Fuchsian Differential Equations

Mario Bonk (UCLA)

Summer Workshop
Zoom Universe, August 10, 2020

Holomorphic function of several complex variables

Let $f\colon \Omega \subseteq \mathbb{C}^2 \to \mathbb{C}$ be a function defined on a region $\Omega \subseteq \mathbb{C}^2$.
Then f is called *holomorphic* if for each $(z_0, w_0) \in \Omega$ the function f has a local power series representation

$$f(z,w) = \sum_{k,n=0}^{\infty} a_{kn}(z - z_0)^k (w - w_0)^n$$

for (z,w) near (z_0, w_0).
This is equivalent to: f holomorphic in each variable with the other variable fixed.
Then partial derivatives exist of all orders and are holomorphic.
All this is also true for holomorphic functions of more than two complex variables.
Reference: Steven Krantz, "Function Theory of Several Complex Variables".

Holomorphic ODEs

Basic reference: Ludwig Bieberbach, "Theorie der gewöhnlichen Differentialgleichungen".

Theorem (Basic existence and uniqueness)

Let $\Omega \subseteq \mathbb{C}^2$ be a region, $f \colon \Omega \to \mathbb{C}$ be holomorphic, and $(z_0, w_0) \in \Omega$. Then there exists a unique holomorphic function $w(z)$ that solves the initial value problem

$$w(z_0) = w_0, \quad w'(z) = f(z, w(z))$$

for z near z_0.

Outline of proof: uniqueness

Uniqueness: The ODE and the initial value determine the derivates of $w(z)$ of all orders at z_0 recursively:

$$w(z_0) = w_0, \quad w'(z_0) = f(z_0, w_0).$$

Now differentiate ODE repeatedly and plug in z_0:

$$w''(z) = f_z(z, w(z)) + f_w(z, w(z))w'(z) \quad \text{(Chain rule!)}$$

$$w''(z_0) = f_z(z_0, w_0) + f_w(z_0, w_0)w'(z_0),$$

etc. So Taylor series

$$w(z) = \sum_{n=0}^{\infty} \frac{w^{(n)}(z_0)}{n!}(z - z_0)^n$$

uniquely determined.

Outline of proof: Existence I

Method I: Use power series "ansatz" $w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, plug into ODE, where $f(z, w)$ is expanded in a power series at (z_0, w_0) and show that one can solve recursively for coefficients $a_0 = w_0$, a_1, a_2, etc.
Proof of convergence of series not hard, but somewhat painful.

Method II: Picard-Lindelöf iteration scheme.
Assume $K := \overline{B}(z_0, R_z) \times \overline{B}(w_0, R_w) \subseteq \Omega$, and $|f|, |f_w| \leq M$ on K.
Want to solve

$$w(z) = w_0 + \int_{z_0}^{z} f(u, w(u))\, du$$

for $z \in D := B(z_0, R)$ with suitable $R > 0$.

Outline of proof: existence II

Use recursion scheme:

$$w_0(z) :\equiv w_0$$

$$w_{n+1}(z) := w_0 + \int_{z_0}^{z} f(u, w_n(u))\, du.$$

Need: $|w_n(z) - w_0| \leq R_w$ for $z \in D = B(z_0, R)$. Proved inductively:

$$|w_{n+1}(z) - w_0| \leq RM \leq R_w$$

for $z \in D$ if $R := \min\{R_z, R_w/M\}$.

NOTE: this second restriction is NOT necessary when $f(z, w)$ defined for all w independent of z, i.e. when $\Omega = U \times \mathbb{C}$ is a cylinder domain.

Outline of proof: existence III

One uses the bound $|f_w| \leq M$ (which implies that f is M-Lipschitz in w with z fixed) to show inductively that

$$|w_n(z) - w_{n-1}(z)| \leq \frac{M^n |z - z_0|^n}{n!} \leq \frac{M^n R^n}{n!}$$

for $z \in D = B(z_0, R)$.
Hence

$$w(z) := \lim_{n \to \infty} w_n(z) = w_0(z) + \sum_{n=1}^{\infty} (w_n(z) - w_{n-1}(z))$$

converges uniformly on D. So w is holomorphic on D, and solves integral equation (pass to limits under integral sign!)) and hence the initial value problem.
NOTE: If Ω is a cylinder domain and f_w is independent of w (so that we have a uniform bound $|f_w| \leq M$), then we get solutions defined on a fixed disk $D = B(z_0, R)$ independent of the initial value w_0.

ODE systems

Theorem (Existence and uniqueness for ODE systems)

Let $\Omega \subseteq \mathbb{C} \times \mathbb{C}^n$ be a region, $f : \Omega \to \mathbb{C}^n$ be holomorphic, and $(z_0, \mathbf{w}_0) \in \Omega$ (here $\mathbf{w}_0 \in \mathbb{C}^n$). Then there exists a unique $\mathbb{C}^n$-valued holomorphic function $\mathbf{w}(z)$ defined near z_0 that solves the initial value problem

$$\mathbf{w}(z_0) = \mathbf{w}_0, \quad \mathbf{w}'(z) = f(z, \mathbf{w}(z))$$

for z near z_0.

Note: if one uses component functions, then the ODE is

$$\begin{aligned} w_1'(z) &= f_1(z, w_1(z), \ldots, w_n(z)), \\ w_2'(z) &= f_2(z, w_1(z), \ldots, w_n(z)), \\ &\vdots \\ w_n'(z) &= f_n(z, w_1(z), \ldots, w_n(z)). \end{aligned}$$

Linear homogeneous ODE systems I

Theorem

Let $U \subseteq \mathbb{C}$ be a region, $A\colon U \to M_n(\mathbb{C})$ be a matrix-valued holomorphic function, $z_0 \in U$, and $\mathbf{w}_0 \in \mathbb{C}^n$. Then there exists a unique $\mathbb{C}^n$-valued holomorphic function $\mathbf{w}(z)$ that solves the initial value problem $\mathbf{w}(z_0) = \mathbf{w}_0$ and

$$\mathbf{w}'(z) = A(z)\mathbf{w}(z) \qquad (LS)$$

for z near z_0. Moreover, here $\mathbf{w}$ is defined on a fixed size disk $B(z_0, R)$ with radius $R = R(z_0) > 0$ independent of $\mathbf{w}_0$.

Note: 1) We think of $\mathbf{w}(z)$ as a column vector

$$\mathbf{w}(z) = \begin{pmatrix} w_1(z) \\ w_2(z) \\ \vdots \\ w_n(z) \end{pmatrix}$$

2) The relevant function $f(z, \mathbf{w}) = A(z)\mathbf{w}$ is defined for all $\mathbf{w}$ (so $\Omega = U \times \mathbb{C}^n$) and has partial w_k-derivatives that do not depend on $\mathbf{w}$. This justifies the existence of a uniform radius $R > 0$ independent of $\mathbf{w}_0$ (but dependent on z_0) such that a solution $\mathbf{w}(z)$ exists on $B(z_0, R)$.

3) Let γ be a path in U starting at z_0 and ending at a point $z_1 \in U$. Then if we start analytic continuation along γ, then $\mathbf{w}(z)$ always remains a solution of LS (follows from uniqueness theorem for holomorphic functions!). Moreover, since solutions exist on fixed size disks (one can actually take the radius uniform along γ), analytic continuation can never run into trouble.

Corollary

A solution $\mathbf{w}(z)$ of LS can analytically be continued along any path γ in U and remains a solution of LS.

1) The monodromy theorem implies: if U is simply connected, then $\mathbf{w}(z)$ is holomorphic on the whole region U.
2) The solution $\mathbf{w}(z)$ of the initial value problem is holomorphic at least on the largest disk with $B(z_0, R) \subseteq U$.

Solution space of the homogeneous linear system

Theorem

Let $U \subseteq \mathbb{C}$ be a region, $A\colon U \to M_n(\mathbb{C})$ be be a matrix-valued holomorphic function, $z_0 \in U$. The $\mathbb{C}^n$-valued holomorphic function $\mathbf{w}(z)$ defined near z_0 that solve

$$\mathbf{w}'(z) = A(z)\mathbf{w}(z) \qquad (LS)$$

form a vector space V of dimension n.

Proof: Standard! Linear combinations of solutions are solutions. So they form a vector space V.
Since a solution is uniquely determined by its initial values, the solutions $\mathbf{w}_1(z), \ldots, \mathbf{w}_n(z)$ with initial values $\mathbf{w}_1(z_0) = \mathbf{e}_1, \ldots,$ $\mathbf{w}_n(z_0) = \mathbf{e}_n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{C}^n$, form a basis of V. $\square$

Analytic continuation of a basis

Let $\mathbf{w}_1(z), \ldots, \mathbf{w}_n(z)$ be a basis of the solution space of LS near z_0 (called a *fundamental system*). We form the matrix $B(z)$ with these column vectors $B(z) = (\mathbf{w}_1(z), \ldots, \mathbf{w}_n(z))$ and set

$$W(z) = \det(B(z)) \quad \text{(Wronski determinant).}$$

Then $\mathbf{w}_1(z), \ldots, \mathbf{w}_n(z)$ form a basis of the solution space of LS at z iff $W(z) \neq 0$.

Theorem

If $W(z_0) \neq 0$ for some $z_0 \in U$, then $W(z) \neq 0$ under any analytic continuation along any path γ in U that starts at z_0 and ends at z. In particular, a basis $\mathbf{w}_1(z), \ldots, \mathbf{w}_n(z)$ of a solution space of LS near z_0 remains a basis at any point z under any analytic continuation.

Proof of theorem

Lemma

Let $M(z)$ be an invertible holomorphic matrix-valued function, and $D(z) = \det(M(z))$. Then

$$D'(z) = trace(M'(z)M^{-1}(z))D(z).$$

Proof: Standard and left as an exercise. First consider the case $D(z) = I_n$ at the point z in question. $\square$

We apply this to $W(z)$ and $B(z)$ (we can do this at least near z_0), because there $B(z)$ is invertible. Note $B'(z) = A(z)B(z)$ and so

$$W'(z) = \mathrm{trace}(B'(z)B^{-1}(z))W(z) = \mathrm{trace}(A(z))W(z).$$

Hence

$$W(z) = W(z_0)\exp\left(\int_{\gamma_{z_0,z}} \mathrm{trace}(A(u))\,du\right) \neq 0$$

along any path $\gamma_{z_0,z}$ in U from z_0 to $z \in U$. $\square$

As before: matrix $B(z)$ with columns given by basis of solution space of LS near z_0. Continue analytically along any loop γ in U starting and ending in z_0. This gives a matrix $B_\gamma(z)$ whose columns form again a basis of the solution space near z_0. Hence

$$B_\gamma(z) = B(z)S_\gamma$$

with $S_\gamma \in GL_n(\mathbb{C})$. Here S_γ is a constant matrix (follows from uniqueness theorem).
Note that S_γ only depends on the homotopy class of γ as follows from the monodromy theorem. This gives an induced map

$$[\gamma] \in \pi_1(z_0, U) \mapsto S_\gamma \in GL_n(\mathbb{C}).$$

This is a homomorphism: if $\tilde{\gamma}$ is another loop and we consider the concatenation $\tilde{\gamma} \cdot \gamma$ (first run through γ and then through $\tilde{\gamma}$):

$$B_{\tilde{\gamma}\cdot\gamma}(z) = (B_\gamma(z))_{\tilde{\gamma}} = (B(z)S_\gamma)_{\tilde{\gamma}} = B_{\tilde{\gamma}}(z)S_\gamma = B(z)S_{\tilde{\gamma}}S_\gamma.$$

Hence

$$S_{\tilde{\gamma}\cdot\gamma} = S_{\tilde{\gamma}}S_\gamma.$$

Let $\tilde{B}$ another matrix obtained from a basis of solution space of LS. Then there exists $M \in GL_n(\mathbb{C})$ such that $\tilde{B}(z) = B(z)M$. Analytic continuation along γ gives

$$(\tilde{B})_\gamma(z) = (B(z)M)_\gamma = B_\gamma(z)M = B(z)S_\gamma M = \tilde{B}(z)M^{-1}S_\gamma M.$$

Hence $\tilde{S}_\gamma = M^{-1}S_\gamma M$. We have proved :

Theorem (Monodromy representation)

Analytic continuation of a matrix $B(z)$ with columns given by a basis of the solution space of LS induces a natural representation

$$\varphi\colon \pi_1(z_0, U) \to GL_n(\mathbb{C}).$$

It is independent of the choice of $B(z)$ up to conjugation by an element $M \in GL_n(\mathbb{C})$.

Higher-order homogeneous linear ODEs I

Consider the n-th order ODE

$$w^{(n)}(z) + p_1(z)w^{(n-1)}(z) + \cdots + p_n(z)w(z) = 0,$$

where $p_1, \ldots, p_n$ are holomorphic functions on some region $U \subseteq \mathbb{C}$. One can reduce this to a linear system: consider

$$\mathbf{u}(z) = \begin{pmatrix} w^{(n-1)}(z) \\ w^{(n-2)}(z) \\ \vdots \\ w(z) \end{pmatrix}$$

and the LS

$$\mathbf{u}'(z) = A(z)\mathbf{u}(z),$$

where

Higher-order homogeneous linear ODEs II

$$A(z) = \begin{pmatrix} -p_1(z) & -p_2(z) & \dots & -p_{n-1}(z) & -p_n(z) \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

One has: existence and uniqueness for solution $w(z)$ with

$$w(z_0) = c_0,\ w'(z_0) = c_1,\ \dots,\ w^{(n-1)}(z_0) = c_{n-1},$$

where $c_0, \dots, c_{n-1} \in \mathbb{C}$ are given. So there exist precisely n linearly independent solutions (fundamental system). One has analytic continuation, monodromy representations, etc.

Consider 2nd order ODE

$$w'' + pw' + qw = 0,$$

where p and q are holomorphic on the punctured disk $U = \{0 < |z| < R\}$. Let $z_0 \in U$. Then $\pi_1(z_0, U) \cong \mathbb{Z}$. So the monodromy representation essentially tells us what happens if a loop γ based at z_0 runs around 0 in positive orientation. This gives us a matrix $S \in GL_2(\mathbb{C})$ ("Umlaufsubstitution"). By change of the basis of the solution space, we can also get any conjugate of S. So we may assume that S has Jordan normal form:

$$S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } S = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda_1, \lambda_2, \lambda \neq 0$.

In any case, there ex. a solution vector such that

$$\begin{pmatrix} w'(z) \\ w(z) \end{pmatrix} \mapsto \lambda \begin{pmatrix} w'(z) \\ w(z) \end{pmatrix}.$$

or equivalently, $w(z) \mapsto \lambda w(z)$ if we run through the loop γ. So there always exists a *multiplicative solution*. Let

$$z^\rho = \exp(\rho \log z).$$

Then if we run through the loop γ, this function changes as $z^\rho \mapsto \exp(2\pi i \rho) z^\rho$. So if we set

$$\rho = \frac{1}{2\pi i} \log(\lambda),$$

then we have the same change $z^\rho \mapsto \lambda z^\rho$.

So $z^{-\rho}w(z)$ does not change at all if we run though the loop γ. This means that it is uniquely defined holomorphic function on the punctured disk U, and so it has a Laurent series expansion.

Conclusion: We have

$$w(z) = z^{\rho} \sum_{n=-\infty}^{\infty} a_n z^n$$

for some convergent Laurent series on U.

Does this help? Can we turn this around, use this as an "ansatz" and insert back into the ODE?

2nd order ODE near an isolated singularity IV

The answer is no! The problem is that in general the Laurent series will have infinitely many coefficients with negative indices. We run into some terrible linear equations for the coefficients that cannot be solved recursively. This problem disappear if we can guarantee that the Laurent expansion has only finitely many negative terms. This is equivalent to the blow-up of the solution at a polynomial rate if we approach 0 radially.

Definition (Stelle der Bestimmtheit)

Consider a linear homogenous ODE and let z_0 be an isolated singularity of the coefficients of the ODE. Then z_0 is a *regular singularity* ("Stelle der Bestimmheit") if for every solution w of the ODE there exists an exponent $M > 0$ such that

$$w(z) = O(|z - z_0|^{-M})$$

as z approaches z_0 along any path with finite total angular variation.

Theorem of Fuchs

Theorem (Sufficient criterion)

Let $A\colon U \to M_n(\mathbb{C})$ be a matrix-valued holomorphic function defined on a punctured neighborhood $U = \{0 < |z| < R\}$ of the origin, and consider the homogenous linear system

$$\mathbf{w}'(z) = A(z)\mathbf{w}(z) \qquad (LS)$$

on U. If the coefficients of A have poles of order at most 1 at 0, then 0 is a regular singularity of the ODE.

Outline of proof

Let $\mathbf{w} \neq \mathbf{0}$ be a solution. Then $u = |\mathbf{w}| > 0$ on U, and so $v = \log u$ is well-defined and smooth on $U \subseteq \mathbb{C} \cong \mathbb{R}^2$. Hence

$$|\nabla v(z)| = \frac{|\nabla u(z)|}{u(z)} \lesssim \frac{|\mathbf{w}'(z)|}{u(z)} \lesssim \frac{|A(z)| \cdot |\mathbf{w}(z)|}{|\mathbf{w}(z)|} \leq \frac{M}{|z|}$$

with a uniform constant $M > 0$. We can assume that the path along which we approach 0 stays in a definite sector Σ based at 0. Then it follows that for $z \in \Sigma$ we have

$$|v(z)| \leq C_1 + M \log(1/|z|)$$

with a constant $M > 0$ independent of the path. Hence

$$|\mathbf{w}(z)| \lesssim |z|^{-M}$$

as $z \to 0$ along the path. $\square$

Theorem of Fuchs for higher order ODEs

Theorem

Consider the linear n-th order homogeneous linear ODE

$$w^{(n)}(z) + p_1(z)w^{(n-1)}(z) + \cdots + p_n(z)w(z) = 0,$$

where $p_1, \ldots, p_n$ *are holomorphic functions on the punctured disk* $U = \{0 < |z| < R\}$*. For* 0 *to be a regular singularity of the ODE it is necessary and sufficient that the singularity of* p_k *at* 0 *is at most a pole of order* k *for* $k = 1, \ldots, n$*.*

Proof

Necessity: we'll skip this.
Sufficiency: one can reduce this to a linear system. Consider

$$\mathbf{u}(z) = \begin{pmatrix} z^{n-1}w^{(n-1)}(z) \\ z^{n-2}w^{(n-2)}(z) \\ \dots \\ w(z) \end{pmatrix}$$

and the LS

$$\mathbf{u}'(z) = A(z)\mathbf{u}(z),$$

where $A(z) =$

$$\frac{1}{z}\begin{pmatrix} (n-1) - zp_1(z) & -z^2p_2(z) & \dots & -z^{n-1}p_{n-1}(z) & -z^np_n(z) \\ 1 & n-2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

This linear system is equivalent to the original ODE. The coefficients have poles of order at most 1 at 0.

Definition

A homogeneous linear ODE is called *Fuchsian* if all of its coefficients are rational functions and all of its (finitely many) singularities (including ∞) are regular singularities.

For ∞ one introduces a new variable $u = 1/z$ and considers the ODE near $u = 0$.

Regular singularity at 0

Consider the 2nd order ODE

$$w''(z) + \frac{1}{z}P(z)w'(z) + \frac{1}{z^2}Q(z)w(z) = 0$$

and assume that the ODE has a regular singularity at 0. Then P and Q are holomorphic at 0. We know that the ODE has at least one multiplicative solution of the form

$$w(z) = z^\rho \sum_{n=0}^{\infty} a_k z^k$$

with $a_0 \neq 0$. How to determine ρ?

Plug "ansatz" into ODE and compare "lowest order terms" (corresponding to $z^{\rho-2}$). This gives the *indicial equation*

$$\rho(\rho-1)+\rho P(0)+Q(0)=0.$$

This is a quadratic equation in ρ and it has two (possibly identical) solutions ρ_1 and ρ_2. Note

$$\rho_1+\rho_2=1-P(0).$$

Case 1: $\rho_1-\rho_2\notin\mathbb{Z}$ (generic case). Then the ODE has two linearly independent multiplicative solutions of the form

$$w_1(z)=z^{\rho_1}h_1(z),$$
$$w_2(z)=z^{\rho_2}h_2(z),$$

where h_1,h_2 are holomorphic near 0 with $h_1(0),h_2(0)\neq 0$.
Idea of proof: Plug "ansatz" into ODE, solve for coefficients recursively, and estimate to show convergence of power series h_k.
The mulipliers $\lambda_k=\exp(2\pi i\rho_k)$ are distinct, and so w_1 and w_2 are linearly independent anf form a fundamental system.

Case 2: $\rho_1 - \rho_2 \in \mathbb{Z}$ (degenerate case). This is much messier! Wlog we can label the roots such that $\rho_1 = \rho_2 + n$, $n \in \mathbb{N}_0$. One can show that the ODE two linearly independent solutions of the form

$$\begin{aligned} w_1(z) &= z^{\rho_1} h_1(z), \\ w_2(z) &= z^{\rho_2} h_2(z) + A \log z \cdot w_1(z), \end{aligned}$$

where h_1, h_2 are holomorphic near 0 with $h_1(0) \neq 0$, and $h_2(0) \neq 0$ or $A \neq 0$.
Idea of proof: One can show that there exists a multiplicative solution w_1 for ρ_1 (the "larger" root). To find w_2 one makes the "ansatz" $w_2 = v w_1$. Then

$$v''(z) + \left(\frac{2w_1'(z)}{w_1(z)} + \frac{P(z)}{z} \right) v'(z) = 0.$$

One can solve for v explicitly by integration. Here a logarithmic term may arise.

Indicial equation III

Actually:

$$
\begin{aligned}
\frac{d}{dz}\log v'(z) &= -\frac{2w_1'(z)}{w_1(z)} - \frac{P(z)}{z} \\
&= \frac{1}{z}(-2\rho_1 - P(0)) + \text{holomorphic} \\
&= \frac{1}{z}(\rho_2 - \rho_1 - 1) + \text{holomorphic.}
\end{aligned}
$$

We used $\rho_1 + \rho_2 = 1 - P(0)$. Integration gives

$$
v'(z) = z^{\rho_2 - \rho_1 - 1} \sum_{k=0}^{\infty} c_k z^k,
$$

where $c_0 \neq 0$. Since $\rho_2 - \rho_1 = -n$, we may get a logarithmic terms by integrating once more (if $c_n \neq 0$; for sure if $n = 0$ (double root)). If not, $w_2 = v w_1$ is a multiplicative solution $w_2(z) = z^{\rho_2} h_2(x)$.

Apparent singularities

An isolated singularity of a linear ODE is called an *apparent singularity* ("scheinbare Singularität" oder "Nebenpunkt") if all solutions are holomorphic near the singularity. Example:

$$w''(z) - \frac{2}{z} w'(z) + \frac{2}{z^2} w(z) = 0.$$

Indicial equation:

$$0 = \rho(\rho - 1) - 2\rho + 2 = \rho^2 - 3\rho + 2 = (\rho - 1)(\rho - 2).$$

So $\rho_1 = 2$ and $\rho_2 = 1$. Actually, $w_1(z) = z^2$ and $w_1(z) = z$ is a fundamental system.

Regular singularity at ∞

Consider the 2nd order ODE

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0$$

with p and q holomorphic in a punctored neighborhood of ∞. Introduce new variable $u = 1/z$. Then

$$w''(u) + \left(\frac{2}{u} - \frac{1}{u^2}p(1/u)\right)w'(u) + \frac{1}{u^4}q(1/u)w(u) = 0.$$

So $u_0 = 0$ is a regular singularity iff

$$p(1/u) = O(u) \text{ and } q(1/u) = O(u^2).$$

Equivalently, $z_0 = \infty$ is a regular singularity iff

$$p(z) = O(1/z) \text{ and } q(z) = O(1/z^2).$$

Theorem

The 2nd order Fuchsian ODEs are precisely those of the form

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0$$

with

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}, \quad q(z) = \sum_{k=1}^{n} \left(\frac{B_k}{(z - a_k)^2} + \frac{C_k}{z - a_k} \right),$$

$$\sum_{k=1}^{n} C_k = 0.$$

Indicial equations

At $z = a_k$ for $k = 1, \ldots, n$:

$$\rho(\rho - 1) + A_k \rho + B_k = 0.$$

At ∞ (for $u = 1/z$):

$$\rho(\rho - 1) + \left(2 - \sum_{k=1}^{n} A_k\right)\rho + \sum_{k=1}^{n}(B_k + a_k C_k) = 0.$$

Note that the sum of all roots (called *critical exponents*) is equal to

$$\sum_{k=1}^{n}(1 - A_k) + \sum_{k=1}^{n} A_k - 1 = n - 1.$$

Examples of 2nd order Fuchsian equations

1. No finite singularity: then $p(z) = q(z) \equiv 0$. So ODE $w''(z) = 0$, and $w(z) = c_1 z + c_2$ is general solution (boring!).
2. One finite singularity: wlog at $a = 0$. Then $p(z) = A/z$, $q(z) = B/z^2$. ODE:

$$w''(z) + \frac{A}{z} w'(z) + \frac{B}{z^2} w(z) = 0.$$

Change of variables: $z = e^u$. Then (for $w(u) := w(z(u))$)

$$w''(u) + a w'(u) + b w(u) = 0.$$

General solution:

$$w(u) = c_1 e^{\rho_1 u} + c_2 e^{\rho_2 u} \quad \text{or} \quad w(u) = c_1 e^{\rho u} + c_2 u e^{\rho u},$$

Equivalently:

$$w(z) = c_1 z^{\rho_1} + c_2 z^{\rho_2} \quad \text{or} \quad w(z) = c_1 z^{\rho} + c_2 \log z \cdot z^{\rho}.$$

Fuchsian equations with three singularities I

Wlog singularities at 0, 1, ∞ (Fuchsian equations invariant under changing z by Möbius transformation!). Then

$$w''(z)+w'(z)\left(\frac{A_0}{z}+\frac{A_1}{z-1}\right)+w(z)\left(\frac{B_0}{z^2}+\frac{B_1}{(z-1)^2}-\frac{C}{z}+\frac{C}{z-1}\right)=0.$$

Let α, α', β, β', γ, γ' be the critical exponents at $0, 1, \infty$. We know that

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$$

Subject to this condition one can prescribe the critical exponents arbitrarily: α, α' determine A_0, B_0, β, β' determine A_1, B_1, and C then is determined by $\gamma\gamma'$.

A solution of this ODE is denoted by

$$w(z) = P\begin{pmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{pmatrix}$$

By considering $v(z) = z^{-\alpha}(z-1)^{-\beta}w(z)$, one can always reduce to $\alpha = 0$ and $\beta = 0$. If one denotes the exponents at ∞ now by α and β and the other exponent at 0 by $1-\gamma$, then one is reduced to

$$w(z) = P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix}$$

For the corresponding equation one has $A_0 = \gamma$, $B_0 = 0$, $A_1 = 1+\alpha+\beta-\gamma$, $B_1 = 0$, $C = \alpha\beta$.

$$w''(z) + w'(z)\left(\frac{\gamma}{z} + \frac{1+\alpha+\beta-\gamma}{z-1}\right) + w(z)\left(-\frac{\alpha\beta}{z} + \frac{\alpha\beta}{z-1}\right) = 0$$

or equivalently

$$w''(z) + \left(\frac{(1+\alpha+\beta)z-\gamma}{z(z-1)}\right)w'(z) + \frac{\alpha\beta}{z(z-1)}w(z) = 0.$$

This is the famous *hypergeometric differential equation* already considered by Euler and Gauss. We have shown: every Fuchsian equation with three singularities can be reduced to the hypergeometric ODE.

Hypergeometric series

What does the general theory tell us:
The critical exponents at 0 are: 0 and $1-\gamma$. So unless $1-\gamma \in \mathbb{N}$, we expect a holomorphic solution at 0 that takes a non-zero value. Indeed, if $1-\gamma \notin \mathbb{N}$, or equivalently $\gamma \neq 0, -1, -2, \ldots$ such a solution is given by the *hypergeometric series*

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} z^n,$$

where $(a)_n := a(a+1)\cdots(a+n-1)$.

Schwarzian derivatives I

Let $w(z)$ be a holomorphic function. Then its *Schwarzian derivative* $\{w, z\}$ is defined as

$$\{w, z\} = \frac{d^2 \log w'(z)}{dz^2} - \frac{1}{2}\left(\frac{d \log w'(z)}{dz}\right)^2.$$

Facts:

1. If $w(z)$ is a Möbius transformation, then $\{w, z\} \equiv 0$.

2. For holomoprhic functions, we have the following chain rule for Schwarzian derivatives

$$\{\varphi \circ w, z\} = \{\varphi, w\}\left(\frac{dw}{dz}\right)^2 + \{w, z\}.$$

In particular: post-composition with a Möbius transformation φ does not change the Schwarzian derivatives of a function w.

Schwarzian derivatives II

Let w_1 and w_2 be a fundamental system of a holomorphic 2nd order homogenous linear ODE

$$w'' + pw' + q = 0.$$

Define $v = w_1/w_2$. Then

$$\{v, z\} = -p' - \frac{1}{2}p^2 + 2q.$$

Proof by straightforward, but tedious computation. Note that the result does nor depend on the choice of the fundamental system! This is easy to understand:

Suppose $\tilde{w}_1$ and $\tilde{w}_2$ is another fundamental system. Then there exist coefficients a, b, c, d with $ad - bc \neq 0$ such that

$$\tilde{v} := \tilde{w}_1/\tilde{w}_2 = \frac{aw_1 + bw_2}{cw_1 + dw_2} = \varphi \circ v,$$

where φ is a Möbius transformation. So

$$\{\tilde{v}, z\} = \{v, z\}.$$

Schwarzian derivatives III

Conversely, suppose we have a given analytic function $S(z)$ and we want the to solve the ODE

$$\{v, z\} = S(z) \qquad \text{(SW)}$$

for v.
The solution of (SW) is unique up to postcomposition with a Möbius transformation (essentially follows chain rule!).
Consider the 2nd order linear homogenous ODE

$$w''(z) + \tfrac{1}{2} S(z) w(z) = 0.$$

If w_1 and w_2 is a fundamental system and $v = w_2/w_1$, then v solves (SW).

Normal form of a Fuchsian ODE

Two Fuchsian ODEs are called *projectively equivalent* iff the ratio of a fundamental system is the same up to Möbius transformation iff the Schwarzian derivatives of the ratios are the same.

Corollary

Let $w'' + pw' + qw = 0$ be a Fuchsian ODE. Then there is precisely one Fuchsian ODE of the form $w'' + \tilde{q}w = 0$ that is projectively equivalent to the original equation.

This equation is called the *normal form* of the Fuchsian equation.

Proof: Take $\tilde{q} = -\frac{1}{2}p' - \frac{1}{4}p^2 + q$. This is the only possible choice, because by our previous discussion both equations will have the same Schwarzian derivatives for ratios of fundamental solutions (namely $2\tilde{q}$).
The normal form is also Fuchsian, because $\tilde{q}$ has at most poles of order 2 in $\mathbb{C}$ and $\tilde{q} = O(1/|z|^2)$ near ∞. $\square$

Schwarzians and solving 2nd order ODEs

The following problems are "equivalent": For given q solve

$$w''(z) + q(z)w(z) = 0 \tag{1}$$

or solve

$$\{u, z\} = 2q(z) \tag{2}$$

Proof: (1)$\rightarrow$ (2): As we discussed: Find a fundamental system w_1, w_2 of (1). Then $u = w_1/w_2$ solves (2).

(2)$\rightarrow$ (1): For $W = w_1' w_2 - w_1 w_2'$, we have $W' \equiv 0$. So wlog $W \equiv 1$. Then

$$u' = (w_1/w_2)' = 1/w_2^2, \text{ equiv. } w_2 = 1/\sqrt{u'}, w_1 = uw_2.$$

Orbifolds

An *orbifold structure* on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is given by a ramification function $n \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Here $\#\{z \in \widehat{\mathbb{C}} : n(z) \geq 2\} < \infty$.

The finitely many points z with $n(z) \geq 2$ are the *singularities* of the orbifold. If $\alpha(z) = \infty$, then z is called a *cusp* or *puncture* and a point z with $2 \leq n(p) < \infty$ a *conic singularity*. The *Euler characteristic* of $\mathcal{O} = (\widehat{\mathbb{C}}, n)$ is defined as

$$\chi(\mathcal{O}) = 2 - \sum_{z \in \widehat{\mathbb{C}}} \left(1 - \frac{1}{n(z)}\right).$$

We say that $\mathcal{O}$ is *hyperbolic* if $\chi(\mathcal{O}) < 0$.

Orbifold uniformization I

Theorem (Orbifold uniformization)

Let $\mathcal{O} = (\widehat{\mathbb{C}}, n)$ be a hyperbolic orbifold. Then there exists a holomorphic branched covering map

$$\Theta \colon \mathbb{D} \to \widehat{\mathbb{C}}_p := \widehat{\mathbb{C}} \setminus \{punctures\}$$

such that

$$\deg(\Theta(u), u) = n(\Theta(u))$$

for all $u \in \mathbb{D}$. The map Θ is unique up to a precomposition with an automorphism of $\mathbb{D}$.

The automorphisms g of $\mathbb{D}$ with $\Theta \circ g = \Theta$ are called the *deck transformations* of $\mathcal{O}$. They form a discrete group $\pi_1(\mathcal{O})$, called the *fundamental group* of $\mathcal{O}$.

We want to study the inverse map $u(z) = \Theta^{-1}(z)$. This is multi-valued, but its branches differ by postcomposition with a Möbius transformations in $\pi_1(\mathcal{O})$. Hence we consider the Schwarzian derivative $\{u, z\} = \{\Theta^{-1}, z\}$ which is well-defined, holomorphic outside the singularities of $\mathcal{O}$, and also independent of the choice of Θ.

Let $a \in \widehat{\mathbb{C}}$ be a conic singularity with $2 \leq n(a) < \infty$ and $\lambda = \lambda_a = 1/n(a) \in (0, 1)$. Wlog $a = 0$ and $u(a) = 0$. One can show that then near $a = 0$ in a suitable conformal coordinates (also called z):

$$u(z) = z^{\lambda}(c_0 + c_1 z + \dots), \quad c_0 \neq 0.$$
$$u'(z) = z^{\lambda-1}(\lambda c_0 + \dots), \quad \lambda c_0 \neq 0.$$

Orbifold uniformization III

Hence

$$\begin{aligned}
\frac{d}{dz}\log u'(z) &= \frac{\lambda-1}{z} + \text{holomorphic},\\
\frac{d^2}{dz^2}\log u'(z) &= \frac{1-\lambda}{z^2} + \text{holomorphic},\\
\{u,z\} &= \frac{d^2}{dz^2}\log u'(z) - \frac{1}{2}\left(\frac{d}{dz}\log u'(z)\right)^2\\
&= \frac{1}{2z^2}(2-2\lambda-(1-\lambda)^2) + \text{lower order terms}\\
&= \frac{1-\lambda^2}{2z^2} + \frac{h}{z} + \text{holomorphic}.
\end{aligned}$$

A similar computation with

$$u(z) = \log(z) + \text{holomorphic}$$

show that this is also true near a puncture with $\lambda = 1/n(a) = 0$.

Orbifold uniformization IV

We have proved:

Theorem

The inverse $u = u(z)$ of the universal orbifold covering map of a hyperbolic orbifold $(\widehat{\mathbb{C}}, n)$ has a Schwarzian derivative $\{u, z\}$ which is a rational function with poles of order 2 at the singularities and no other poles. Near a singularity $a \in \mathbb{C}$ we have

$$\{v, z\} = \frac{1 - \lambda_a^2}{2(z-a)^2} + \textit{lower order terms},$$

where $\lambda_a = 1/n(a)$.

For $a = \infty$ the same is true if we change coordinates $z = 1/t$ and consider $t = 0$.

The point ∞ as a singularity I

$$\{u,t\} = \{u,z\}\left(\frac{dz}{dt}\right)^2 = \{u,z\} \cdot \frac{1}{t^4}.$$

So we need $\{u,z\} = O(t^2) = O(1/z^2)$ near ∞ and $\{u,z\}$ must vanish at least to 2nd order.
The general form of $\{u,z\}$ is

$$\{u,z\} = \sum_{n(a)\geq 2} \frac{1-\lambda_a^2}{2(z-a)^2} + \frac{P(z)}{\prod_{n(a)\geq 2}(z-a)}.$$

Here P is a polynomial. Let m be the number of singularities in $\mathbb{C}$. Then P can have at most degree $m-2$. Let $P(z) = cz^{m-2} + \dots$ be the leading term. Plug in $z = 1/t$ and multiply by $1/t^2$ and look at the coefficient of $1/t^2$ (the leading term), or equivalently look at the coefficient for $1/z^2$. Then:

$$\tfrac{1}{2}(1-\lambda_\infty^2) = \sum_{a\in\mathbb{C}} \tfrac{1}{2}(1-\lambda_a^2) + c.$$

Hence

$$c = \tfrac{1}{2}\left(-m + 1 - \lambda_\infty^2 + \sum_{a\in\mathbb{C}} \lambda_a^2\right).$$

Conclusion: If there is a singularity at ∞, then

$$\{u, z\} = \sum_{n(a)\geq 2} \frac{1-\lambda_a^2}{2(z-a)^2} + \frac{P(z)}{\prod_{n(a)\geq 2}(z-a)},$$

where $P(z) = cz^{m-2} + \ldots$ is a polynomial of degree $m-2$ with the coefficient c as above.

Orbifolds with three singularities

Wlog at $0, 1, \infty$. Then $m = 2$ and P is a polynomial of degree $m - 2$, i.e., constant, and so $\{u, z\}$ is uniquely determined. Let $\lambda = \lambda_0$, $\mu = \lambda_1$, and $\nu = \lambda_\infty$. Then

$$\{u, z\} = \frac{1 - \lambda^2}{2z^2} + \frac{1 - \mu^2}{2(z-1)^2} + \frac{1 - \lambda^2 - \mu^2 + \nu^2}{2z(1-z)}.$$

Up to a factor $1/2$ this is the coefficient $\widetilde{q}$ of the normal form of the hypergeometric ODE. Here

$$\alpha = \frac{1}{2}(1 - \lambda - \mu + \nu),\ \beta = \frac{1}{2}(1 - \lambda - \mu - \nu),\ \gamma = 1 - \lambda.$$

Conversely, $\lambda = 1 - \gamma,\ \mu = \gamma - \alpha - \beta,\ \nu = \alpha - \beta$. If an orbifold has three singularities, then its universal orbifold covering map can be expressed by solutions of the hypergeometric ODE.

Suppose we have punctures at 0, 1, ∞, and $a \neq 0, 1, \infty$. Then we have $m = 3$, and so $P(z) = cz + B$. We have $\lambda_0 = \lambda_1 = \lambda_\infty = \lambda_a = 0$, and so $c = -1$. We see that

$$\{u, z\} = \frac{1}{2z^2} + \frac{1}{2(z-1)^2} + \frac{1}{2(z-a)^2} + \frac{-z+B}{z(z-1)(z-a)}.$$

Here B is an *a priori* undetermined parameter, called *accessory parameter*. Note that each point a determines a unique accessory parameter $B = B(a)$, but it is a hard (and in general unsolved) problem how to compute accessory parameters in this and similar cases.

Lemma

Let $P \subseteq \widehat{\mathbb{C}}$ be a finite set with $\#P \geq 3$, $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, $P' = g^{-1}(P)$, and $\varphi\colon \mathbb{D} \to \widehat{\mathbb{C}} \setminus P'$ be the universal covering map. Then $\psi = R \circ \varphi$ is the universal covering map $\psi\colon \mathbb{D} \to \widehat{\mathbb{C}} \setminus P$. Moreover, for Schwarzian derivatives we have the formula

$$\{\varphi^{-1}, t\} = \{\psi^{-1}, z\}(R(t)) \cdot R'(t)^2 + \{R, t\}.$$

Proof: Note that $\psi = R \circ \varphi$ is unramified over $\widehat{\mathbb{C}} \setminus P$. So ψ is the holomorphic universal cover of $\widehat{\mathbb{C}} \setminus P$. For (local) inverses we have $\psi^{-1} = \varphi^{-1} \circ R^{-1}$, or, $\varphi^{-1} = \psi^{-1} \circ R$. Now apply chain rule for Schwarzians. □

So if we know the Schwarzian for ψ^{-1}, we can compute it for φ^{-1} by pulling back with R.

Finding accessory parameters I

We know that the Schwarzian of the inverse of the uniformizing map for $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is given by

$$S(z) = \frac{1}{2z^2} + \frac{1}{2(z-1)^2} + \frac{1}{2z(1-z)}.$$

Let $R(z) = z^2$. Then $R^{-1}\{0, 1, \infty\} = \{0, 1, -1, \infty\}$. So the Schwarzian for the inverse of the uniformization of $\widehat{\mathbb{C}} \setminus \{0, 1, -1, \infty\}$ is

$$\begin{aligned} \{u, z\} &= S(R(z))R'(z)^2 + \{R, z\} \\ &= \frac{1}{2z^2} + \frac{1}{2(z-1)^2} + \frac{1}{2(z+1)^2} - \frac{1}{(z-1)(z+1)}. \end{aligned}$$

So the accessory parameter is $B = 0$ for $a = -1$.

Finding accessory parameters II

Note: For every set of four points on $\widehat{\mathbb{C}}$ there exists a non-trivial group of Möbius transformations that leave the set invariant and interchanges points of two pairs. This group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ ("Kleinsche Vierergruppe"). To see this, you may wlog assume that the points are $\{0, 1, a, \infty\}$. Then $g(z) = a/z$ is a non-trivial Möbius transformation with $0 \leftrightarrow \infty$ and $1 \leftrightarrow a$.
So the Schwarzian (of the inverse of) the uniformizing map of $\{0, 1, a, \infty\}$ must be invariant under pull-back by such a Möbius transformation.
If the four-point set has additional symmetries, then this leads to a non-trivial condition for the accessory parameter. For example, $\{-1, 0, 1, \infty\}$ the non-trival symmetry given by $1/z$ (the "generic" symmetry is $-1/z$.) This is another way to compute $B = B(-1) = 0$. Similarly, one can compute the accessory parameters for $a = 1/2$, 2, $\frac{1}{2}(1 \pm i\sqrt{3})$.

The Heun equation I

The *Heun equation* is the general Fuchsian equation with four singular points. Let's assume that the singularities are at $e_1, e_2, e_3 \in \mathbb{C}$, and at ∞. In its normal form, the sum of its critical exponents is equal to 1 at each singularity. Moreover, the total sum of all critical exponents must be equal to 2. This leads to the following general table for critical exponents with arbitrary $\lambda_1,\ \lambda_2\ , \lambda_3,\ \lambda_\infty \in \mathbb{C}$:

$$\begin{array}{cccc} e_1 & e_2 & e_3 & \infty \\ \frac{1}{2}(1+\lambda_1) & \frac{1}{2}(1+\lambda_2) & \frac{1}{2}(1+\lambda_3) & -\frac{1}{2}(1-\lambda_\infty) \\ \frac{1}{2}(1-\lambda_1) & \frac{1}{2}(1-\lambda_2) & \frac{1}{2}(1-\lambda_3) & -\frac{1}{2}(1+\lambda_\infty) \end{array}$$

Set $P(z) = (z - e_1)(z - e_2)(z - e_3)$. Then with the given table of exponents, the normal form of the Heun equation is

$$w''(z) + \frac{A(z) + B}{4P(z)} w(z) = 0,$$

where

$$\begin{aligned} A(z) = (1 - \lambda_1^2)\frac{P'(e_1)}{z - e_1} + (1 - \lambda_2^2)\frac{P'(e_2)}{z - e_2} + (1 - \lambda_3^2)\frac{P'(e_3)}{z - e_3} \\ + (1 - \lambda_\infty^2)(z - \tfrac{1}{3}(e_1 + e_2 + e_3)). \end{aligned}$$

Here B is an accessory parameter. Interesting cases:

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_\infty = 0$. Arises from Schwarzian derivatives $\{u, z\}$ of inverse of universal orbifold covering map of sphere with punctures at e_1, e_2, e_3, ∞.

Connection to Lamé equation

By introducing a new variable,

$$t = \frac{1}{2}\int_{z_0}^{z} \frac{d\zeta}{\sqrt{(\zeta - e_1)(\zeta - e_2)(\zeta - e_3)}},$$

or equivalently,

$$z = \wp(t), \text{ where } \wp'(t)^2 = 4(\wp(t) - e_1)(\wp(t) - e_2)(\wp(t) - e_3),$$

some special cases of the Heun equation can be reduced to the *Lamé equation*

$$w''(t) + (A\wp(t) + B)w(t) = 0.$$

The Heun equation III

Another interesting case is if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_\infty = 1$. Then the equation (the *MRW equation*) can be written simply as

$$w''(z) + \frac{B}{P(z)} w(z) = 0.$$

It arises according to Marshall-Rohde-Wang if one considers the Riemann map of the complementary components of Jordan curves that pass through the given points and are Loewner energy minimizers in their isotopy class rel. $\{e_1, e_2, e_3, \infty\}$. Then the equation has (projective) $PSL_2(\mathbb{R})$ monodromy, i.e., the Möbius transformations arising from *Umlaufsubstitutionen* around the singularities leave a circle (say the unit circle) invariant.
Conjecture (Rohde): For given $\{e_1, e_2, e_3, \infty\}$ the accessory parameters B for which this Heun equation has $PSL_2(\mathbb{R})$-monodromy are in one-to-one correspondence with isotopies classes of Jordan curves through $\{e_1, e_2, e_3, \infty\}$.

If one runs though a loop γ, say based, at z_0, that avoids the singularities of 2nd order ODE, then the ratio $u = w_1/w_2$ changes to

$$\tilde{u} = \tilde{w}_1/\tilde{w}_2 = \frac{aw_1 + bw_2}{cw_1 + dw_2} = M_\gamma \circ u.$$

where M_γ is a Möbius transformations. This gives the *projective monodromy representation*

$$\varphi\colon \pi_1(z_0, U) \to \mathrm{M\ddot{o}b}(\widehat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C}). \tag{3}$$

One can represent the Möbius transformation by a matrix $A_\gamma \in \mathrm{SL}_2(\mathbb{C})$. Up to a factor this matrix is the same as the monodromy matrix S_γ discussed earlier.
For u's arising from uniformization problems, these Möbius transformation form a discrete group leaving a circle invariant, i.e., they form a discrete group conjugate to subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (a *Fuchsian group*).

Projective Monodromy II

General Problem: Can one decide from the (Fuchsian) ODE whether we have $\mathrm{PSL}_2(\mathbb{R})$ monodromy, i.e., whether its (projective) monodromy group is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (or, equivalently, leaves a circle invariant)?

Consider case of Heun equation with singularities $\{0, 1, a, \infty\}$: The Möbius transformations running around the singularities can be represented by $A_0, A_1, A_a, A_\infty \in SL_2(\mathbb{C})$ with

$$A_0 A_1 A_a A_\infty = I_2.$$

A necessary condition for matrices to generate a group conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is that the traces of the matrices and the traces of any product of them are real.
For the converse direction one often only has to check traces of products of two.

Set $P(z) = z(z-1)(z-a)$. The MRW equation is

$$w''(z) + \frac{B}{P(z)} w(z) = 0$$

with an accessory parameter B.

Problem: For which $B \neq 0$ do we have $\mathrm{PSL}_2(\mathbb{R})$ monodromy?

Critical exponents are 0 and 1 at $0, 1, a$, and -1 and 0 at ∞. Consider a singularity, say $z = 0$. General theory tell us that that there is a unique fundamental system of the form

$$\begin{aligned} w_1(z) &= z + \text{holomorphic} \\ w_2(z) &= 1 + \text{holomorphic} + c w_1(z) \log z \end{aligned}$$

Here $c \neq 0$ for $B \neq 0$ (actually c is proportional to B). So

$$u(z) = w_2(z)/w_1(z) = c \log z + \frac{1}{z} + \text{holomorphic}.$$

Monodromy of the MRW equation

So projective monodromy is the Möbius transformation $z \mapsto z + 2\pi ic$, corresponding to the matrix

$$A_0 = \begin{pmatrix} 1 & 2\pi ic \\ 0 & 1 \end{pmatrix}.$$

This shows: the Möbius transformations A_0, A_1, A_a, A_∞ are *parabolic*, equiv. all traces $= \pm 2$.

Fundamental system of MRW ODE near ∞

$$w_1(z) = 1/z + O(1/z^2)$$
$$w_2(z) = 1 + O(1/z^2) - 2Bw_1(z)\log z$$

So near ∞ :

$$u(z) = w_2(z)/w_1(z) = z - 2B\log z + O(1).$$

Monodromy: $z \mapsto z - 4\pi i B$

Lemma

Two parabolic Möbius transformations with different fixed points represented by matrices $A, B \in \mathrm{SL}_2(\mathbb{R})$ have a common invariant circle if and only if $\mathrm{trace}(AB) \in \mathbb{R}$.

Proof: wlog

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Then $\mathrm{trace}(AB) = 2 + c$.
The only possible invariant circle is the a real axis. It is invariant under both A and B iff $c \in \mathbb{R}$ iff $\mathrm{trace}(AB) \in \mathbb{R}$. $\square$