_2,\dots,j_n}$, where the union is taken over all tuples with $j_k \in \{0,1\}$ for $k = 1, \dots, n$. Induction also shows that the sets $E_{j_1,\dots,j_n}$ are disjoint and compact for fixed n. Using the fact that $|g_j'| \leq \lambda$ in E, we get that $\operatorname{diam} E_{j_1,\dots,E_n} \leq \lambda^n \operatorname{diam} E \to 0$, as $n \to \infty$. Every connected component of K_f has to be contained in a connected component of $f^{-n}(E)$ for every n, so its diameter must be 0. This shows that K_f is totally disconnected. By our general results, K_f is compact and perfect, showing that K_f is a Cantor set. □

For an illustration of the sets constructed in the proof for the case $c = 4i$, see Figure 5.

Moving to a non-polynomial example, we can at least explain why Schröder and Cayley had trouble trying to explain the dynamics of Newton's method for cubic polynomials.

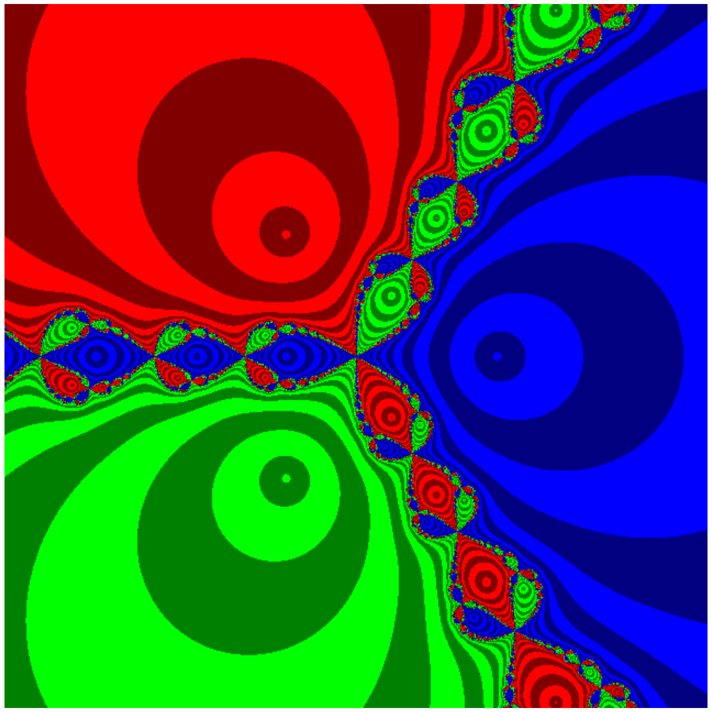


FIGURE 6. Newton's method for $f(z) = z^3 - 1$, showing the three basins of attraction of the roots of f in different colors. All three basins have the Julia set as the common boundary.

Theorem 8.3. *Let $f(z) = (z - z_1)(z - z_2)(z - z_3)$ be a cubic polynomial with distinct roots $z_k \in \mathbb{C}$, and let $N_f(z) = z - \frac{f(z)}{f'(z)}$ be the associated Newton's method. Denote by A_k the set of points whose orbit under Newton iteration converges to z_k. Then A_1, A_2, A_3 are disjoint open sets with $\partial A_1 = \partial A_2 = \partial A_3 = \mathcal{J}(N_f)$.*

Proof. Since f has simple roots at z_k, the Newton's method N_f has super-attracting fixed points at z_k, and A_k is the basin of attraction of the super-attracting fixed point z_k under N_f. Then Corollary 7.21 gives that $\partial A_k = \mathcal{J}(f)$ for $k = 1, 2, 3$. □

The proof of this is really simple (now that we have the tools), but the result is remarkable: Every point in $\mathcal{J}(f)$ is the common boundary point of three disjoint open sets, making its topology quite complicated. This is a marked contrast with the case of Newton's method for quadratic polynomials, where the basins are halfplanes. For an illustration of an example, see Figure 6.

9. Periodic points

9.1. **Julia sets and periodic points.** In this section, we will study fixed points and periodic points, as well as their relation to Fatou and Julia sets, in more detail. Julia defined the Julia set as the closure of the set of all repelling periodic points. The following result is one major ingredient on the road to show that the two definitions are equivalent.

Theorem 9.1. *The Julia set $\mathcal{J}(f)$ is contained in the closure of the set of periodic points.*

Remark. This can be rephrased as saying that periodic points are dense in the Julia set. Since attracting and super-attracting periodic points are always in the Fatou set, we can further say that the Julia set is contained in the closure of the set of all repelling and indifferent periodic points. In order to show that the Julia set actually equals the closure of the set of repelling periodic orbits, Fatou showed that the number of indifferent periodic orbits is always finite.

Proof. Passing to an iterate does not change the Julia set or the set of periodic points, so we may assume that $\deg f \geq 3$. Let $U \subseteq \hat{\mathbb{C}}$ be an open set with $U \cap \mathcal{J}(f) \neq \emptyset$. We claim that U contains at least one periodic point of f. There are only finitely many critical values, and the Julia set has no isolated points, so there exists a regular value $z_0 \in U \cap \mathcal{J}(f)$. This means that $f^{-1}(\{w\}$ has at least three distinct preimages $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and f has three local analytic inverse branches g_j defined in an open disk V centered at z_0 with $V \subset U$, such that $g_j(z_0) = z_j$ for $j = 1, 2, 3$. By choosing the radius small enough, we can make sure that the images $g_j(V)$ are mutually disjoint. We know that the family of iterates $\{f^n\}$ is not normal in V, since $z_0 \in V \cap \mathcal{J}(f)$. By Montel's theorem for omitted functions (Corollary 6.13), there exists $n \geq 1$, $j \in \{1, 2, 3\}$, and $z \in V$ such that $f^n(z) = g_j(z)$. This implies that $f^{n+1}(z) = f(g_j(z)) = z$, so that z is a periodic point. □

9.2. **Local dynamics at attracting and super-attracting fixed points.** Next we will study the local dynamics near attracting, repelling, and super-attracting fixed points and periodic points. The indifferent case is significantly harder and will be handled later. The results in this subsection will have important consequences on the dynamics in attracting and super-attracting basins. For the local dynamics, we will assume that we have fixed points at zero. For the general case of periodic points we can then pass to an iterate and conjugate with an appropriate Möbius transformation. For the following local theorems we only assume that our functions are analytic in some neighborhood of zero.

Theorem 9.2 (Kœnigs linearization theorem). *Let $f(z) = \lambda z + a_2 z^2 + \dots$ be analytic with $|\lambda| \notin \{0, 1\}$. Then there exists a unique analytic map $\phi(z) = z + b_2 z^2 + \dots$ and a constant $r > 0$ such that $\phi(f(z)) = \lambda\phi(z)$ for $|z| < r$.*

In other words, the following diagram commutes, where U and V are neighborhoods of zero, and all maps fix zero.

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \Big\downarrow \phi & & \Big\downarrow \phi \\ \mathbb{C} & \xrightarrow{w \mapsto \lambda w} & \mathbb{C} \end{array}$$

Proof. If ϕ is such a conjugacy for f, and if f^{-1} is the local inverse of f, then ϕ conjugates f^{-1} to $w \mapsto \lambda^{-1} w$. This shows that it is enough to show the claim of the theorem for the attracting case $0 < |\lambda| < 1$, the repelling case then follows by passing to the inverse.

Let us fix $f(z) = \lambda z + a_2 z^2 + \dots$ with $0 < |\lambda| < 1$. For the uniqueness let us assume that we have $\phi(z) = z + b_2 z^2 + \dots$ and a constant $r > 0$ such that $\phi(f(z)) = \lambda\phi(z)$ for $|z| < r$. By possibly making r smaller, we can assume that $|f(z)| \leq \mu |z|$ for $|z| < r$, where $\mu = |\lambda|^{2/3}$ satisfies $\mu^2 < |\lambda| < \mu < 1$. We can also assume that there is a constant C such that $|\phi(z) - z| \leq C|z|^2$ for $|z| < r$. By induction, $|f^n(z)| \leq \mu^n |z| < r$ for all $|z| < r$, so in particular the disk of radius r is forward-invariant, and another easy induction shows $\phi(f^n(z)) = \lambda^n \phi(z)$ for $|z| < r$. Fixing z with $|z| < r$, and writing $z_n = f^n(z)$, we know that

$|z_n| \le \mu^n r$ and so

$$|\phi(z) - \lambda^{-n} f^n(z)| = |\lambda|^{-n} |\phi(z_n) - z_n| \le |\lambda|^{-n} C |z_n|^2 \le C \left(\frac{\mu^2}{|\lambda|}\right)^n$$

By our choice of μ, we have that $\mu^2/|\lambda| < 1$, so $\phi(z) = \lim_{n\to\infty} \lambda^{-n} f^n(z)$, showing uniqueness of ϕ.

The uniqueness proof also guides the way to showing existence. Let $f(z) = \lambda z + a_2 z^2 + \ldots$ be analytic with $0 < |\lambda| < 1$. We pick $\mu = |\lambda|^{2/3}$ as above, and we find constant $r > 0$ and $C > 0$ such that $|f(z)| \le \mu |z|$ and $|f(z) - \lambda z| \le C|z|^2$ for $|z| < r$. Now we let $\phi_n(z) = \lambda^{-n} f^n(z)$ for $|z| < r$, and we claim that these functions form a uniform Cauchy sequence in $\mathbb{D}_r$. The estimate is similar to the one above. Fixing z with $|z| < r$, and writing $z_n = f^n(z)$, we get $|z_n| \le \mu^n r$ and so

$$\begin{aligned} |\phi_{n+1}(z) - \phi_n(z)| &= |\lambda^{-n-1} f(z_n) - \lambda^{-n} z_n| = |\lambda|^{-n-1} |f(z_n) - \lambda z_n| \\ &\le |\lambda|^{-n-1} C |z_n|^2 \le Cr|\lambda|^{-1} \left(\frac{\mu^2}{|\lambda|}\right)^n \end{aligned}$$

Since the successive differences are uniformly dominated by a convergent geometric series, the functions ϕ_n converge uniformly to some limit ϕ in $\mathbb{D}_r$. This limit function is again analytic, and since $\phi_n(0) = 0$ and $\phi_n'(0) = 1$ for all n, we get the same normalization for the limit, so that $\phi(z) = z + b_2 z^2 + \ldots$. Since $\phi_n(f(z)) = \lambda^{-n} f^{n+1}(z) = \lambda \phi_{n+1}(z)$, passing to the limit $n \to \infty$ gives $\phi(f(z)) = \lambda \phi(z)$. □

We will later need a version of this theorem for analytic families of functions. We say that a function of two complex variables $F(t, z)$ is *analytic* in some open set $U \subset \mathbb{C}^2$ iff it can be expanded into a locally convergent power series $F(t, z) = \sum_{j,k=0}^{\infty} a_{j,k}(t - t_0)^j (z - z_0)^k$ about any $(t_0, z_0) \in U$. This easily follows from the Cauchy formula if F is continuous and analytic in each variable separately. There is a miraculous result by Hartogs that continuity automatically follows from analyticity in each variable, but in our applications it will usually be easy to check continuity of F directly.

Theorem 9.3 (Kœnigs linearization theorem with parameters)**.** *Let $F(t, z) = f_t(z) = \lambda_t + a_2(t) z^2 + \ldots$ be an analytic function of (t, z) for $|t - t_0| < \rho$ and $|z| < R$. Assume that $0 < |\lambda_{t_0}| < 1$. Then there exists $\rho_1 > 0$, $r > 0$ and a unique analytic map $\Phi(t, z) = \phi_t(z) = z + b_2(t) z^2 + \ldots$ such that $\phi_t(f_t(z)) = \lambda_t \phi_t(z)$ for $|t - t_0| < \rho_1$ and $|z| < r$.*

Proof. This theorem is proved by carefully checking that everything in the proof of Theorem 9.2 for $f = f_{t_0}$ still works uniformly for f_t, as long as $|t - t_0|$ is sufficiently small. The resulting function family of function $\phi_t(z)$ will be uniform limit of analytic functions in (t, z), so it will be analytic as a function of (t, z). Details are left to the reader. □

In the case of a super-attracting fixed point $f(z) = a_m z^m + a_{m+1} z^{m+1} + \ldots$ with $m \ge 2$, $a_m \ne 0$, the first observation is that the linearly conjugate map $g(z) = \lambda f(\lambda^{-1} z)$ has power series expansion $g(z) = a_m \lambda^{-m+1} z^m + \ldots$. If we choose $\lambda = a_m^{1/(m-1)}$, where we can pick any branch of the $m - 1$-st root, we get $g(z) = z^m + \ldots$, so we may as well assume that $a_m = 1$ to begin with.

Theorem 9.4 (Böttcher's Theorem)**.** *Let $f(z) = z^m + a_{m+1} z^{m+1} + \ldots$ be analytic with $m \ge 2$. Then there exists a unique analytic map $\phi(z) = z + b_2 z^2 + \ldots$ and a constant $r > 0$ such that $\phi(f(z)) = [\phi(z)]^m$ for $|z| < r$.*

In other words, the following diagram commutes, where U and V are neighborhoods of zero, and all maps fix zero.

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \Big\downarrow \phi & & \Big\downarrow \phi \\ \mathbb{C} & \xrightarrow{w \mapsto w^m} & \mathbb{C} \end{array}$$

Proof. The idea is very similar to the case of an attracting fixed point. We obtain the linearizing map ϕ as a limit of maps ϕ_n which are the composition of the n-th forward iterate of f with the n-th backward iterate of the normal form. In the super-attracting case, the map ϕ will be the unique limit of the maps $\phi_n(z) = [f^n(z)]^{1/m^n}$. However, there are some added technical difficulties in this case related to the m^n-th root appearing in the definition of ϕ_n.

For the uniqueness, assume that we have $r > 0$ and a map $\phi(z) = z + \ldots$ such that $\phi(f(z)) = [\phi(z)]^m$ for $|z| < r$. We may also assume that r is chosen small enough so that $|f(z)| \le \frac{|z|}{2}$ for $|z| < r$, so that the disk of radius r is forward-invariant. We may further assume that

$$\frac{1}{2} \le \left|\frac{\phi(z)}{z}\right| \le 2 \quad \text{for } 0 < |z| < r.$$

In particular, this implies that $\phi(z) \neq 0$ for $0 < |z| < r$, so that the function $G(z) = \log|\phi(z)|$ is continuous (and harmonic) for $0 < |z| < r$. The functional equation for ϕ then implies $G(f(z)) = mG(z)$, and induction gives $G(f^n(z)) = m^n G(z)$ for $0 < |z| < r$. (Note that we avoid difficulties of the complex m^n-th root by passing to absolute values, and we simplify the algebra a little bit by passing to the logarithm.) Fixing $z \in \mathbb{D}_r$ and writing $z_n = f^n(z)$, we get

$$\left|G(z) - \frac{\log|f^n(z)|}{m^n}\right| = \frac{|G(z_n) - \log|z_n||}{m^n} = \frac{1}{m^n}\left|\log\left|\frac{\phi(z_n)}{z_n}\right|\right| \le \frac{\log 2}{m^n},$$

showing that $G(z) = \lim\limits_{n\to\infty} \frac{\log|f^n(z)|}{m^n}$ is uniquely determined for $0 < |z| < r$. If $\phi(z) = z + \ldots$ and $\tilde{\phi}(z) = z + \ldots$ both satisfy the functional equation near zero, then there exists $r > 0$ such that $G(z) = \log|\phi(z)| = \log|\tilde{\phi}(z)|$ for $0 < |z| < r$, so $h(z) = \frac{\tilde{\phi}(z)}{\phi(z)} = 1 + \ldots$ is analytic in $|z| < r$, with $|h(z)| = 1$ for $|z| < r$. This implies that $h \equiv 1$, so $\phi(z) = \tilde{\phi(z)}$ for $|z| < r$, showing uniqueness.

For the existence part, we will work with the complex logarithm of ϕ, so we need to be a little more careful to justify its existence and specify the branch. We write $f(z) = z^m g(z)$ with $g(z) = 1 + a_{m+1}z + \ldots$. Now we pick $r \in (0, 1/4)$ such that $|g(z) - 1| < 1/2$ for $|z| < r$. Then $|f(z)| = |z|^m |g(z)| \le 2r^{m-1}|z| < \frac{|z|}{2}$ for $|z| < r$. By induction we see that $f^n(z) = z^{m^n} g_n(z)$ with g_n analytic and $g_n(0) = 1$ for all n. We define $\phi_n(z) = [f^n(z)]^{1/m^n} = z g_n(z)^{1/m^n}$, where we choose the branch of the m^n-th root of g_n which fixes 1, so that $\phi_n(z) = z + \ldots$. We could use the Monodromy Theorem to show that the ϕ_n are all analytic and well-defined on $\mathbb{D}_r$, but we prefer to switch to the logarithmic coordinate $w = \log z$ at this point, and define $F(w) = \log f(e^w) = mw + \operatorname{Log} g(e^w)$ for $\operatorname{Re} w < \log r$. Let us denote this halfplane by $H_r = \{w \in \mathbb{C} \colon \operatorname{Re} w < \log r\}$. Note that the values of $g(z)$ for $|z| < r$ are contained in the disk of radius $1/2$ centered at 1, so the principal branch of $\operatorname{Log} g(z)$ is defined and analytic for $|z| < r$, with $|\operatorname{Log} g(z)| < \log 2$ for $|z| < r$. This shows that $|F(w) - mw| < \log 2$ for $w \in H_r$. Since f maps the punctured disk $\mathbb{D}_r^* = \mathbb{D}_r \setminus \{0\}$ into itself, the map F maps the halfplane H_r

into itself. The logarithmic coordinate is illustrated by the following commutative diagram.

$$\begin{tikzcd}
H_r \arrow[r, "F"] \arrow[d, "\exp"] & H_r \arrow[d, "\exp"] \\
\mathbb{D}_r^* \arrow[r, "f"] & \mathbb{D}_r^*
\end{tikzcd}$$

We now define $\psi_n(w) = \frac{F^n(w)}{m^n}$. An easy induction proof shows that $\lim\limits_{\operatorname{Re} w \to -\infty} (F^n(w) - m^n w) = 0$, so $\lim\limits_{\operatorname{Re} w \to -\infty} (\psi_n(w) - w) = 0$. The maps ψ_n are analytic in H_r, and if we fix $w \in H_r$ and write $w_n = F^n(w)$, we get

$$|\psi_{n+1}(w) - \psi_n(w)| = \left| \frac{F(w_n)}{m^{n+1}} - \frac{w_n}{m^n} \right| = \frac{|F(w_n) - m w_n|}{m^{n+1}} \leq \frac{\log 2}{m^n},$$

so $\{\psi_n\}$ is a uniform Cauchy sequence in H_r, hence $\psi_n \to \psi$ uniformly for some analytic limit ψ. We know that $\psi_n(F(w)) = \frac{F^{n+1}(w)}{m^n} = m\psi_{n+1}(w)$ for all n. Passing to the limit gives

$$\psi(F(w)) = m\psi(w) \quad \text{for } w \in H_r. \tag{9.1}$$

By definition we have $F(w + 2\pi i) = F(w) + 2\pi i m$, so $F^n(w + 2\pi i) = F^n(w) + 2\pi i m^n$, and $\psi(w + 2\pi i) = \psi(w) + 2\pi i$. This shows that $\phi(z) = \exp[\psi(\log z)]$ defines an analytic function for $0 < |z| < r$, and that it is the uniform limit of $\phi_n(z) = \exp[\psi_n(\log z)]$ as $n \to \infty$. We know that $\lim\limits_{\operatorname{Re} w \to -\infty} (\psi_n(w) - w) = 0$, so $\lim\limits_{\operatorname{Re} w \to -\infty} \frac{\exp[\psi_n(w)]}{\exp w} = 1$, which implies that $\lim\limits_{z \to 0} \frac{\phi_n(z)}{z} = 1$. This means that every ϕ_n has a removable singularity at 0, with $\phi_n'(0) = 1$. Then it is easy to see that ϕ_n converges uniformly to ϕ in the whole disk $\mathbb{D}_r$, and that $\phi(0) = 0$ and $\phi'(0) = 1$. The functional equation (9.1) then becomes

$$\phi(f(z)) = \phi(z)^m \tag{9.2}$$

The construction in this proof is probably best illustrated in a longer commutative diagram as follows.

$$\begin{tikzcd}
U^* \arrow[r, "w \mapsto w^m"'] & U^* \\
\hat{U} \arrow[u, "\exp"] \arrow[r, "w \mapsto mw"'] & \hat{U} \arrow[u, "\exp"] \\
H_r \arrow[u, "\psi"] \arrow[r, "F"'] \arrow[d, "\exp"] & H_r \arrow[u, "\psi"] \arrow[d, "\exp"] \\
\mathbb{D}_r^* \arrow[r, "f"'] & \mathbb{D}_r^*
\end{tikzcd}$$

where $\hat{U} = \psi(H_r)$ contains some left halfplane, and $U^* = \exp \hat{U}$ contains a punctured disk about zero. The map $\phi = \exp \circ \psi \circ \log$ is the map from the bottom to the top in this diagram and conjugates f to $w \mapsto w^m$ in the punctured disk $\mathbb{D}_r^*$. This conjugacy then extends to a conjugacy from $\mathbb{D}_r$ to U, without the punctures. $\square$

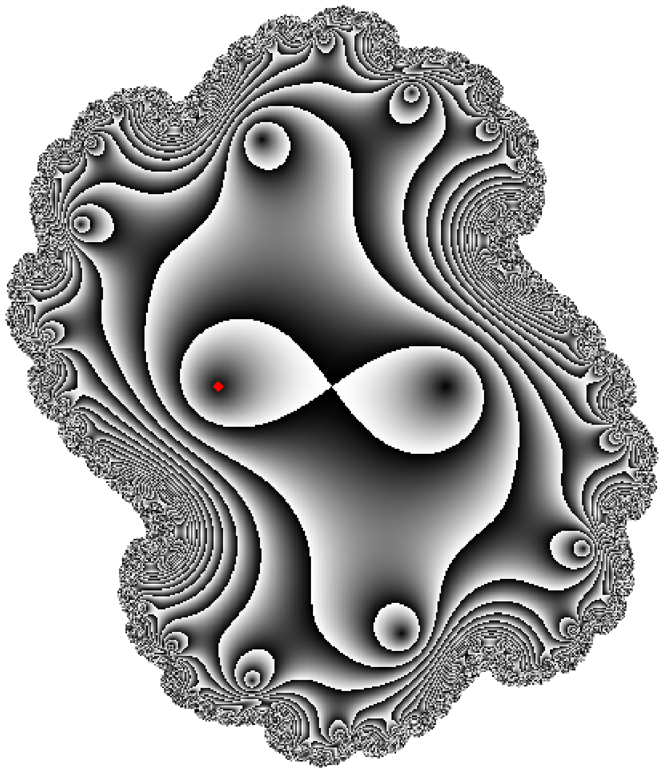


FIGURE 7. Attractive basin of $z_0 = 0$ (red dot) for the map $f(z) = \lambda z(1-z)$ with $\lambda = 0.6 + 0.3i$. Shading is done according to level lines of the absolute value of the linearizing map $|\phi|$. The critical point $c = 1/2$ is at the center of the picture, and the left lobe of the figure 8 passing through it is $\psi(\mathbb{D}_R)$, the image of the maximal disk under the inverse of ϕ. In this case, $R \approx 0.2071$.

9.3. **Attracting and super-attracting basins.** The local normal forms in the attracting and super-attracting case allow us say a little more about attracting and super-attracting basins for rational maps. The super-attracting case is particularly important for analyzing polynomials dynamics, as ∞ in that case is always a super-attracting fixed point.

Theorem 9.5 (Attracting basins)**.** *Let $z_0 \in \mathbb{C}$ be an attracting fixed point of the rational map f, and let A and A^* be its basin and immediate basin of attraction under f. Then there exists a unique analytic map $\phi : A \to \mathbb{C}$ with $\phi(z_0) = 0$, $\phi'(z_0) = 1$, and $\phi(f(z)) = \lambda\phi(z)$ for all $z \in A$, where $\lambda = f'(z_0)$ is the multiplier of the fixed point. Furthermore, the local inverse $\psi = \phi^{-1}$ with $\psi(0) = z_0$ extends meromorphically to a maximal disk $\mathbb{D}_R$ and continuously to its closure $\overline{\mathbb{D}}_R$ such that $\psi(\overline{\mathbb{D}}_R) \subset A^*$ and such that $\psi(\partial\mathbb{D}_R)$ contains a critical point of f.*

The statement of the theorem is illustrated by an example in Figure 7.

Remark. This still works for $z_0 = \infty$, but then one has to choose a different normalization, e.g., $\phi(z) = \frac{1}{z} + \dots$ near ∞.

This theorem has a remarkable corollary on the maximal number of attracting and super-attracting basins.

Corollary 9.6. *Let f be a rational map of degree $d \geq 2$. Then the immediate basin of every attracting and super-attracting cycle contains a critical point. Furthermore, if n_1 and n_2 are the number of attracting and super-attracting cycles of f, respectively, then $n_1 + n_2 \leq 2d - 2$.*

Proof of Corollary 9.6. Every super-attracting cycle contains a critical point, so the same is true for its basin. If $Z = (z_0, \ldots, z_{q-1})$ is an attracting cycle, then f^q has attracting fixed points $z_0, \ldots, z_{q-1}$, so by the theorem there exists a critical point $c_0 \in A^*_{f^q}(z_0)$. By the chain rule, $(f^q)'(c_0) = \prod_{k=0}^{q-1} f'(c_k)$ with $c_k = f^k(c_0)$, so there exists some k such that $f'(c_k) = 0$. Then $c_k \in A^*_{f^q}(z_k) \subseteq A^*_f(Z)$, so the immediate basin of Z contains the critical point c_k. Since basins of different attracting and super-attracting cycles are disjoint, the number of distinct attracting and super-attracting cycles is bounded by the total number of critical points, which for a rational map of degree d is $2d - 2$. □

Remark. Note that the proof shows that $n_1 + n_2$ is really bounded by the total number of distinct critical points. E.g., for the polynomial $f(z) = z^d + c$ with $d \geq 2$, the only two critical points are 0 and ∞, where ∞ is a super-attracting fixed points, so there can be at most one attracting or super-attracting cycle in $\mathbb{C}$. Another very practical consequence of this theorem is an effective algorithm to find all attracting and super-attracting cycles by calculating the orbits of all critical points. Naturally, it is numerically difficult to impossible to identify attracting or super-attracting cycles with very high period, so this algorithm has some limitations. However, it works very well to find all attractors of relatively small periods.

Proof of Theorem 9.5. Using a suitable conjugacy we may assume that $z_0 = 0$, and that $\infty \notin A$. Then Theorem 9.2 gives $r > 0$ and a unique local linearizing map $\phi(z) = z + \ldots$ with $\phi(f(z)) = \lambda\phi(z)$ for $|z| < r$. The functional equation can be written as $\phi(z) = \lambda^{-1}\phi(f(z))$, and by induction we get $\phi(z) = \lambda^{-n}\phi(f^n(z))$ for all $n \geq 1$ and $|z| < r$. We can now use this to extend ϕ to the whole basin A as follows. Given any $z \in A$, there exists a minimal $n = n(z) \geq 0$ such that $|f^n(z)| < r$. Then we define this extension as $\hat{\phi}(z) = \lambda^{-n}\phi(f^n(z))$. By definition, $\hat{\phi}$ agrees with ϕ on $\mathbb{D}_r$. With the same constant n, we also get that $\lambda^{-(n+k)}\phi(f^{n+k}(z)) = \lambda^{-n}\lambda^{-k}\phi(f^k(f^n(z))) = \lambda^{-n}\phi(f^n(z)) = \hat{\phi}(z)$ for any $k \geq 0$. This shows that $\hat{\phi}(z) = \lambda^{-m}\phi(f^m(z))$ whenever $|f^m(z)| < r$, not only for the minimal such m. Given $z_0 \in A$ and a disk D whose closure $\overline{D}$ is still contained in A, we get that $f^m \to 0$ uniformly on $\overline{D}$, so there exists m such that $f^m(D) \subseteq \mathbb{D}_r$. Then $\hat{\phi}(z) = \lambda^{-m}\phi(f^m(z))$ on D, so $\hat{\phi}$ is analytic on D, and it satisfies $\hat{\phi}(f(z)) = \lambda^{-m}\phi(f^{m+1}(z)) = \lambda\lambda^{-(m+1)}\phi(f^{m+1}(z)) = \lambda\hat{\phi}(z)$. This map $\hat{\phi}$ is the claimed analytic continuation of ϕ to the whole basin A. Uniqueness follows from the uniqueness of the local conjugacy and the functional equation $\hat{\phi}(f^m(z)) = \lambda^m\hat{\phi}(z)$.

Now let $\psi(z) = z + \ldots$ be the local inverse of ϕ in a neighborhood of 0. Then $\psi(\lambda w) = f(\psi(w))$ for $|w|$ sufficiently small. By the permanence principle, any analytic continuation to some disk $\mathbb{D}_R$ will still satisfy this functional equation, and since $f^n(\psi(w)) = \psi(\lambda^n w) \to \psi(0) = 0$ for $n \to \infty$, any such continuation will map $\mathbb{D}_R$ into the basin A. Since the image $\psi(\mathbb{D}_R)$ is connected, it will be contained in the immediate basin A^*. If ψ had an analytic continuation to the whole plane, then ψ would be an entire function into A, omitting the whole Julia set. Since the Julia set contains more than 3 points, Picard's theorem would imply that ψ is constant, contradicting $\psi'(0) = 1$. This shows that there exists a maximal disk $\mathbb{D}_R$ to which ψ extends analytically.

Let $U = \psi(\mathbb{D}_R)$, so that $\phi(U) = \mathbb{D}_R$. Then $f(\overline{U}) = \overline{f(U)} = \overline{\psi(\mathbb{D}_{\lambda R})} \subset U$, so $\overline{U} \subset A^*$. Near 0, we know that $\phi \circ \psi(w) = w$, and $\psi \circ \phi(z) = z$. By the permanence principle, these identities are still valid for $w \in \mathbb{D}_R$ and $z \in U$, respectively. This shows that ϕ maps U conformally onto $\mathbb{D}_R$, with inverse ψ. Now ϕ is still defined in a neighborhood of $\overline{U}$, and differentiating the functional equation $\phi(f(z)) = \lambda\phi(z)$ gives $\phi'(f(z))f'(z) = \lambda\phi'(z)$. If $z_0 \in \partial U$ is not a critical point of f, then this shows that $\phi'(z_0) = \lambda^{-1}\phi'(f(z_0))f'(z_0) \neq 0$, since $f(z_0) \in U$, and ϕ has no critical points in U. This shows that ϕ has a local inverse in a neighborhood of $w_0 = \phi(z_0)$, mapping w_0 to z_0, and hence ψ extends analytically to a neighborhood of w_0. If f had no critical points on ∂U, then this argument would show that ψ extends analytically to a strictly larger disk, contradicting the fact that R was maximal.

The fact that ψ extends continuously to the boundary could be proved by showing that ∂U is a Jordan curve, but there is a more elementary approach. Let $\{w_n\}$ be a sequence in $\mathbb{D}_R$ with $w_n \to w_\infty \in \partial\mathbb{D}_R$. Then $f(\psi(w_n)) = \psi(\lambda w_n) \to \psi(\lambda w_0) = z_1$ as $n \to \infty$, so any convergent subsequence of $\psi(w_n)$ has a limit in $Z_0 = f^{-1}(\{z_1\})$. This shows that the set of all accumulation points of $\psi(w)$ as $w \to w_\infty$ is a subset of the finite set Z_0. However, this set is also connected, so it can only consist of one point $z_0 \in Z_0$. The same argument shows that the extension defined by $\psi(w_\infty) = z_0$ is continuous. □

For super-attracting basins, the local conjugacy to $w \mapsto w^m$ given by Theorem 9.4 does not necessarily extend to the whole basin. If we were to try the same idea as in the attracting case, we would end up with $\phi(z) = (\phi(f^n(z)))^{1/m^n}$, and there are obvious difficulties with the analyticity of the m^n-th root. However, the absolute value of ϕ always has a unique extension to the basin.

Theorem 9.7 (Super-attracting basins)**.** *Let $z_0 \in \mathbb{C}$ be a super-attracting fixed point of the rational map f, and let A be its basin of attraction under f. Let ϕ be a Böttcher map given by Theorem 9.4, satisfying $\phi(z_0) = 0$, $|\phi'(z_0)| = 1$, and $\phi(f(z)) = \phi(z)^m$ in a neighborhood of z_0. Then $p = |\phi|$ extends uniquely to a continuous map $p : A \to [0, 1)$ satisfying $p(f(z)) = p(z)^m$.*

Remark. Since we are not requiring a particular normalization of f near z_0, the Böttcher function is only unique up to multiplication with $(m-1)$-th roots of unity, and there is no preferred normalization of $\arg\phi'(z_0)$. Also, in many cases it will be more convenient to work with $G(z) = \log p(z)$, which is harmonic in A, except for logarithmic singularities on the grand orbit of the fixed point z_0. In particular, in the polynomial case with $z_0 = \infty$, the function G is the Green's function of $A_f(\infty)$.

Proof. For every $z \in A$ there exists n such that $f^n(z)$ is in the domain of ϕ. Then we define $p(z) = |\phi(f^n(z))|^{1/m^n} \in [0, \infty)$. It is easy to check (exactly like in the attracting case) that this gives the unique continuous extension p of $|\phi|$ to the whole basin A which satisfies $p(f(z)) = p(z)^m$. Also note that $f^n(z) \to z_0$, so $p(z)^{m^n} = p(f^n(z)) \to 0$ as $n \to \infty$, for any $z \in A$, which implies that $0 \le p(z) < 1$ for $z \in A$. □

In some important cases the map ϕ actually extends to a conformal map of the immediate basin A^* onto the unit disk $\mathbb{D}$.

Theorem 9.8. *Let $z_0 \in \mathbb{C}$ be a super-attracting fixed point of the rational map f, and let A^* be its immediate basin of attraction under f. Let ϕ be a Böttcher map given by Theorem 9.4, satisfying $\phi(z_0) = 0$, $|\phi'(z_0)| = 1$, and $\phi(f(z)) = \phi(z)^m$ in a neighborhood of z_0, and let ψ be its local inverse. Then one of the following mutually exclusive cases occurs.*

1. *There are no critical points of f other than z_0 in A^*, and the map ϕ extends to a conformal map $\phi : A \to \mathbb{D}$ satisfying $\phi(f(z)) = \phi(z)^m$.*
2. *There exists a maximal radius $R \in (0, 1)$ such that ψ extends meromorphically to $\mathbb{D}_R$. In this case, $U = \psi(\mathbb{D}_R)$ is compactly contained in A^*, and ∂U contains a critical point of f. Furthermore, ψ extends to a continuous map $\psi : \overline{\mathbb{D}}_R \to \overline{U}$.*

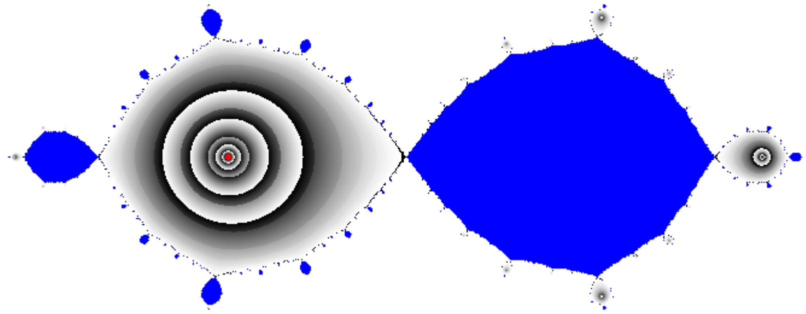


FIGURE 8. Super-attractive basin of $z_0 = 0$ (red dot) for the map $f(z) = 3z^2 - 2z^3$. Shading is done according to level lines of the absolute value of the Böttcher map $|\phi|$, extended to the whole basin. Here ϕ maps the immediate basin conformally onto $\mathbb{D}$, and the sharp contrast lines in the immediate basin are the preimages of circles $|w| = 2^{-k}$ under ϕ. The blue domains are the basin of the other super-attracting fixed point $z_1 = 1$.

Remark. Note that in the second case it is not necessarily true that U is a Jordan domain, so ψ does not always extend to a homeomorphism of the closures. (Picture!)

Proof. As in the attracting case, by conjugation we may assume that $z_0 = 0$, and that $\infty \notin A$. Let $R \in (0, 1]$ be the maximal radius such that ψ extends analytically to $\mathbb{D}_R$. Then the functional equation $\psi(w^m) = f(\psi(w))$ still holds for $|w| < R$. In particular, $f^n(\psi(w)) = \psi(w^{m^n}) \to \psi(0) = z_0$ as $n \to \infty$, so $\psi(\mathbb{D}_R)$ is a connected subset of the basin of attraction of z_0, hence $\psi(\mathbb{D}_R) \subseteq A^*$. We also know that $p(\psi(w)) = |w|$ for $|w|$ small, so for any $|w| < R$ there exists $n \geq 0$ such that $p(\psi(w^{m^n})) = |w|^{m^n}$. Applying the functional equations for p and ψ then gives $p(\psi(w))^{m^n} = p(f^n(\psi(w))) = p(\psi(w^{m^n})) = |w|^{m^n}$, so $p(\psi(w)) = |w|$ for all $w \in \mathbb{D}_R$.

Differentiating the functional equation gives $mw^{m-1}\psi'(w^m) = f'(\psi(w))\psi'(w)$. If ψ has a critical point w with $0 < |w| < 1$, then $\psi'(w^{m^n}) = 0$ for all $n \geq 0$. This is a sequence of zeros of ψ' accumulating at $w = 0$, so this would imply $\psi \equiv 0$, contradicting $|\psi'(0)| = 1$. This shows that $\psi'(w) \neq 0$ for all $|w| < R$.

We claim that ψ is one-to-one on $\mathbb{D}_R$. Assume to the contrary that there exist $w_1, w_2 \in \mathbb{D}_R$ with $w_1 \neq w_2$ and $\psi(w_1) = \psi(w_2)$. Since ψ has no critical points, there exists such a pair with $|w_1| = |w_2|$ minimal. However, ψ is an open mapping, so there exist w_1' and w_2' with $|w_1'| = |w_2'| < |w_1$ and $\psi(w_1') = \psi(w_2')$, which is a contradiction.

If $R = 1$, then ψ maps $\mathbb{D}$ conformally into A^*. As shown above, $p(\psi(w)) = |w|$, so $p(z) \to 1$ as $z \to \partial A'$. Since p is a continuous function of A into $[0, 1)$, this shows that $\partial A' \cap A = \emptyset$. This shows that A' is a connected component of A, hence $A' = A^*$. As an immediate consequence, ψ^{-1} is an analytic continuation of ϕ to a conformal map from A^* onto $\mathbb{D}$.

If $R < 1$, then $f(A') = f(\psi(\mathbb{D}_R)) = \psi(\mathbb{D}_{R^m})$ is contained in the compact set $K = \psi(\overline{\mathbb{D}}_{R^m}) \subset A^*$, so A' is contained in the compact set $f^{-1}(K) \subseteq A$. Since A' is connected, its closure is contained in the immediate basin, i.e. $\overline{A'} \subseteq A^*$.

If we assume that $\partial A'$ does not contain a critical point, then there are local analytic branches of f^{-1} near every point of $z_2 = f(z_1)$ for $z_1 \in \partial A'$, mapping z_2 to z_1. Writing $w_2 = \psi^{-1}(z_2)$, we know that $|w_2| = R^m$, and so we can choose a disk $D_2 = D_r(w_2)$ with $0 < r < R - R^m$, such that $D_2 \subset \mathbb{D}_R$. We let $U_2 = \psi(D_2)$, and we let U_1 be the component

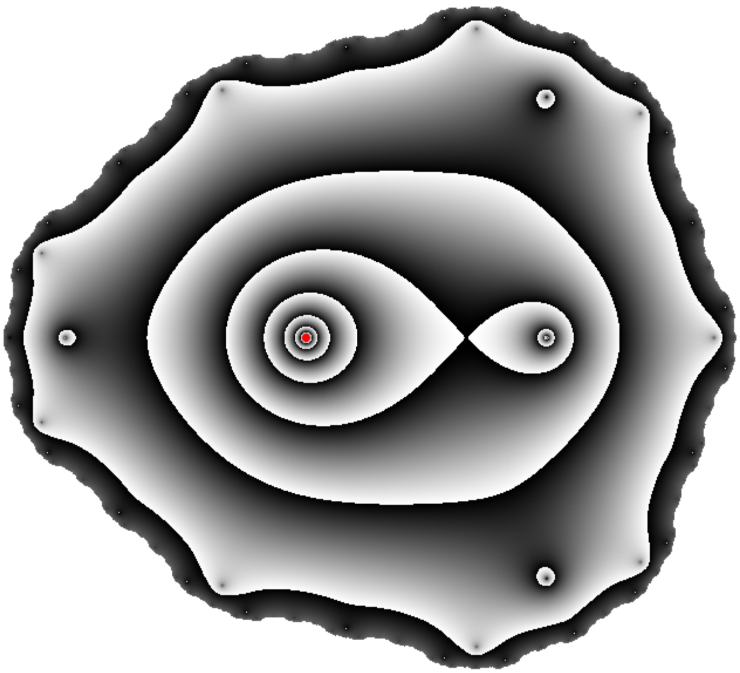


FIGURE 9. Super-attractive basin A of $z_0 = 0$ (red dot) for the map $f(z) = z^2 - 2z^3$. Shading is done according to level lines of the absolute value of the Böttcher map $|\phi|$, extended to the whole basin. Here $\psi = \phi^{-1}$ extends to a maximal disk $\mathbb{D}_R$ (with $R \approx 0.1887$) and maps it conformally onto a domain U (the left lobe of the central figure 8) compactly contained in A. The boundary ∂U contains the critical point $c = 1/3$, and the sharp contrast lines in the immediate basin are the the level lines $|\phi(z)| = 2^{-k}R$ (using the extension of $|\phi|$ to A.) In this case A is simply connected, but ϕ is not the conformal map from A onto $\mathbb{D}$.

of $f^{-1}(U_2)$ which contains z_1. By possibly choosing r smaller, we can guarantee that f maps U_1 conformally onto U_2, so that we have a local analytic inverse $f^{-1} : U_2 \to U_1$. By assumption, $z_1 \in \partial A'$, so $U_1 \cap A' \neq \emptyset$, and the functional equation $\psi(w^m) = f(\psi(w))$ turns into $\psi(w) = f^{-1}(\psi(w^m))$ for $w \in \psi^{-1}(U_1 \cap A')$. As $w \to w_2$, we get that $\psi(w) \to w_1$, where w_1 is one of the solutions of $w_1^m = w_2$. Then $\psi(w) = f^{-1}(\psi(w^m))$ gives a local analytic continuation of ψ to a neighborhood of w_1. Since we assumed that $\partial A'$ has no critical points, this argument shows that ψ has an analytic continuation in the neighborhood of any point on $\partial\mathbb{D}_R$. This implies that ψ extends analytically to a larger disk, contradicting the choice of R. □

10. A closer look at polynomial dynamics

In this section we will apply the results on super-attracting fixed points and basins to the basin of ∞ for polynomials of degree $d \geq 2$. By conjugation with a linear map, we may assume that we have a monic polynomial $f(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$, so that we have a canonical normalization of the Böttcher map ϕ. The results in the previous section give us $\phi(z) = \frac{1}{z} + O(1/z)$ near ∞, conjugating f near ∞ to $w \mapsto w^d$ near 0. However, in the context of polynomials it is often more convenient to work with the reciprocal of ϕ which still conjugates f in a neighborhood of ∞ to $w \mapsto w^d$ (since this power map commutes with the inversion $w \mapsto 1/w$.) Using this observation, an immediate corollary of Theorems 9.4, 9.7, and 9.8 is the following.

Theorem 10.1. *Let $f(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$ be a monic polynomial of degree $d \geq 2$ with basin of infinity $A = A_f(\infty)$. Then the following statements hold:*

(1) *There exists a unique analytic map $\phi(z) = z + b_0 + b_1 z^{-1} + \ldots$ in some neighborhood of ∞, and a constant $R > 0$ such that $\phi(f(z)) = \phi(z)^d$ for $|z| > R$.*
(2) *The function $G(z) = \log|\phi(z)|$ has a unique continuous extension to A which satisfies $G(f(z)) = d \cdot G(z)$ for all $z \in A$. This continuous extension is a positive harmonic function in A.*
(3) *If A contains no critical points of f, then ϕ extends to a conformal map from A onto $\Delta = \{w \in \mathbb{C} \colon |w| > 1\}$.*
(4) *If A contains a critical point of f, then there exists a minimal $R > 1$ such that the local inverse $\psi = \phi^{-1}$ extends analytically to $\Delta_R = \{w \in \mathbb{C} \colon |w| > R\}$. In this case, $U = \psi(\Delta_R) = \{z \in A \colon G(z) > \log R\}$ satisfies $\overline{U} \subset A$, the map ψ extends continuously to ∂U, and ∂U contains a critical point of f.*

Remark. With a little bit more work, one can show that G is explicitly given by $G(z) = \lim_{n\to\infty} d^{-n} \log|f^n(z)|$, and that $\lim_{z \to J(f)} G(z) = 0$, so that G has a continuous extension to the whole plane $\mathbb{C}$ by $G(z) = 0$ for $z \in K(f)$. The map G is the *Green's function* of A, or the Green's function of the polynomial f.

In the case where A does not contain critical points, one might hope that the inverse $\psi = \phi^{-1}$ extends continuously to the closure, so that it maps the unit circle onto $\partial A = J(f)$. By Theorem 6.10, this happens iff $J(f)$ is locally connected. In that case, $\psi(w^d) = f(\psi(w))$ still holds for $|w| = 1$, so the dynamics of f on $J(f)$ are semi-conjugate to the dynamics of $w \mapsto w^d$ on the unit circle by a continuous map. In other words, the topological dynamical system $f|_{J(f)}$ is the quotient of $w \mapsto w^d$ by the equivalence relation given by $w \sim w'$ iff $\psi(w) = \psi(w')$. It turns out that in many (but not all) cases the Julia set is actually locally connected, and the possible equivalence relation induced by ψ and their relationship to the dynamical behavior have been extensively studied, in particular for the case of quadratic polynomials.

Definition 10.2. The level lines of the Green's function $E_u = \{z \in \mathbb{C} \colon G(z) = u\}$ for $u > 0$ are called *equipotentials* (of level u), and the inverse images of concentric rays under the Böttcher map, $R_t = \{z \in A \colon \arg f(z) = 2\pi t\}$, are called *external rays (of external angle t)*.

Note that it is customary to write external angles as multiples of 2π. In the case where A contains no critical points, we can also write $R_t = \psi(\{re^{2\pi it} \colon r > 1\})$. If A contains critical points, external rays R_t can be extended as gradient flow lines of the Green's function, as long as they do not hit a critical point of G (which are exactly the backward orbits of critical points of f in A.)

Also note that f maps equipotentials to equipotentials and external rays to external rays, or more precisely $f(E_u) = E_{du}$ and $f(R_t) = R_{dt}$. This means that f acts both on potentials and on external angles as multiplication by d, in the case of potentials on the positive real line $(0, \infty)$, in the case of external angles on the circle $\mathbb{R}/\mathbb{Z}$.

11. Parabolic fixed points

The study of the local dynamics near parabolic fixed points is considerably more technical than the one near attracting, repelling, and super-attracting fixed points. Much of the fundamental results go back to the works of Leau and Fatou in the early 1900's. Since parabolic fixed points are unstable under perturbation, this is the beginning of the study of the bifurcation locus in parameter space. It turns out that many of the local tools developed for parabolic fixed points can be enhanced to give striking results on perturbations. Beginning with Shishikura's work in the 1980's, this has led to several deep results, e.g., Shishikura's proof that the boundary of the Mandelbrot set has Hausdorff dimension 2, and more recently Buff and Chéritat's proof of the existence of polynomial Julia sets of positive Lebesgue measure.

11.1. **Local dynamics at parabolic fixed points.** Everything in this subsection is again completely local, even though we will mostly apply it to rational functions and possibly their local inverses. There is a special case if some iterate of f is the identity, which we will exclude. As in previous cases, we may assume that we have a fixed point at $z = 0$, so $f(0) = 0$, $f'(0) = e^{2\pi i p/q}$ with integers $p, q \in \mathbb{Z}$ satisfying $q \geq 1$. Here we can yet again pass to an iterate to make our life easier, namely $(f^q)(z) = z + a_2 z^2 + \dots$. We will assume that f^q is not the identity, so it has a fixed point of some finite multiplicity ≥ 2 at $z = 0$. Computer experiments or a little thought show that the dynamical behavior crucially depends on the multiplicity of the fixed point. We will write $f(z) = z + a_{m+1} z^{m+1} + \dots = z(1 + a_{m+1} z^m + \dots)$ where $m \geq 1$ and $a_{m+1} \neq 0$, so that the multiplicity of the fixed point at zero is $m + 1$. For convenience, we can linearly conjugate f to $\lambda f(\lambda^{-1} z) = z(1 + \lambda^{-m} a_{m+1} z^m + \dots) = z(1 - z^m + \dots)$ where $\lambda = (-a_{m+1})^{1/m}$, for any choice of the m-th root. From here on, in this subsection we will assume that f has this normalization if not otherwise stated.

Let $\alpha_j = \frac{2\pi j}{m}$ be the arguments of the m-th roots of unity, and let $\beta_j = \alpha_j + \frac{\pi}{m}$ be the arguments of the m-th roots of -1. If $z = re^{i\alpha_j}$ with $r > 0$ small, then $f(z) = z(1 - r^m + \dots) \approx z(1 - r^m)$. Similarly, for $z = re^{i\beta_j}$ with $r > 0$ sufficiently small we get that $f(z) \approx z(1 + r^m)$. This shows that f almost preserves the rays of arguments α_j and β_j, and that its dynamical behavior is attracting in the α_j directions and repelling in the β_j directions. Accordingly, we call α_j the *attracting directions* and β_j the *repelling directions* of f at the fixed point 0. (In the non-normalized case, the attracting and repelling directions are rotated by some constant argument.)

We say that a forward orbit $z_n = f^n(z_0)$ *converges non-trivially* to 0 iff $\lim_{n\to\infty} z_n = 0$, but $z_n \neq 0$ for all n. For such an orbit we will say that it *converges along (the direction)* α iff $\lim_{n\to\infty} \arg z_n = \alpha$. And for the purposes of the following lemma we define sectors

$$A_{j,\delta,\varepsilon} = \left\{ z \in \mathbb{C} \colon 0 < |z| < \delta,\ |\arg z - \alpha_j| < \frac{\pi - \varepsilon}{m} \right\}$$

for $j \in \{1, \dots, m\}$, $\delta > 0$, and $0 < \varepsilon < \pi$.

Theorem 11.1. *Let $0 < \varepsilon < \varepsilon' < \pi/6$. Then there exist constants δ' and δ and simply connected domains P_j such that $0 < \delta' < \delta$, $A_{j,\delta',\varepsilon'} \subseteq P_j \subseteq A_{j,\delta,\varepsilon}$ such that*

1. *$f(P_j) \subseteq P_j$, and f is univalent on P_j,*
2. *$f^n \to 0$ uniformly on P_j,*
3. *$\arg f^n(z) \to \alpha_j$ on P_j, and*
4. *if $z_n = f^n(z)$ is any orbit converging to 0 along α_j, then there exists n_0 such that $z_n \in P_j$ for $n \geq n_0$.*

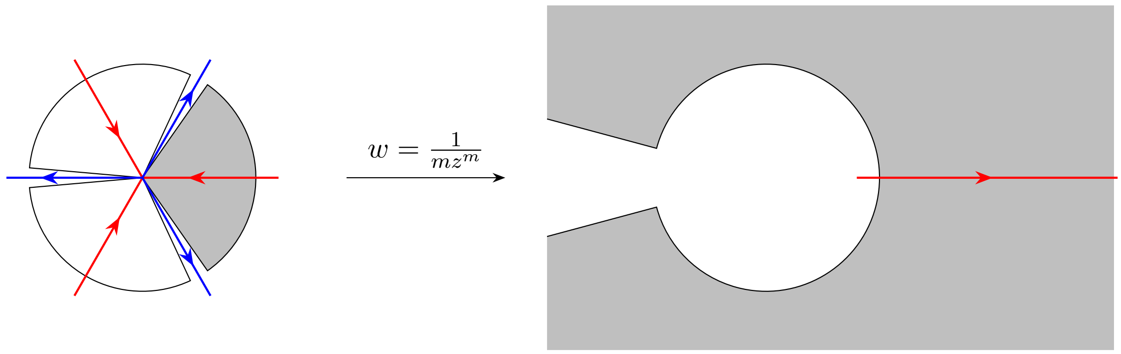


FIGURE 10. On the left: Sketch of attracting sectors $A_{j,\delta,\varepsilon}$ centered around attracting directions α_j in red, for the case of $m = 3$. The repelling directions β_j are indicated in blue. The radius of the sectors is δ, and the line segments bounding them have arguments $\beta_j \pm \varepsilon/m$. On the right: Image B of the attracting sector $A_{0,\delta,\varepsilon}$ under the coordinate transformation $w = \frac{1}{mz^m}$, with the image of the attracting direction now being the positive real line.

In particular, every attracting direction attracts almost a symmetric sector of opening angle $2\pi/m$. The convergence of the argument to α_j is not uniform in any sector of the form $A_{j,\delta,\varepsilon}$, since all its images under iterates of f will still have the same opening angle at 0. Any domain P_j satisfying the conditions of the theorem is called an *attracting petal* for f in the direction α_j at 0. (This notion is not quite invariant under conjugation with Möbius transformations, but the exact definition of what is and what is not a parabolic petal is not too important at this point.) We similarly get attracting petals for the local inverse f^{-1}, centered about the repelling directions of f. These are called *repelling petals* for f. Note that adjacent attracting and repelling petals overlap (since $\varepsilon' < \pi/2$) and that a union of attracting and repelling petals covers a punctured neighborhood of the fixed point.

Proof. Conjugation by multiplication with an m-th root of unity cyclically permutes the attracting directions, so it is enough to show the claim for $j = 0$. We pass to a new coordinate system using the substitution $w = \phi(z) = \frac{1}{mz^m}$, so that $z = \phi^{-1}(w) = (mw)^{-1/m}$. Ignoring for the moment any problems with well-definedness and different branches of the m-th root, we can formally calculate the conjugate function in this new coordinate as

$$F(w) = \phi(f(\phi^{-1}(w))) = \frac{1}{mf\left((mw)^{-1/m}\right)^m} = \frac{1}{w^{-1}\left(1-(mw)^{-1}+O\left(|w|^{-\frac{m+1}{m}}\right)\right)^m}$$
$$= w\left(1 + w^{-1} + O\left(|w|^{-\frac{m+1}{m}}\right)\right) = w + 1 + O\left(|w|^{-\frac{1}{m}}\right)$$

where the $O(.)$ term really stands for a Puiseux series (i.e., a power series in the variable $w^{-1/m}$) which converges in some neighborhood of ∞ and has only terms of order at least as high as indicated.

Since f is tangent to the identity, there exists $\delta > 0$ such that f is univalent on $\mathbb{D}_\delta$ and that $|\arg f(z) - \arg z| = \left|\arg \frac{f(z)}{z}\right| < \varepsilon$ whenever $0 < |z| < \delta$. This shows that if $z \in A_{j,\delta,\varepsilon}$, then $f(z) \neq 0$, and $|\arg f(z)| < \frac{\pi}{m}$. The map $\phi(z) = \frac{1}{mz^m}$ maps the sector $|\arg z| < \frac{\pi}{m}$ conformally onto the slit plane $S = \mathbb{C}\backslash(-\infty, 0])$, so the map F as defined above is well-defined

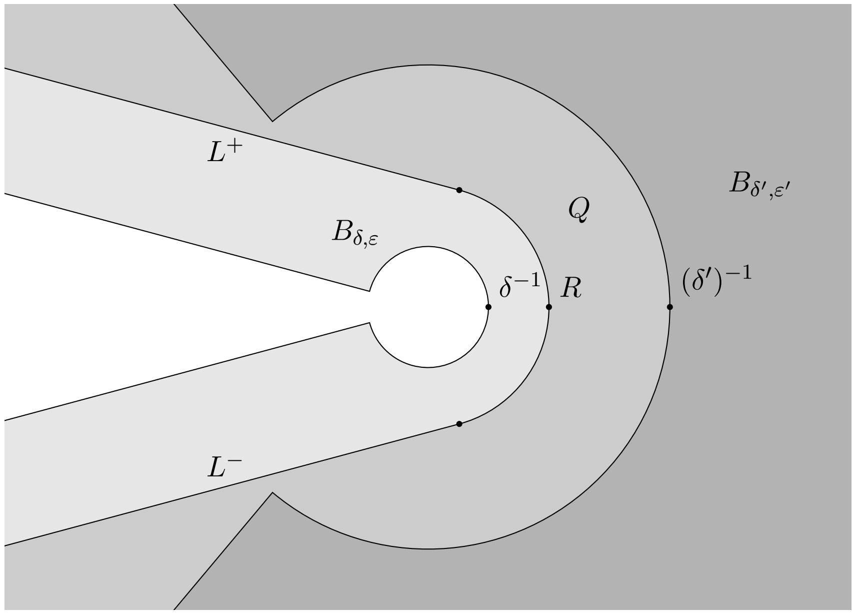


FIGURE 11. Simply connected nested domains $B_{\delta,\varepsilon} \subset Q \subset B_{\delta',\varepsilon'}$ appearing in the proof of Theorem 11.1. Also indicated are the rays $L^{\pm}$ tangent to the circle $|w| = R$, as well as the points of tangency. Given $B_{\delta,\varepsilon}$, the domain Q is a forward-invariant subdomain. Given $\varepsilon' \in (0, \varepsilon)$, there always is $\delta' > 0$ such that $B_{\delta',\varepsilon'} \subset Q$, as indicated. Geometrically, this just means that any keyhole contour $\partial B_{1,\varepsilon'}$ with a steeper slope than the boundary of $B_{\delta,\varepsilon}$ and Q, i.e., with $\varepsilon' > \varepsilon$, can be scaled up by some constant $(\delta')^{-1} > 0$ to fit into Q. (In this picture, ε' is larger than $\pi/6$, but that is not essential.)

as $F = \phi \circ f \circ \phi^{-1}$ in the domain $B_{\delta,\varepsilon} = \phi(A_{j,\delta,\varepsilon}) = \{w \in \mathbb{C}\colon |w| > \delta^{-1},\ |\arg w| < \pi - \varepsilon\}$, mapping it into S.

Note that $\lim_{|w|\to\infty}(F(w) - w) = 1$ uniformly in $B_{\delta,\varepsilon}$, so that there exists $R \geq \delta^{-1}$ such that for all $w \in B_{\delta,\varepsilon}$ with $|w| > R$ we have that

$$|F(w) - w - 1| < \sin\varepsilon$$

We assumed $0 < \varepsilon < \pi/6$, so $0 < \sin\varepsilon < 1/2$, and

$$\operatorname{Re} F(w) > \operatorname{Re} w + \frac{1}{2}. \tag{11.1}$$

From elementary trigonometry we get

$$|\arg(F(w) - w)| < \varepsilon. \tag{11.2}$$

Now consider the circle S of radius R and the two tangent rays $L^{\pm}$ to S of arguments $\pm(\pi-\varepsilon)$ which start at the point of tangency and are contained in the upper and lower halfplane, respectively. Let Q be the connected component of $\mathbb{C} \setminus (S \cup L^+ \cup L^-)$ which contains the interval (R, ∞) (i.e., the unbounded domain to the right of the circle and the two tangent rays.) For an illustration of this construction, see Figure 11.

We claim that

(1) $F(Q) \subseteq Q$;
(2) $F^n(w) \to \infty$ uniformly on Q;

(3) for every $w \in Q$, $\arg F^n(w) \to 0$;
(4) there exists $\delta' > 0$ such that $B_{\delta',\varepsilon'} \subseteq Q$.

The first claim follows directly from (11.2) since for every $w_0 \in Q$, the whole sector $\{w \in \mathbb{C} : |\arg(w - w_0)| < \varepsilon\}$ is contained in Q. Letting $w_n = F^n(w_0)$, equation (11.1) gives $\operatorname{Re} w_n \geq \operatorname{Re} w_0 + n/2 \to \infty$. This shows that $\operatorname{dist}(0, F^n(Q)) \geq \operatorname{dist}(-n/2, Q) \to \infty$, showing the second claim. This implies that $\lim_{n\to\infty}(w_n - w_{n-1}) = 1$, so the same is true for the average of this sequence of differences, namely

$$\lim_{n\to\infty} \frac{w_n - w_0}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (w_k - w_0) = 1$$

Since $w_0/n \to 0$, we get that $w_n/n \to 1$, so

$$\lim_{n\to\infty} \arg w_n = \lim_{n\to\infty} \arg \frac{w_n}{n} = \arg 1 = 0$$

which was our third claim. The fourth claim is an easy geometry exercise as illustrated in Figure 11.

Passing back to the original variable $z = \phi^{-1}(w)$, and letting $P_j = \phi^{-1}(Q)$, we see that $A_{j,\delta',\varepsilon'} \subseteq P_j \subseteq A_{j,\delta,\varepsilon}$, that $f(P_j) \subseteq P_j$, and that $f^n \to 0$ uniformly in P_j, nontrivially, with $\arg f^n(z) \to \alpha_j$ for $z \in P_j$. This shows the first three claims in the theorem. (Univalence of f on P_j follows from the fact that we chose δ small enough so that f is univalent on the disk $\mathbb{D}_\delta$.) The last claim follows from the forward-invariance of P_j and the fact that P_j contains a small sector neighborhood of the ray of argument α. □

Corollary 11.2. *If an orbit $\{z_n\}$ converges non-trivially to* 0*, then it converges along an attracting direction α_j for some j.*

Proof. Assume that $z_n = f^n(z_0)$ is an orbit converging non-trivially to 0. Then there exists a subsequence $\{z_{n_k}\}$ with $|z_{n_k+1}/z_{n_k}| < 1$, so that

$$\left|\arg\left(1 - \frac{z_{n_k+1}}{z_{n_k}}\right)\right| < \frac{\pi}{2}.$$

Since $f(z) = z(1 - z^m + O(|z|^{m+1}))$, we get that $1 - z_{n+1}/z_n = z_n^m(1 + O(|z_n|))$. We know that $\arg(1 + O(|z_n|)) \to 0$ so that there exists k_0 such that for all $k \geq k_0$ we have

$$\left|\arg z_{n_k}^m\right| < \frac{3\pi}{4}.$$

This shows that $|\arg z_{n_k} - \alpha_j| < \frac{3\pi}{4m}$ for some j (which might a priori depend on k). Using Theorem 11.1 with $\varepsilon' = \pi/4$ we find a $\delta' > 0$ such that all orbits of points in $A_{j,\delta',\varepsilon'}$ converge to 0 along α_j. Then there exists some $k \geq k_0$ such that $|z_{n_k}| < \delta'$, so that $z_{n_k} \in A_{j,\delta',\varepsilon'}$ for some j. This implies that $z_n \to 0$ along α_j. □

It turns out that the local dynamics in attracting (and repelling) petals is also conjugate to a very simple normal form, namely a translation, as proved by Fatou. The proof in this case is slightly more involved than in the attracting and super-attracting case, and the explicit ways to numerically calculate the conjugacy converge very slowly. The functional equation $\phi(f(z)) = \phi(z) + 1$ is known as *Abel's functional equation.*

Theorem 11.3 (Fatou coordinates)**.** *If P is an attracting petal for f, then there exists a conformal map ϕ mapping P into the plane such that $\phi(f(z)) = \phi(z) + 1$ for all $z \in P$. Furthermore, the image $\phi(P)$ intersects every horizontal line $L_t = \{w \in \mathbb{C}\colon \operatorname{Im} w = t\}$. The map ϕ is unique up to post-composition with a translation.*

In other words, the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ \mathbb{C} & \xrightarrow{w\mapsto w+1} & \mathbb{C} \end{array}$$

The fact that the image $\phi(P)$ intersects every horizontal line shows that the orbit space $P/\sim$ where the equivalence relation is generated by $z \sim f(z)$, is conformally equivalent to the parabolic cylinder $\mathbb{C}/\mathbb{Z}$ and not any hyperbolic cylinder $\mathbb{H}/\mathbb{Z}$, or $S_a/\mathbb{Z}$, where $S_a = \{w \in \mathbb{C} \colon 0 < \operatorname{Im} w < a\}$ is a horizontal strip.

Proof. Using the same coordinate transformation $w = \frac{1}{mz^m}$ as before, we get $F(w) = w+1+O(|w|^{-1/m})$ and the forward-invariant image of the petal Q. It turns out that one can prove that the maps $\psi_n(w) = F^n(w) - F^n(w_0)$ converge locally uniformly to some conformal limit ψ satisfying $\psi(F(w)) = \psi(w) + 1$. The details of this proof can be found in the textbooks of Milnor [Mil06] and Steinmetz [Ste93].

As an alternative proof, one can use the uniformization theorem on the orbit space $P/\sim$. This proof (and a similar one for the attracting and repelling normal forms) will be added later to these notes. A draft is available on the course website. $\square$

11.2. **Parabolic basins.** These results show that every parabolic fixed point with attracting directions $\alpha_1, \ldots, \alpha_m$ has associated basins of attraction $A_1, \ldots, A_m$, where orbits converge to the fixed point along these attracting directions.

Definition 11.4. Let f be a rational map of degree $d \geq 2$, and let z_0 be a fixed point of f with multiplicity $m+1 \geq 2$ and attracting directions $\alpha_1, \ldots, \alpha_m$. Then the *parabolic basin of attraction in the direction* α_j is defined as the set of all $z \in \hat{\mathbb{C}}$ for which $f^n(z) \to z_0$ non-trivially with $\arg(f^n(z) - z_0) \to \alpha_j$. We will denote it by $A_f(z_0, \alpha_j)$. The corresponding *immediate parabolic basin* $A_f^*(z_0, \alpha_j)$ is defined as the connected component of $A_f(z_0, \alpha_j)$ which contains the attracting sectors defined in Theorem 11.1. Alternatively, it is the unique forward-invariant connected component of $A_f(z_0, \alpha_j)$.

Theorem 11.5. *Let f be a rational function of degree $d \geq 2$ with a parabolic basins $A_j = A_f(z_0, \alpha_j)$ for $j = 1, \ldots, m$. Then the basins A_j are mutually disjoint open subsets of the Fatou set $\mathcal{F}(f)$ with $\partial A_j \subset \mathcal{J}(f)$ for all j. In particular, the connected components of A_j are Fatou components.*

Proof. By definition, the A_j are mutually disjoint. Let $P_j \subset A_j$ be parabolic attracting petals for $j = 1, \ldots, m$ for f. If $z \in A_j$, then there exists some $n_0 \geq 0$ such that $f^{n_0}(z) \in P_j$. Since P is open, there exists some open neighborhood U of z such that $f^{n_0}(U) \in P_j$. This implies that $f^n \to 0$ uniformly on U, so that $z \in \mathcal{F}(f)$. This argument shows that each A_j is an open subset of $\mathcal{F}(f)$.

If $z_1 \in \partial A_j$, then either there exists $n \geq 0$ such that $f^n(z_1) = z_0$, in which case $z_1 \in \mathcal{J}(f)$ by Theorem 7.14 and complete invariance of the Julia set. Otherwise, $f^n(z_1)$ can not converge to z_0 since non-trivial convergence to z_0 only takes place in the basins A_j. Then there exists $\varepsilon > 0$ and a sequence $n_k \to \infty$ such that $|f^{n_k}(z_1) - z_0| \geq \varepsilon$. Any neighborhood of z_1 contains points $z \in A_j$ for which $f^{n_k}(z) \to 0$, so no subsequential limit of the sequence $\{f^{n_k}\}$ can be continuous in any neighborhood of z_1, showing that $z_1 \in \mathcal{J}(f)$.

Lastly, this shows that any connected component of A_j is both open and (relatively) closed in $\mathcal{F}(f)$, so it is also a connected component of $\mathcal{F}(f)$. $\square$

This local analysis yields another immediate corollary on general parabolic fixed points whose derivative is some root of unity.

Corollary 11.6. *Let f be a rational map of degree $d \geq 2$ with a parabolic fixed point $f(z_0) = z_0$ with multiplier $\lambda = f'(z_0) = e^{2\pi i p/q}$ with $p, q \in \mathbb{Z}$ coprime, $q \geq 1$. Then the q-th iterate f^q has a fixed point of multiplicity $m+1 \geq 2$ at z_0, where $m = kq$ is a positive integer multiple of q. If $A_0, \ldots, A_{kq-1}$ denotes the immediate parabolic attractive basins of f^q in cyclic order, then f permutes the basins according to $f(A_j) = A_{j+pk \pmod{q}}$. In particular, the immediate basins form k periodic cycles of period q of Fatou components.*

Proof. Since f^q has the same Fatou set as f, the map f permutes the immediate attractive basins A_j. This means that multiplication with $f'(z_0)$ permutes the attractive directions, mapping the argument α_j to $\alpha_j + 2\pi p/q \pmod{2\pi}$. This immediately implies the claim in the Corollary. □

Just as in the case of attracting fixed points, the local conjugacy in parabolic petals can be extended to the whole basin and as a consequence yields the existence of a critical point in every invariant parabolic basin.

Theorem 11.7 (Parabolic basins)**.** *Let A be a parabolic basin with associated immediate parabolic basin $A^* \subseteq A$ and parabolic petal $P \subset A^*$. Then the local conjugacy $\phi : P \to \mathbb{C}$ given by Theorem 11.3 extends uniquely to an analytic map $\phi : A \to \mathbb{C}$ satisfying $\phi(f(z)) = \phi(z)+1$. Furthermore, the local inverse $\psi = \phi^{-1}$ mapping $\phi(P)$ to P extends analytically to a maximal right halfplane $H_R = \{w \in \mathbb{C} \colon \operatorname{Re} w > R\}$ and continuously to its closure $\overline{H}_R$ (in $\mathbb{C}$) such that $\phi(\overline{H_r}) \subset A^*$ and such that $\psi(\partial H_r)$ contains a critical point of f.*

Before proving this theorem, let us state a remarkable corollary improving our previous count of attracting and super-attracting basins.

Corollary 11.8. *Let f be a rational map of degree $d \geq 2$. Then every periodic cycle of immediate basins of attracting, super-attracting, or parabolic periodic points contains a critical point. If n_1, n_2, and n_3 are the number of periodic cycles of attracting, super-attracting, and parabolic basins, then $n_1 + n_2 + n_3 \leq 2d - 2$.*

Proof of Corollary 11.8. We already know that every invariant attracting, super-attracting, or parabolic basin contains a critical point. Passing from this to cycles works exactly as in the proof of Corollary 9.6, using the chain rule, and the upper bound follows from the fact that f has $2d-2$ critical points. □

Proof of Theorem 11.7. The main ideas here are very similar to those in the proof of Theorem 9.5. First of all, the functional equation $\phi(f(z)) = \phi(z) + 1$ implies that

$$\phi(z) = \phi(f^n(z)) - n \tag{11.3}$$

for all $n \geq 1$, which can be used both to extend ϕ analytically to the whole basin A (choosing n large enough such that $f^n(z) \in P$), and for showing uniqueness of this extension.

The inverse $\psi : \phi(P) \to P$ of ϕ satisfies the functional equation $\psi(w+1) = f(\psi(w))$. If $\psi(w)$ is not a critical point of f, then $\psi(w) = f^{-1}(\psi(w+1))$ with a suitably chosen local inverse branch f^{-1}. This shows that ψ has an analytic continuation along any path γ as long as $\psi(w)$ is not a critical point of f for this continuation. Any such analytic continuation satisfies $\phi(\psi(w)) = w$, so the only possible singularities of analytic continuations of ψ are at points $\phi(c)$ for critical points c of f. Since the image of ψ under any such analytic continuation is connected and thus contained in the immediate basin A^*, we can be even more precise in only considering critical points $c \in A^*$. If there were no critical points of f in A^*, then ψ would have an analytic continuation along any path in $\mathbb{C}$, so by the Monodromy Theorem it would extend to an entire function $\psi : \mathbb{C} \to A^*$. However, since ψ omits the whole Julia set, and the Julia contains more than two points, ψ would have to be constant by Picard's Theorem. This contradicts the fact that ψ is conformal in $\phi(P)$. Now if $c_1, \dots, c_r$ are the critical points of f in A^*, and if $R_k = \operatorname{Re} \phi(c_k)$, then ψ extends analytically to the halfplane $H = \{w \in \mathbb{C} \colon \operatorname{Re} w > \max_k R_k\}$, again by the Monodromy Theorem. This shows that the domain of ψ contains a halfplane, and that it is not the whole plane, so there is a maximal halfplane H_R to which ψ extends analytically. (In the case $r > 1$, we might have $H \neq H_R$.) The fact that ψ extends to a homeomorphism of the closures and that $\psi(\partial H_R)$ contains a critical point of f is exactly analogous to the same step in the proof of Theorem 9.5. $\square$

12. Invariant Fatou components

So far all the invariant Fatou components we have seen were immediate basins of superattracting, attracting, and parabolic fixed points. It turns out that there are two more classes of invariant Fatou components, and both are *rotation domains.*

Definition 12.1. An invariant Fatou component $U = f(U)$ is a *rotation domain* if there exists a conformal map $\phi : U \to V$ conjugating f to an irrational rotation, i.e., there exists $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational, and $\phi(f(z)) = \lambda \phi(z)$ for all $z \in U$.

In other words, the following diagram commutes.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"] & U \arrow[d, "\phi"] \\ V \arrow[r, "w \mapsto \lambda w"] & V \end{tikzcd}$$

This is obviously the same diagram and the same functional equation as in the case of attracting fixed points. However, the behavior of the iterates here is completely different. First of all, in this case ϕ is actually a conformal map, so it is invertible and it conjugates f to the linear map $w \mapsto \lambda w$ on the whole domain U. As a consequence, $f|_U$ is invertible, and U does not contain critical points of f. Also, it is easy to see that V has to be rotationally symmetric since for every $w \in V$ we must also have $\lambda^n w \in V$ for all $n \geq 0$, and this set of points is dense in the circle of radius $r = |w|$. Hence, V can only be the plane, a disk, or an annulus $A_{r,R} = \{w \in \mathbb{C} \colon r < |w| < R\}$. If $V = \mathbb{C}$, then ϕ^{-1} would be an entire function omitting the Julia set, and by Picard's theorem it would be constant, so that is not possible. If $V = A_{0,R}$, then ϕ^{-1} would have a removable singularity (or a pole, since we are working in the sphere), and $\phi^{-1}(0)$ would be an isolated point in the Julia set, which is not possible either. For the same reason we get that $R < \infty$, and by rescaling ϕ we see that the only possibilities for V are $V = \mathbb{D}$ or $V = A_{1,R}$ for some $R \in (1, \infty)$. In the first case, U is simply connected, in the second case it is doubly connected.

Definition 12.2. Let f be a rational map with a rotation domain U. Then U is a *Siegel disk* if U is simply connected and an *Arnold-Herman ring* if U is doubly connected.

Note that a Siegel disk always contains an irrationally indifferent fixed point $z_0 = \phi^{-1}(0)$, whereas an Arnold-Herman ring contains no fixed points.

The classification of invariant Fatou components was started by Fatou (and Julia?), but the classification of rotation domains was only completed later by Cremer in the 1930's, using the Uniformization Theorem. The existence of rotation domains was proved even later than that, first by Siegel in the 1940's for the simply connected case, and later by Arnold in the 1960's for the doubly connected case. Arnold was not really interested in complex dynamics and did not put his result in this context, and Herman later expanded Arnold's results and analyzed rotation domains from the complex dynamical point of view, hence they bear his name.

Theorem 12.3 (Classification of invariant Fatou components)**.** *If f is a rational map of degree $d \geq 2$ and $U = f(U)$ is an invariant component of the Fatou set, then it is of one of the following five types:*

(1) *U is an immediate attracting basin,*
(2) *U is an immediate super-attracting basin,*
(3) *U is an immediate parabolic basin,*
(4) *U is a Siegel disk, or*
(5) *U is an Arnold-Herman ring.*

Remark. Obviously, any periodic Fatou domain $U = f^q(U)$ is also of one of these types since we can always pass to the iterate f^q which fixes U and has the same Fatou set. Sullivan showed in the 1980's that rational maps have no *wandering domains*, i.e., every Fatou domain eventually maps into a periodic cycle of Fatou domains. In a sense, this completely classifies the possible dynamical behavior of rational functions on the Fatou set. Even better, although rotation domains do not contain critical points, Shishikura in the 1980's succeeded in finding sharp bounds on the number of periodic cycles of Fatou components. (We already know that every basin needs at least one critical point, and Shishikura showed that each Siegel disk also need one critical point, and each Arnold-Herman ring needs two.)

As another side note, in the dynamics of entire transcendental functions (such as $f(z) = \lambda e^z$ or $f(z) = \lambda \sin z$) there is one more class of invariant Fatou domains, called *Baker domains*. These look almost like parabolic basins, but all orbits in U converge to the essential singularity at ∞. However, in the case of transcendental dynamics there actually do exist wandering domains, and counting the number of periodic cycles of Fatou domain or wandering domains is only possible in certain restricted subclasses.

Proof. TO BE ADDED. □

13. Irrationally indifferent fixed points

13.1. **Topological stability implies analytic linearizability.** In this subsection, we will consider the local dynamics near irrationally indifferent fixed points, without assuming that our maps are globally defined rational maps. By conjugation we may assume that we have a map with an irrationally indifferent fixed point at 0, analytic in the unit disk. A striking result is that local stability of the fixed point implies analytic linearizability, i.e., analytic conjugacy to an irrational rotation.

Theorem 13.1. *Let $f(z) = \lambda z + O(z^2)$ be analytic in $\mathbb{D}$ with $\lambda = e^{2\pi i \alpha}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume that there exists $r > 0$ such that $|f^n(z)| < 1$ for all $z \in \mathbb{D}_r$. Then there exists a unique analytic map $\phi(z) = z + O(z^2)$ such that $\phi(f(z)) = \lambda\phi(z)$ whenever $|z| < r$ and $|f(z)| < r$.*

In other words, the following diagram commutes, where $U = \mathbb{D}_r \cap f^{-1}(\mathbb{D}_r)$ and all maps fix zero.

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & \mathbb{D}_r \\ \big\downarrow \phi & & \big\downarrow \phi \\ \mathbb{C} & \xrightarrow{w \mapsto \lambda w} & \mathbb{C} \end{array}$$

The functional equation is the same as in the attracting and repelling case, but obviously the local dynamical behavior is very different. Orbits of $w \mapsto \lambda w$ are dense on the circle with radius $\rho = |w|$, so the orbits of f in a neighborhood of zero are dense subsets of the analytic curves $\phi^{-1}(\partial \mathbb{D}_\rho)$. If such a map ϕ exists, we will say that f is *linearizable* at 0. (Note that the theorem shows that the existence of a homeomorphism ϕ satisfying the functional equation automatically implies the existence of an analytic linearizing map. In other words, topological and analytic linearizability are equivalent.)

Proof. The proof of this theorem is relatively easy. The assumption given imply that $|f^n(z)| < 1$ for all $|z| < r$ and $n \in \mathbb{N}$, so the maps $\lambda^{-n} f^n(z) = z + \dots$ are always bounded by 1 for $|z| < r$, so they form a normal family. However, these do not necessarily converge, and subsequential limits will not satisfy the desired functional equation. This changes when one considers the averages

$$\phi_n(z) = \frac{1}{n} \sum_{k=1}^{n} \frac{f^k(z)}{\lambda^k},$$

which are still uniformly bounded by $|\phi_n(z)| < 1$ and have the desired normalization $\phi_n(z) = z + \dots$. The definition immediately gives

$$\begin{aligned} |\phi_n(f(z)) - \lambda \phi_n(z)| &= \frac{1}{n} \left| \sum_{k=1}^{n} \frac{f^{k+1}(z)}{\lambda^k} - \sum_{k=1}^{n} \frac{f^k(z)}{\lambda^{k-1}} \right| = \frac{1}{n} \left| \sum_{k=1}^{n} \frac{f^{k+1}(z)}{\lambda^k} - \sum_{k=0}^{n-1} \frac{f^{k+1}(z)}{\lambda^k} \right| \\ &= \frac{1}{n} \left| \frac{f^{n+1}(z)}{\lambda^n} - f(z) \right| \le \frac{1}{n} \left(\frac{|f^{n+1}(z)|}{|\lambda|^n} + |f(z)| \right) \le \frac{2}{n} \end{aligned}$$

By Montel's theorem, $\{\phi_n\}$ is a normal family in $\mathbb{D}_r$, so there exists a locally uniform subsequential limit $\phi(z) = \lim_{k \to \infty} \phi_{n_k}(z)$. Passing to the limit in the string of inequalities given above, we get that $\phi(f(z)) - \lambda \phi(z) = 0$ for $z \in U = \mathbb{D}_r \cap f^{-1}(\mathbb{D}_r)$, so $\phi(z) = z + \dots$ is the desired linearizing map.

There are different ways to prove uniqueness, and the following points to one very important approach to solving the functional equation via power series. Writing

$$f(z) = \lambda z \left(1 + \sum_{k=1}^{\infty} a_k z^k \right) \quad \text{and} \quad \phi(z) = z \left(1 + \sum_{n=1}^{\infty} b_n z^n \right),$$

we get

$$\phi(f(z)) = \lambda z \left(1 + \sum_{k=1}^{\infty} a_k z^k \right) \left[1 + \sum_{n=1}^{\infty} b_n \lambda^n z^n \left(1 + \sum_{k=1}^{\infty} a_k z^k \right)^n \right] = \lambda z \left(1 + \sum_{n=1}^{\infty} c_n z^n \right)$$

where $c_n = b_n \lambda^n + p_n(a_1, \dots, a_n, b_1, \dots, b_{n-1})$ with polynomials p_n. With this, the functional equation $\phi(f(z)) = \lambda \phi(z)$ turns into

$$\lambda z \left(1 + \sum_{n=1}^{\infty} c_n z^n \right) = \lambda z \left(1 + \sum_{n=1}^{\infty} b_n z^n \right).$$

Comparing coefficients we get $b_n = c_n = b_n\lambda^n + p_n(a_1, \dots, a_n, b_1, \dots, b_{n-1})$, which leads to recursion formulas of the form

$$b_n = \frac{1}{1-\lambda^n} p_n(a_1, \dots, a_n, b_1, \dots, b_{n-1}) \tag{13.1}$$

This shows that the coefficients b_n are uniquely determined, so the linearizing map is unique if it exists □

Note that the equation (13.1) also suggests a way to find ϕ, as follows: The coefficients b_n are recursively determined, and so there is a formal power series solution ϕ to $\phi(f(z)) = \lambda\phi(z)$. In order to show that this actually gives a linearization, one has to prove that this series has a positive radius of convergence. This is not at all easy, but it was the strategy successfully applied by Siegel in 1942 for the first proof of the existence of Siegel disk. It turns out that the growth of the coefficients b_n depends crucially on number-theoretic properties of α. If α is "very well approximated" by rational numbers, then the denominators $1 - \lambda^n$ are too small too often, and the resulting formal series has a zero radius of convergence. On the other hand, if α is "badly approximated" by rational numbers, Siegel was able to prove that the radius of convergence is positive. We will get to the exact definitions of the corresponding number-theoretic conditions and improvements of Siegel's result later in this section.

13.2. **Cremer points.** We will start by presenting a historically earlier and easier to prove result about the non-existence of Siegel disks in certain cases, due to Cremer in the 1930's. Before stating it, let us first state and prove some preliminary results. We use the notation $d(t) = \operatorname{dist}(t, \mathbb{Z}) = \inf_{p\in\mathbb{Z}} |t - p|$ for $t \in \mathbb{R}$.

Lemma 13.2. *For every $t \in \mathbb{R}$ we have*

$$4d(t) \le |e^{2\pi i t} - 1| \le 2\pi d(t).$$

In particular, for $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}$, and for every $q \in \mathbb{N}$ we have

$$4d(q\alpha) \le |\lambda^q - 1| \le 2\pi d(q\alpha).$$

Proof. Since $\lambda^q = e^{2\pi i q\alpha}$, the second string of inequalities directly follows from the first with $t = q\alpha$. In order to prove the first statement, we can use the fact that both $d(t)$ and $e^{2\pi i t}$ are $\mathbb{Z}$-periodic, so we may assume that $t \in (-1/2, 1/2]$. In that case, $d(t) = |t|$, and the inequality $|e^{2\pi i t} - 1| \le 2\pi|t|$ follows directly from integrating e^z over the arc of the unit circle from 1 to $e^{2\pi i t}$. The inequality $4|t| \le |e^{2\pi i t} - 1|$ follows directly from an easy geometric argument (sketch things in the complex plane) or some straight-forward trigonometric identities. Details are left to the reader. □

We say that a set $A \subset \mathbb{R}$ is a *residual* set if A contains a countable intersection of open and dense subsets of $\mathbb{R}$. By Baire's theorem, every residual set is uncountable and dense. Obviously, the intersection of countably many residual sets is again residual. We will also use this notion for subsets of the unit circle, where the same Baire results are true.

Lemma 13.3. *Assume that for each $q \in \mathbb{N}$ we have a positive increasing continuous functions $L_q : [0, \infty) \to [0, \infty)$ with $L_q(0) = 0$. Then the set*

$$A_L = \left\{ \alpha \in \mathbb{R} \colon \liminf_{q\to\infty} L_q(d(q\alpha)) = 0 \right\}$$

is residual.

Proof. We can rewrite the definition of A_L as

$$A_L = \left\{ \alpha \in \mathbb{R} \colon \forall m \in \mathbb{N} \quad \forall q_0 \in \mathbb{N} \quad \exists q \geq q_0 \quad \exists p \in \mathbb{Z} : \; L_q(|q\alpha - p|) < \frac{1}{m} \right\}$$

which we can turn into unions and intersections as follows

$$A_L = \bigcap_{m=1}^{\infty} \bigcap_{q_0=1}^{\infty} \bigcup_{q=q_0}^{\infty} \bigcup_{p=-\infty}^{\infty} U_{L,p,q,m}$$

where

$$U_{L,p,q,m} = \left\{ \alpha \in \mathbb{R} \colon L_q(|q\alpha - p|) < \frac{1}{m} \right\}$$

is an open interval containing p/q. This shows that

$$U_{q_0,m} = \bigcup_{q=q_0}^{\infty} \bigcup_{p=-\infty}^{\infty} U_{L,p,q,m}$$

is an open set containing all rational numbers, so it is open and dense. The set A_L is the intersection of all of these sets, so it is a countable intersection of open dense sets. □

For the following corollary, we write $\mathbb{T}$ for the unit circle in the complex plane.

Corollary 13.4. *Under the same assumptions, the set*

$$S_L = \left\{ \lambda \in \mathbb{C} \colon \liminf_{q \to \infty} L_q(|\lambda^q - 1|) = 0 \right\}$$

is a residual subset of $\mathbb{T}$. In particular, S_T itself is uncountable and dense in $\mathbb{T}$.

Proof. This follows directly from Lemmas 13.2 and 13.3. □

With this as preparation, we can finally state Cremer's results on non-existence of Siegel disks for certain rotation numbers.

Theorem 13.5 (Cremer)**.** *Let*

$$\mathcal{C}_d = \left\{ \lambda \in \mathbb{T} \colon \liminf_{q \to \infty} |\lambda^q - 1|^{1/(d^q-1)} = 0 \right\}$$

for $d \geq 2$, and let $f(z) = \lambda z + a_2 z^2 + \ldots + a_{d-1} z^{d-1} + z^d$ be a polynomial of degree $d \geq 2$ with $\lambda \in \mathcal{C}_d$. Then f is not linearizable at 0.

By Corollary 13.4, the sets $\mathcal{C}_d$ are residual subsets of the unit circle, so we immediately get the following corollary for the intersection $\mathcal{C}_\infty = \bigcap_{d=2}^{\infty} \mathcal{C}_d$.

Corollary 13.6. *There exists a residual subset $\mathcal{C}_\infty$ of the unit circle such that no polynomial $f(z) = \lambda z + a_2 z^2 + \ldots + a_{d-1} z^{d-1} + z^d$ of degree $d \geq 2$ is linearizable at 0.*

As an exercise, find an explicit description of $\mathcal{C}_\infty$, in the most simple form possible.

Proof of Theorem 13.5. The idea is to show that under the assumptions in the theorem, there is a simple obstruction to linearizability, namely a sequence of periodic points converging to the fixed point 0. I.e., there is a sequence of periods $q_n \to \infty$, and a sequence of points $z_n = f^{q_n}(z_n)$ with $z_n \neq 0$ and $\lim_{n \to \infty} z_n \to 0$. If f were linearizable, there would be a neighborhood of 0 in which f has no other periodic points except for 0 itself.

The equation for periodic points of (not necessarily minimal) period q is

$$0 = f^q(z) - z = (\lambda^q - 1)z + \ldots + z^{d^q}$$

By the Fundamental Theorem of Algebra, this factors as

$$0 = z \prod_{j=1}^{d^q-1} (z - z_q^{(j)})$$

where $z_q^{(1)}, \ldots, z_q^{(d^q-1)}$ are the q-periodic points other than 0. Comparing the linear coefficients and taking absolute values gives

$$\prod_{j=1}^{d^q-1} \left| z_q^{(j)} \right| = |\lambda^q - 1|,$$

so there exists j_q such that

$$0 < \left| z_q^{(j_q)} \right| \leq |\lambda^q - 1|^{1/(d^q-1)}.$$

By assumption, there exists $q_k \to \infty$ such that $|\lambda^{q_k} - 1|^{1/(d^q-1)} \to 0$. This shows that $z_k = z_{q_k}^{(j_{q_k})} \to 0$. Each z_k is q_k-periodic with $z_k \neq 0$, so f can not be linearizable at zero. □

In honor of Cremer, irrationally indifferent fixed points for which a local linearization does not exist are called *Cremer points*.

13.3. **Existence of Siegel disks.** Siegel's original 1942 proof of the existence of Siegel disk is technically quite involved. Here we will present a much simpler proof due to Yoccoz in 1987. It does not give explicit sufficient number-theoretic conditions, and it only works for quadratic polynomials, so it does not quite replace Siegel's proof.

Theorem 13.7. *For almost all $\alpha \in \mathbb{R}$, the quadratic polynomial $f_\lambda(z) = \lambda(z - z^2)$, where $\lambda = e^{2\pi i\alpha}$, has a Siegel disk centered at* 0.

Yoccoz's proof relies on two results from classical complex and harmonic analysis, one due to Fatou, the other due to the Riesz brothers. Both of these can be found in many classical books on these topics, e.g., the one by Rudin [Rud66]. We denote by $H^\infty(\mathbb{D})$ the space of all bounded analytic functions $f : \mathbb{D} \to \mathbb{C}$. (This is one of the *Hardy spaces* $H^p(\mathbb{D})$, a family of Banach spaces of analytic functions. They can be identified with the subspace of functions in $L^p(\mathbb{T})$ for which all the negative Fourier coefficients vanish.)

Theorem 13.8 (Fatou)**.** *If $f \in H^\infty(\mathbb{D})$, then the radial limit*

$$f^*(z) = \lim_{r \to 1} f(rz)$$

exists for almost every $z \in \mathbb{T}$.

Here the limit is taken along the interval $0 < r < 1$, and the measure on the circle $\mathbb{T}$ is one-dimensional length measure. Even more than stated is true: f is both the Poisson integral and the Cauchy integral of f^*. However, the fact that we need in our proof is provided by the following (also classical) theorem

Theorem 13.9 (F. and M. Riesz)**.** *If $f \in H^\infty(\mathbb{D})$ is not the constant zero function, then the radial limit satisfies $f^*(z) \neq 0$ for almost every $z \in \mathbb{T}$.*

We are going to take these theorems for granted, but it should be stressed that they are both not too hard to prove with a little knowledge of L^p spaces and some analysis of the Poisson kernel.

Proof of Theorem 13.7. We have that $f'_\lambda(z) = \lambda(1-2z)$ has a unique critical point at $z = 1/2$. We will use our detailed knowledge of the local dynamics near attracting fixed points from Theorem 9.5, as well as the analytic dependence of the linearizing function on λ, which follows from the parametrized version of Kœnigs linearization theorem, Theorem 9.3.

For $0 < |\lambda| < 1$, let $\phi_\lambda(z) = z + \dots$ be the linearizing function satisfying $\phi_\lambda(f_\lambda(z)) = \lambda\phi_\lambda(z)$ near 0, and let $\psi_\lambda = \phi_\lambda^{-1}$ be its local inverse. Then ψ_λ extends uniquely to a largest disk $\mathbb{D}_{r(\lambda)}$ of radius $r(\lambda) = |\phi_\lambda(1/2)| > 0$ (since $\psi^{-1}(\mathbb{D}_{r(\lambda)})$ has a critical point on its boundary, and there is only one, namely $z = 1/2$.) We define $h(\lambda) = \phi_\lambda(1/2)$. For $|\lambda| \geq 1/2$ and $|z| \geq 4$ we get that $|f(z)| = |\lambda||z||1 - z| \geq \frac{3}{2}|z|$, so the basin of attraction of zero is contained in $\mathbb{D}_4$ for $1/2 \leq |\lambda| < 1$. This shows that for these λ, the inverse of the linearizing map ψ_λ maps the disk $\mathbb{D}_{r_\lambda}$ into $\mathbb{D}_4$. By the Schwarz Lemma applied to the map $\frac{1}{4}\psi_\lambda(r(\lambda)w))$ we get that $|h(\lambda)| = r(\lambda) \leq 4$ for $1/2 \leq |\lambda| < 1$. We remember that we found the linearizing maps as $\phi_\lambda(z) = \lim_{n\to\infty} \lambda^{-n} f_\lambda^n(z)$, so in order to get a more constructive grip on h, we define

$$h_n(\lambda) = \frac{f_\lambda^n(1/2)}{\lambda^n}.$$

Then $h_n \to h$ locally uniformly for $0 < |\lambda| < 1$, and we get the following recursive relation:

$$\begin{aligned} h_{n+1}(\lambda) &= \frac{f_\lambda^{n+1}(1/2)}{\lambda^{n+1}} = \frac{f_\lambda(f_\lambda^n(1/2))}{\lambda^{n+1}} = \frac{f_\lambda(\lambda^n h_n(\lambda))}{\lambda^{n+1}} \\ &= \frac{\lambda \cdot \lambda^n h_n(\lambda)(1 - \lambda^n h_n(\lambda))}{\lambda^{n+1}} = h_n(\lambda)(1 - \lambda^n h_n(\lambda)) \end{aligned}$$

In particular, this shows that $h_{n+1}(\lambda) - h_n(\lambda) = O(\lambda^n)$, so that the coefficients of $1, \lambda, \dots, \lambda^{n-1}$ are the same for all h_{n+k} for $k \geq 0$. It also shows that each h_n extends analytically to the unit disk, and since $h_n(\lambda) \to h(\lambda)$ uniformly for $|\lambda| = 1/2$, and $|h(\lambda)| \leq 4$ for $|\lambda| = 1/2$, we get that there exists n_0 such that $|h_n(\lambda)| \leq 1$ for $|\lambda| = 1/2$ and $n \geq n_0$. Then the maximum principle shows that this still holds for $|\lambda| \leq 1/2$, and Montel's theorem shows that $\{h_n\}$ is a normal family in the unit disk, since it is uniformly bounded. This implies that h itself extends analytically to the unit disk with $h_n \to h$ locally uniformly in $\mathbb{D}$.

Calculating a few approximations, we get

$$\begin{aligned} h_0(\lambda) &= \frac{1}{2} \\ h_1(\lambda) &= \frac{1}{4} \\ h_2(\lambda) &= \frac{1}{4}\left(1 - \frac{\lambda}{4}\right) = \frac{1}{4} - \frac{\lambda}{16} \\ h_3(\lambda) &= \left(\frac{1}{4} - \frac{\lambda}{16}\right)\left(1 - \lambda^2\left(\frac{1}{4} - \frac{\lambda}{16}\right)\right) = \frac{1}{4} - \frac{\lambda}{16} - \frac{\lambda^2}{16} + \frac{\lambda^3}{32} - \frac{\lambda^4}{256} \end{aligned}$$

and using a computer algebra system we easily get some more coefficients:

$$h(\lambda) = \frac{1}{4} - \frac{\lambda}{16} - \frac{\lambda^2}{16} - \frac{\lambda^3}{32} - \frac{9\lambda^4}{256} - \frac{\lambda^5}{256} - \frac{7\lambda^6}{256} + \frac{3\lambda^7}{512} - \frac{29\lambda^8}{2048} - \frac{\lambda^9}{512} + O(\lambda^{10})$$

Now h is a bounded non-constant analytic function in the unit disk, so by the theorems of Fatou and F. and M. Riesz the radial limit $h^*(\lambda) = \lim_{t\to 1} h(t\lambda)$ exists and is non-zero for almost every $\lambda \in \mathbb{T}$. Let $\lambda = e^{2\pi i\alpha}$ be such a point and let $r^*(\lambda) = |h^*(\lambda)|$. Then for every $r \in (0, r^*(\lambda))$ there exists $\delta \in (0, 1/2)$ such that $r(t\lambda) > r$ for all $t \in (1 - \delta, 1)$. Then the conformal maps $\psi_{t\lambda}$ map $\mathbb{D}_r$ into $\mathbb{D}_4$, so they form a uniformly bounded, hence normal, family. Let $t_k \to 1$ be a sequence such that $\psi_{t_k\lambda} \to \psi_\lambda$ locally uniformly. Then $\psi_\lambda(w) = w + \dots$

is analytic for $|w| < r$, and as a non-constant locally uniform limit of conformal maps it is conformal by Hurwitz's theorem. Passing to the limit in the functional equation $\psi_{t_k\lambda}(t_k\lambda w) = f_{t_k\lambda}(\psi_{t_k\lambda}(w))$ we get $\psi_\lambda(\lambda w) = f_\lambda(\psi_\lambda(w))$ for $|w| < r$. (The convergence is uniform in every strictly smaller disk, so we can pass to the limit even though the argument also depends on k.) This shows that ψ_λ conjugates the rotation $w \mapsto \lambda w$ to f_λ in a neighborhood of zero, so the inverse $\phi_\lambda = \psi_\lambda^{-1}$ is a linearizing map for f_λ. This shows that f_λ has a Siegel disk centered at 0. □

Remark. If we have a simply connected domain $U \subsetneq \mathbb{C}$ and a marked "center" $z_0 \in U$, then by the Riemann Mapping Theorem there is a unique conformal map $\psi : \mathbb{D} \to U$ with $\psi(0) = z_0$ and $R = \psi'(0) > 0$. The number R is the *conformal radius* of (U, z_0), and it measures the "size" of U with respect to z_0 in a conformally invariant way. The proof above shows that the conformal radius of the Siegel disk is bounded below by $r^*(\lambda) = \lim_{t\to 1} r(t\lambda)$. In fact, Yoccoz showed more: The radial limit r^* of r exists at every point on the unit circle, and it equals the conformal radius of the Siegel disk of f_λ.

13.4. **Results and open question about rotation domains.** We are not going to go into depth on the various techniques and results about Siegel disks and Arnold-Herman rings, but here are a few highlights of results and open questions.

In order to state the results, we need a little bit of number theory. Every irrational number α has a unique continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. The truncated continued fractions give a sequence of rational approximations as

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}}}$$

where $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$ are coprime. These can also be defined as the sequence of best rational approximations of α, i.e., $|\alpha - p_n/q_n| \leq |\alpha - p/q|$ for all rational approximations p/q with $0 < q < q_{n+1}$. The sequences $\{p_n\}$ and $\{q_n\}$ satisfy the recursion formulas

$$p_{n+1} = a_n p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1}$$

with $p_{-1} = 1$, $q_{-1} = 0$, $p_{-2} = 0$, $q_{-2} = 1$. They satisfy

$$\frac{p_{2k}}{q_{2k}} < \alpha < \frac{p_{2k+1}}{q_{2k+1}}$$

and

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n+1}}.$$

We also have the estimates

$$\frac{1}{2q_n q_{n+1}} < \frac{1}{q_n(q_{n+1} + q_n)} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

(This in particular implies that $|\alpha - p_n/q_n| < q_n^{-2}$.) For the golden mean $\alpha = \frac{1+\sqrt{5}}{2}$ we have that $a_n = 1$ for all n and that the denominators q_n are given by the Fibonacci sequence (as are the numerators p_n, only index-shifted by 1.) For a general irrational number α, the denominators q_n grow at least as fast as the Fibonacci sequence, so they grow at least geometrically.

A number is *Diophantine with exponent* τ iff there exists $c > 0$ such that $|\alpha - p/q| \geq cq^{-\tau}$ for all rational approximations p/q, and we denote this subset of the irrational numbers as D_τ. Equivalently, a number α belongs to D_τ iff its continued fraction denominators satisfy $q_{n+1} \leq cq_n^{\tau-1}$ for some constant $c > 0$, or if $a_{n+1} \leq cq_n^{\tau-2}$. In particular, $D_\tau = \emptyset$ for $\tau < 2$, and D_2 contains exactly those irrationals which have bounded continued fraction coefficients. All D_τ are uncountable for $\tau \geq 2$, and while D_2 has measure zero, all D_τ for $\tau > 2$ have full Lebesgue measure. Quadratic irrationals are exactly those numbers which have eventually periodic continued fraction expansions, thus belonging to D_2, and all algebraic irrationals are in D_τ for every $\tau > 2$ by the Siegel-Thue-Roth theorem. The set of *Diophantine numbers* is $D_\infty = \bigcup_{\tau \geq 2} D_\tau$.

Theorem 13.10 (Siegel)**.** *If $\alpha \in D_\infty$ and $\lambda = e^{2\pi i\alpha}$, then every analytic function $f(z) = \lambda z + O(z^2)$ is linearizable at* 0*.*

In particular, this implies linearizability for almost all α, and for all algebraic irrationals α.

This theorem was later improved by Brjuno and Rüssmann in the 1970's. In order to state this sharper version, we define the set of *Brjuno numbers* $\mathcal{B}$ as the set of all irrationals α such that $\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty$, where q_n are the denominators of the convergents p_n/q_n of α, as before. It is not hard to show that $D_\infty \subsetneq \mathcal{B}$, so that the following theorem is strictly stronger than Siegel's theorem.

Theorem 13.11 (Brjuno, Rüssmann)**.** *If $\alpha \in \mathcal{B}$ and $\lambda = e^{2\pi i\alpha}$, then every analytic function $f(z) = \lambda z + O(z^2)$ is linearizable at* 0*.*

Brjuno proved this theorem by an ingenious improvement of Siegel's method of estimating the growth of the coefficients of the formal linearizing power series, whereas Rüsmann used KAM techniques, employing a version of Newton's iteration in function spaces, to show that the linearizing map exists. Both (lengthy and technical) proofs can be found in Steinmetz's textbook [Ste93].

In 1987, Yoccoz gave a new proof of this theorem, using conformal renormalization techniques. His method also enabled him to show that the condition $\mathcal{B}$ was best possible, i.e., that for every $\alpha \notin \mathcal{B}$, $\lambda = e^{2\pi i\alpha}$ there is a non-linearizable function $f(z) = \lambda z + O(z^2)$. Furthermore, using quasiconformal techniques, he was able to show that linearizability of the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$ implies linearizability of every analytic function $f(z) = \lambda z + O(z^2)$ with the same linear part. Combining these two results, we get the following theorem.

Theorem 13.12 (Yoccoz)**.** *If $\alpha \in \mathbb{R} \setminus \mathcal{B}$ and $\lambda = e^{2\pi i\alpha}$, then the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$ is not linearizable.*

Combining the results of Brjuno, Rüssmann, and Yoccoz, we see that the general question of local linearizability is almost completely solved, in that the precise sufficient number-theoretic condition for linearizability of analytic functions $f(z) = e^{2\pi i\alpha}z + O(z^2)$ is known to be $\mathcal{B}$. Obviously, there are plenty of examples of linearizable local functions $f(z) = e^{2\pi i\alpha}z + O(z^2)$ with $\alpha \notin \mathcal{B}$, since we can just define $f(z) = \phi^{-1}(\lambda\phi(z))$ for any analytic function $\phi(z) = z + O(z^2)$. Whether such a function f can possibly extend to a polynomial or rational function of degree ≥ 2 is a famous open problem with several partial results. Please come and talk to me if you want to know more about this.

14. Critical points, rotation domains, and Cremer points

Obviously, rotation domains can not contain critical points, and for Cremer points there is not even an associated Fatou domain, so the relationship between critical points and these objects has to be different than for attracting and parabolic basins. It turns out that it is a little more complicated, but that it is almost possible in practice to use critical orbits to locate rotation domains and Cremer points as well. In order to explain the connection, we first have to have a closer look at critical orbits and limit functions of inverse iterates.

14.1. **Post-critical set and limit functions of inverse iterates.** The *post-critical set* of a rational map f is defined as

$$P(f) = \overline{\{f^n(c) \colon c \text{ critical point}, n \geq 1\}},$$

i.e., the topological closure of the forward orbits of all critical points of f. The set $P(f)$ appears in many contexts in complex dynamics. Most prominently, there is a vast theory of *post-critically finite* maps f, i.e., those rational maps for which $P(f)$ is a finite set. Equivalently, these are maps for which all critical points are either periodic or preperiodic points. This theory was started by Thurston, who studied topologically rational maps (branched coverings of the sphere to itself) which are post-critically finite (now called *Thurston maps*), and determined under which conditions these maps can be realized by an actual rational maps. The fundamental definitions and a proof of Thurston's result can be found in [DH93].

The post-critical set is closed and forward-invariant (in the sense that $f(P(f)) \subseteq P(f)$), so its complement $V = \hat{\mathbb{C}} \setminus P(f)$ is open and backward-invariant, in the sense that $U = f^{-1}(V) \subseteq V$. If $P(f)$ has more than two points (which is always true except for maps conjugate to $f(z) = z^{\pm d}$), then every component of U and V carries a hyperbolic metric, and $f : U \to V$ is a covering map, so it is a local hyperbolic isometry. The inclusion $\iota : U \to V$, $\iota(z) = z$ is analytic, so it is a weak contraction with respect to the hyperbolic metrics by the Schwarz lemma. This implies that f is weakly expanding at z with respect to the hyperbolic metric of V, as long as both $z \in V$ and $f(z) \in V$. In many cases, one can show that this weak expansion is actually a strict expansion, and this makes the post-critical set so very useful in studying the dynamics of f. We will get to some applications of these techniques later.

Inverse iterates are local inverses of iterates, i.e., analytic branches g_n of f^{-n} in some domain U. Note that for any given domain U and integer $n \geq 1$, there might not exist any such branch at all. Note that if such a branch g_n exists, then $f^{n-k} \circ g$ is an analytic branch of f^{-k} for all $0 \leq k \leq n$. If there is an infinite sequence of inverse branches, we have the following result.

Theorem 14.1. *Let f be a rational map of degree $d \geq 2$, let $U \subseteq \hat{\mathbb{C}}$ be a domain, and let g_n be an analytic branch of f^{-n} in U. Then $\{g_n\}$ is a normal family. Furthermore, if $U \cap \mathcal{J}(f) \neq \emptyset$, then any limit function $g = \lim_{k\to\infty} g_{n_k}$ is constant.*

Proof. Let $V_n = g_n(U)$. Then f^n maps V_n univalently onto U, so V_n does not contain any critical point of f^n. Let C denote the set of critical points of f, and let C_n be the critical points of f^n. Since $(f^n)'(z) = \prod_{k=0}^{n-1} f'(f^{k-1}(z))$, we get that $C_n = \bigcup_{k=0}^{n-1} f^{-k}(C)$. This is an increasing sequence of sets whose union $C_\infty = \bigcup_{n=1}^{\infty}$ consists of the backward orbit of the critical points (including the critical points themselves.) A rational map of degree $d \geq 2$

always has at least two critical points, since otherwise it would need one critical point of multiplicity $2d-2$, where the local degree would be $2d-1>d$, contradicting the fact that the local degree is always bounded by the global degree d.

If C_∞ is a finite set, then every critical point would have to be exceptional, so we have the situation where f has two exceptional points and it is conjugate to $F(z)=z^{\pm d}$ by Theorem 7.17. In this case, the assertion of the theorem is easy to check directly, since inverse branches are branches of $G_n(z)=z^{\pm 1/d^n}$. Any infinite sequences of branches of these roots is normal, and limit functions are constants on the unit circle (which is the Julia set of F.)

If C_∞ is an infinite set, then C_2 contains at least three points, and so every g_n for $n\geq 2$ omits C_2, hence $\{g_n\}$ is a normal family by Montel's theorem.

In order to show the second claim, assume that $z_0\in U\cap\mathcal{J}(f)$, and that $g_{n_k}\to g$ in U with g non-constant. Let $V=g(U)$ and $w_0=g(z_0)$. Then V is a neighborhood of w_0, and by locally uniform convergence and Hurwitz's theorem there exists a neighborhood W of w_0 and $k_0\in\mathbb{N}$ such that $g^{n_k}(U)\supseteq W$ for $k\geq k_0$. This implies that f^{n_k} is univalent on W for $k\geq k_0$, and since $n_k\to\infty$, we conclude that $f^n(W)\cap C=\emptyset$ for all n, so $f^n(W)\cap C_\infty=\emptyset$. Again we distinguish the case where C_∞ contains only two points and the claim of the theorem can be checked by hand, and the other case where C_∞ is an infinite set, and Montel's theorem gives us that $\{f^n\}$ is normal in W, so $W\subseteq\mathcal{F}(f)$. However, $w_0=\lim_{k\to\infty}g_{n_k}(z_0)$, and since each $g_{n_k}(z_0)\in\mathcal{J}(f)$ (by invariance of $\mathcal{J}(f)$), and the Julia set is closed, we get that $w_0\in\mathcal{J}(f)$, contradicting $w_0\in W\subset\mathcal{F}(f)$. □

Remark. Non-constant limit functions of inverse iterates can obviously occur as limits of inverses in rotation domains. It is also true that this is the only possibility of non-constant limits of inverse iterates, although the proof above does not show this.

14.2. **Relation between post-critical set, rotation domains, and Cremer points.**

Theorem 14.2. *Let f be a rational map of degree $d\geq 2$ with a Cremer point z_0, i.e., an irrationally indifferent fixed point $z_0\in\mathcal{J}(f)$. Then z_0 is a non-isolated point in the post-critical set $P(f)$. In particular, there exists a critical point c such that z_0 is contained in the closure of the forward orbit $\overline{\{f^n(c)\colon n\geq 1\}}$.*

Proof. Let $\lambda=f'(z_0)$ be the multiplier of z_0. Assume the assertion in the theorem is false. Then there exists $r>0$ such that $P(f)\cap D_r(z_0)\subseteq\{z_0\}$. For every $n\geq 1$, there exists a local inverse g_n of f^n with $g_n(z_0)=z_0$ and $g_n'(z_0)=\lambda^{-n}$. This local inverse can be analytically continued along any path in $D_r(z_0)\setminus\{z_0\}$, so it extends analytically to $D_r(z_0)$ by the Monodromy Theorem. By Theorem 14.1, $\{g_n\}$ forms a normal family, and every limit function $g=\lim_{k\to\infty}g_{n_k}$ is constant. However, $g(z_0)=\lim_{k\to\infty}g_{n_k}(z_0)=z_0$, and $|g'(z_0)|=\lim_{k\to\infty}|g_{n_k}'(z_0)|=1$, so g is not constant, which gives the desired contradiction. The second claim in the theorem follows easily from this and the fact that there are only finitely many critical points. □

Theorem 14.3. *Let f be a rational map of degree $d\geq 2$ with a Siegel disk or Arnold-Herman ring U. Then $\partial U\subseteq P(f)$.*

Proof. The proof here is a slight modification of the previous one, just a little more technical. Assume that the assertion is false. Then there exists $z_0\in\partial U\setminus P(f)$, so there exists a disk $D=D_r(z_0)$ with $D\cap P(f)=\emptyset$. Then $D\cap U\neq\emptyset$, so there exists a disk $D_1\subset D\cap U$. We know that each f^n maps U univalently onto itself, so there exist branches g_n of f^{-n} mapping D_1 into U. By the same argument as in the previous proof, this branch extends to an analytic function in the disk D, and these $\{g_n\}$ form a normal family. Since $z_0 \in D \cap \mathcal{J}(f)$, every limit function has to be constant by Theorem 14.1. However, we know that any limit function of the sequence $\{g_n\}$ is non-constant on D_1 (since it is the restriction of a map conjugate to a rotation), giving us our desired contradiction. □

15. Hyperbolic Julia sets

Another very nice application of the hyperbolic metric is the characterization of *hyperbolic* Julia sets. There are various equivalent definitions of hyperbolicity in complex dynamics, here is one of them.

Definition 15.1. A rational map f is *hyperbolic* if the post-critical set is contained in the Fatou set, i.e., if $P(f) \subseteq \mathcal{F}(f)$.

Here is an equivalent characterization.

Theorem 15.2. *A rational map is hyperbolic iff every critical point is contained in an attracting or super-attracting basin. Hyperbolic rational maps do not have parabolic points, rotation domains, Cremer points, or wandering domains.*

Proof. If every critical point is in an attracting or super-attracting basin, then all critical orbits are contained in the Fatou set, and the only possible accumulation points are attracting or super-attracting periodic orbits, also contained in the Fatou set, so $P(f) \subseteq \mathcal{F}(f)$. Conversely, if $P(f) \subseteq \mathcal{F}(f)$, and if c is any critical point, then c can not be contained in a parabolic basin, because in that case, the forward orbit of c would accumulate on a parabolic cycle in the Julia set, contradicting the assumption. Any rotation domain would have to have a boundary contained in the intersection $\mathcal{J}(f) \cap P(f)$, which would contradict $P(f) \subseteq \mathcal{F}(f)$, so they do not exist for hyperbolic rational maps. Similarly, Cremer points would have to be both in the Julia set and the post-critical set, so they do not occur for hyperbolic maps. Wandering domains do not exist for any rational map by Sullivan's theorem, but it is quite easy to show that limit functions in wandering domains would have to be constant functions in the Julia set, so even without invoking Sullivan's theorem it is easy to see that they can not occur for hyperbolic maps. In the end, this shows that the whole Fatou set contains only attracting and super-attracting basins, so all critical points have to be contained in those. □

The most important property of hyperbolic rational maps is that they are expanding on their Julia sets, in the following sense.

Theorem 15.3. *Let f be a rational map with $\infty \in \mathcal{F}(f)$. Then f is hyperbolic iff there exist constants $c > 0$ and $\lambda > 1$ such that $|(f^n)'(z)| \geq c\lambda^n$ for all $z \in \mathcal{J}(f)$ and $n \geq 1$.*

Remark. Note that $\mathcal{F}(f) \neq \emptyset$ for hyperbolic f, since all critical points of f are in the Fatou set, so by conjugation we can always achieve $\infty \in \mathcal{F}(f)$. If $\mathcal{J}(f) = \hat{\mathbb{C}}$, and if $c \in \mathbb{C}$ is any critical point whose forward orbit does not contain ∞, then $|(f^n)'(c)| = 0$ for all $n \geq 1$, so a map whose Julia set is the whole sphere can not be expanding in the sense of the theorem.

Note also that a suitably modified version of the theorem is true in the case where $\infty \in \mathcal{J}(f)$, using the derivative with respect to the spherical metric instead of the Euclidean metric.

Proof. Assuming that f is hyperbolic with post-critical set $P(f)$, Theorem 15.2 shows that $P(f)$ is a closed countable set, so the complement $U = \hat{\mathbb{C}} \setminus P(f)$ is open, non-empty, and connected, i.e., a domain. It is an easy exercise to show that $P(f)$ contains at least three points unless f is conjugate to $F(z) = z^{\pm d}$. The assertion of the theorem is easy to check for F, and it is invariant under conjugation (with possibly different constants c and λ). Otherwise, U is a hyperbolic domain, and there is at least one critical point whose backward orbit is infinite, so by the Schwarz lemma for hyperbolic metrics, the map $f : f^{-1}(U) \to U$ is a strict expansion with respect to the hyperbolic metric of U. Since $\mathcal{J}(f) \subset U$ is a compact invariant set, we get strict expansion with a uniform constant with respect to the hyperbolic metric. Again, since it is a compact subset of U, hyperbolic and Euclidean metrics are comparable, which implies the claim. Details are left to the reader for now.

Conversely, if there exist $c > 0$ and $\lambda > 1$ such that $|(f^n)'(z)| \geq c\lambda^n$ for all $z \in \mathcal{J}(f)$, then $\mathcal{J}(f)$ can not contain either critical points or parabolic points. Critical points in the Fatou set can not have accumulation points of their orbits in the Julia set unless they are contained in parabolic domains, so f can not have rotation domains. (Here we are invoking Sullivan's theorem that f can not have wandering domains. There should be an easy argument that wandering domains can not exist for maps with this expanding property, but right now I can not think of one.) This shows that every critical point is contained in an attracting or super-attracting basin, so f is hyperbolic. □

16. Hyperbolicity and quadratic polynomials

Returning once again to the class of quadratic polynomials $f_c(z) = z^2 + c$, with Julia set J_c, filled-in Julia set K_c, basin of infinity $A_c(\infty)$, post-critical set $P_c = \overline{\{f^n(0) \colon n \geq 1\}}$, and Mandelbrot set $M = \{c \in \mathbb{C} \colon 0 \in K_c\}$, we can finally motivate the name "hyperbolicity conjecture" in Conjecture 4.3. Since 0 is the only critical point of f_c in the complex plane, the quadratic polynomial f_c is hyperbolic iff $0 \in A_f(\infty)$ or there exists some attracting or super-attracting cycle in $\mathbb{C}$. For $c \in \partial M$, we have that 0 has a bounded orbit under f_c, and that f_c has no attracting or super-attracting cycle in $\mathbb{C}$ (since this would still be true in a neighborhood of c, so it would imply that c is in the interior of M), which shows that f_c is not hyperbolic for $c \in \partial M$. We saw that f_c is hyperbolic for $c \notin M$, so the hyperbolicity conjecture is that f_c is always hyperbolic for c in the interior of the Mandelbrot set. In particular, this would imply that the set of hyperbolic parameters is dense, i.e., that every f_c can be approximated by hyperbolic polynomials. (It is still an open question whether ∂M has zero or positive Lebesgue measure, so this would not quite show that almost every f_c is hyperbolic.) The same question can be asked for polynomials or rational functions of any degree.

Conjecture 16.1 (General Hyperbolicity Conjecture)**.** *Let $\mathcal{P}_d$ and $\mathcal{R}_d$ be the space of polynomials and rational maps of degree $d \geq 2$, respectively. Let $\mathcal{P}_d^h$ and $\mathcal{R}_d^h$ be the subsets of hyperbolic polynomials and rational maps of degree d, respectively. Then $\mathcal{P}_d^h$ is dense in $\mathcal{P}_d$, and $\mathcal{R}_d^h$ is dense in $\mathcal{R}_d$.*

Here the topology on the space of polynomials and rational maps is the one induced by the coefficients, or equivalently the topology of uniform convergence with respect to the spherical metric. While this conjecture is still open for any $d \geq 2$, it should be noted that Rees has shown in [Ree86] that the set of non-hyperbolic rational maps always has positive Lebesgue measure in parameter space, for any $d \geq 2$. (There is no canonical Lebesgue measure on parameter space, but there is a canonical Lebesgue measure equivalence class of mutually absolutely continuous measures, so that the notions of positive and zero measure make sense. It really means that if one picks $2d+2$ random coefficients for a rational map f according to some absolutely continuous probability measure supported on $\mathbb{C}^{2d+2}$, then there is a positive probability that f is non-hyperbolic. In fact, Rees showed that with positive probability $\mathcal{J}(f) = \hat{\mathbb{C}}$.)

Hyperbolic maps have particularly nice dynamics and Julia sets, and we will illustrate this with a couple of results on quadratic polynomials. A very powerful tool in general dynamical systems is to employ a "coding" of points by a sequence of symbols to turn them into "symbolic dynamical systems". Here is one of the basic definitions from the field of symbolic dynamics. Various applications and background can be found in most textbooks on dynamical systems.

Definition 16.2. Let A be a finite set with $n \geq 2$ elements. The *one-sided shift space over A* is $\Sigma = A^{\mathbb{N}} = \{(a_1, a_2, a_3, \ldots) \colon a_n \in A\}$, equipped with the product topology generated by the discrete topology on A. The *one-sided shift on A* is $\sigma : \Sigma \to \Sigma$ defined by $\sigma((a_1, a_2, a_3, \ldots)) = (a_2, a_3, a_4, \ldots)$.

The set A is often called the *alphabet* or the space of *symbols*, and we will usually choose $A = \{1, 2, \ldots, n\}$, and call the resulting map σ the *one-sided shift on n symbols.* The set Σ is a Cantor set, and while there is no canonical metric on Σ, one simple choice of metric generating the topology of Σ is $d((a_1, a_2, a_3, \ldots), (b_1, b_2, b_3, \ldots)) = 2^{-n_0}$ where $n_0 = \inf\{n \in \mathbb{N} \colon a_n \neq b_n\}$.

A good model to keep in mind for the one-sided shift on two symbols is the map $f(x) = 3x \pmod 1$ on the standard Cantor set $X = \{\sum_{n=1}^{\infty} a_n 3^{-n} \colon a_n \in \{0, 2\}\}$ represented as the numbers whose ternary expansion has only digits 0 and 2. Writing $x = 0.a_1 a_2 a_3 \ldots$ out in ternary, we get that $f(x) = 0.a_2 a_3 a_4 \ldots$, so the digits are shifted to the left. The metric given above is not the same as the one induced by the real line, but these two metrics are Hölder equivalent and induce the same topology.

Theorem 16.3. *If $c \notin M$, then $J_c = K_c$ is a Cantor set. Furthermore, the dynamics of f_c on J_c are topologically conjugate to the one-sided shift on two symbols.*

Proof. TO BE ADDED. □

In fact, a similar theorem is true for any polynomial f of degree $d \geq 2$ for which all critical points are in the basin of ∞. It is not too hard to modify the proof to see that $J_f = K_f$ is a Cantor set and that f on J_f is topologically conjugate to a one-sided "subshift of finite type" (a shift on a space where only certain transitions $a_n a_{n+1}$ are allowed) over some alphabet A. It is also true, but a little harder to show that f is in fact conjugate to the one-sided shift over d symbols.

If f_c is a hyperbolic polynomial with $c \in M$, then we also get a very nice result.

Theorem 16.4. *If $c \in M$ is a hyperbolic parameter, then J_c and K_c are locally connected. Furthermore, there is a continuous map ψ from the unit circle $\mathbb{T}$ onto J_c such that $f(\psi(w)) = \psi(w^2)$ for $w \in \mathbb{T}$.*

Remark. The map ψ will in general not be invertible, so it is not always a conjugacy. A continuous map like this is called a *semi-conjugacy*, and the semi-conjugate system (in this case f on J_c) is then called a *factor* of the original dynamical system (in this case $F(w) = w^2$.) The map $F(w) = w^2$ is itself a factor of the one-sided shift over two symbols, via the semi-conjugacy $(a_1, a_2, a_3, \ldots) \mapsto \exp\left(2\pi i \sum_{n=1}^{\infty} 2^{-n} a_n\right)$. The idea here is that the factor f on J_c is the quotient of $F : \mathbb{T} \to \mathbb{T}$ by the equivalence relation $w_1 \sim w_2$ iff $\psi(w_1) = \psi(w_2)$. So if we understand the dynamics of F and this equivalence relation, we understand f on J_c, at least topologically. For a more thorough understanding of symbolic dynamics, factors, etc., any advanced textbook or class on dynamical systems will help.

Proof. TO BE ADDED. □

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