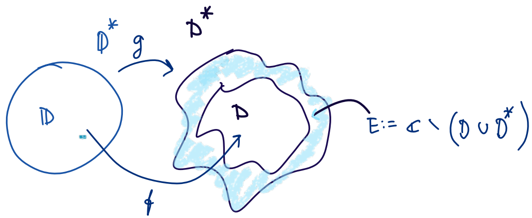
KTH ROYAL INSTITUTE OF TECHNOLOGY

Generalized Grunsky

Inverse Fuchsian Grunsky 2020

Set-up

Up to now: Grunsky matrix and inequality for $g \in \Sigma$.



Disjoint pair of univalent functions (f, g):

$f : \mathbb{D} \to \mathbb{C}, \quad g : \mathbb{D}^* \to \mathbb{C}$ univalent with expansions

$$f(z) = az + a_2 z^2 + \ldots, (a \neq 0) \quad g(z) = z + b_0 + b_1 z^{-1} + \ldots$$

Disjoint means $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \varnothing$.

'Standard' Grunsky uses no information about the inner domain $D := f(\mathbb{D})$.

Goal: generalize Grunsky to use information about the **inner mapping** f.

Example: area theorem $\sum_{n=1}^{\infty} n\,|a_n|^2 + \sum_{n=1}^{\infty} n\,|b_n|^2 \leq 1$.

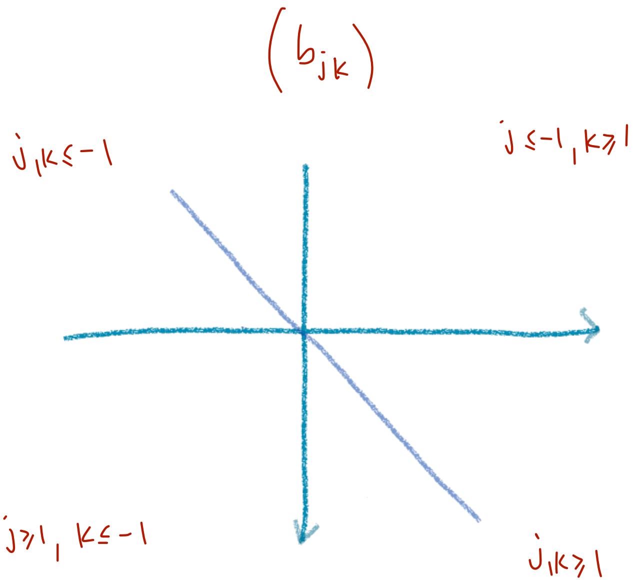
Generalized Grunsky coefficients

We define Grunsky coefficients $(b_{kl})_{k,l\in\mathbb{Z}}, b_{kl} \in \mathbb{C}$ as follows (**warning**: our b_{kl} are Duren's γ_{kl} and $\beta_{kl}^{Duren} = k b_{kl}$)

- $$\log \frac{g(z)-g(w)}{z-w} = -\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{kl}\, z^{-k} w^{-l}, \qquad |z|,|w| > 1$$
- $$\log \frac{f(z)-f(w)}{z-w} = -\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{-k,-l}\, z^{k} w^{l}, \qquad |z|,|w| < 1$$
- $$\log \frac{g(z)-f(w)}{z} = -\sum_{k=0}^{\infty}\sum_{l=1}^{\infty} b_{-k,l}\, z^{-l} w^{k}, \qquad |w| < 1 < |z|$$
- Moreover:

  $b_{k,-l} = b_{-k,l}, k \geq 1, l \geq 0,$ so symmetric: $b_{kl} = b_{lk}, j, k \in \mathbb{Z}$.

Bi-infinite Grunsky matrix



- $b_{00} = -\log a$
- $\log \frac{z}{g(z)} = \sum_{k=1}^{\infty} b_{0,k}\, z^{-k}$
- $\log \frac{z}{f(z)} = \sum_{k=0}^{\infty} b_{0,-k}\, z^{k}$
- $b_{j,k},\ j,k \geq 1$: G_g
- $b_{-j,-k},\ j,k \geq 1$: $G_{a/f(1/z)}$

Generalized Grunsky matrices

Rescale and collect in four **infinite matrices** associated to the **disjoint pair** (f, g):

$$B_1 = (\sqrt{kl}\, b_{-k,-l})_{k,l\geq 1}; \quad B_2 = (\sqrt{kl}\, b_{-k,l})_{k,l\geq 1}$$

$$B_3 = (\sqrt{kl}\, b_{k,-l})_{k,l\geq 1}; \quad B_4 = (\sqrt{kl}\, b_{kl})_{k,l\geq 1}$$

(Note, the zero-index entries are omitted.)

As in Peter's talk, the matrices B_1, B_2, B_3, B_4 give rise to **linear operators** on $\ell^2 = \{z = (z_n)_{n=1}^{\infty} : \sum |z_n|^2 < \infty\}$ that are **contractions** (by the theorem). Alternatively, write $B = [B_1, B_2; B_3, B_4]$ which acts on $\ell^2 \oplus \ell^2$.

Why?

Use more available information, more symmetric setting ...

Bi-infinite Grunsky matrix contains information about the two mapping functions simultaneously. Particularly interesting when omitted set has zero area.

Estimates: Relations between coefficients. Estimates involving both mappings. Quantify one-sided estimates (as in the generalized area theorem).

Operators: When there is equality, generalized Grunsky inequality expresses transfer of information between outside and inside. Relation to composition operators and conformal welding ... **topic of discussion.**

Note: Different generalizations use other information, e.g., Garaberdian-Schiffer.

Statement: generalized Grunsky inequality

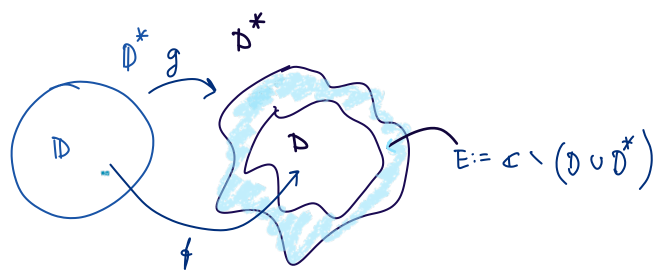
Theorem [Hummel '72]: Let (f, g) be a **disjoint** pair.

Let $m \in \mathbb{N}_0$, and take $\lambda_k \in \mathbb{C}, k = -m, \ldots, m$, not all zero. Then:

$$\sum_{k=1}^{\infty} k \left| \sum_{l=-m}^{m} b_{kl}\lambda_l \right|^2 + \sum_{k=1}^{\infty} k \left| \sum_{l=-m}^{m} b_{-k,l}\lambda_l \right|^2$$

$$\leq \sum_{k=1}^{m} \frac{1}{k}\left(|\lambda_k|^2 + |\lambda_{-k}|^2\right) + 2\Re\left(\bar{\lambda}_0 \sum_{l=-m}^{m} b_{0,l}\lambda_l\right)$$

Equality holds iff the **area** of $\mathbb{C} \setminus (f(\mathbb{D}) \cup g(\mathbb{D}^*)) = 0$.



Generalized Grunsky operators

Many things can be written **very neatly** in this language.

- Example from Takhtajan-Teo: Generalized Grunsky when $\lambda_0 = 0$ and the omitted set has zero area (i.e. when there is equality) can then (after polarization) be formulated as (B_k^* is the adjoint operator):

$$B_1 B_1^* + B_2 B_2^* = I, \qquad B_3 B_1^* + B_4 B_2^* = 0;$$

$$B_1 B_3^* + B_2 B_4^* = 0, \qquad B_3 B_3^* + B_4 B_4^* = I\,.$$

($B_1 \sim f, B_4 \sim g$ ...) or simply $BB^* = I$.

- Operators can be realized in different ways that lead to other proofs and insights. This story is continued in Ilia's talk...

- Clear expression of symmetries. Paths to, e.g., characterizations (Yilin-Tim).

Proof, Step 0

The proof uses **Faber polynomials**. Where do they come from?

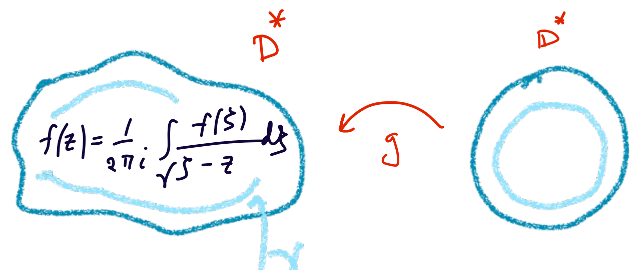
Problem: Ω Jordan domain (with some regularity). **Represent** $f \in H(D) \cap C(\overline{D})$

$$f(z) = \sum_{j=0}^{\infty} c_j \, \Phi_j(z), \qquad z \in D\,.$$

polynomial basis $((\Phi_j)_{j \geq 0})$ depends only on Ω.

Faber [1903]: change variables in Cauchy formula using $g : \mathbb{D}^* \to D^*$ and expand near ∞.

$$\frac{g'(w)}{g(w) - z} = \sum_{j=0}^{\infty} \Phi_j(z)\, w^{-(j+1)}$$



Faber polynomials, recap

Interesting relations. E.g. Schiffer [1940s], appear in many places.

Defining relations (large $|w|$ expansion)

$$\frac{g'(w)}{g(w)-z} = \sum_{j=0}^{\infty} \Phi_j(z)\, w^{-(j+1)}, \Phi_0(z) \equiv 1, \quad \log \frac{g(w)-z}{w} = -\sum_{j=1}^{\infty} \frac{1}{j} \Phi_j(z) w^{-j}$$

Recall the **key formulas**: (Compare expansions with definition of Grunsky coefficients.)

$$\Phi_k(f(z)) = k \sum_{l=0}^{\infty} b_{k,-l} z^l, \quad k \geq 1. \quad (1)$$

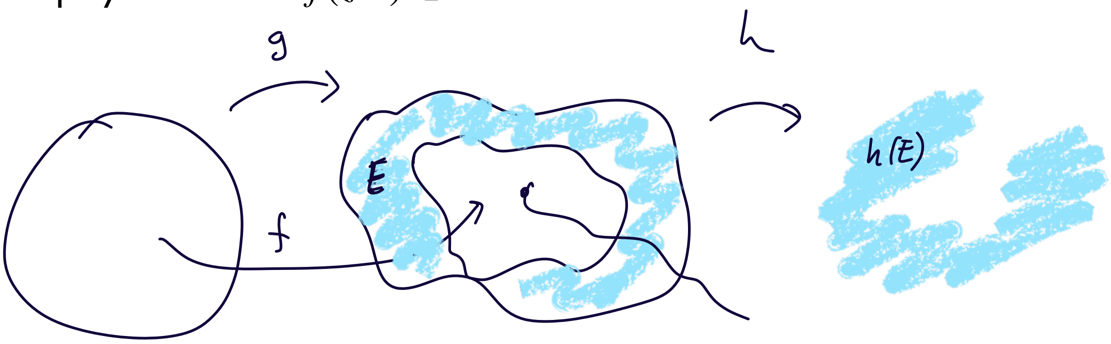
$$\Phi_k(g(z)) = z^k + k \sum_{l=1}^{\infty} b_{kl} z^{-l}, \quad k \geq 1. \quad (2)$$

Proof, Step 1

Idea: compute the **area** of the omitted set E under a carefully chosen function. The estimate then comes from positivity. The function is:

$$h(z) = \sum_{k=1}^{m} \frac{\lambda_k}{k} \Phi_k(z) + \sum_{k=1}^{m} \frac{\lambda_{-k}}{k} \Phi_{-k}(a/z) - \lambda_0 \log z$$

It is holomorphic in $\mathbb{C} \setminus \beta$, a branch cut. Here $\Phi_{-k}(z)$ is the k:th Faber polynomial of $a/f(z^{-1}) \in \Sigma$.



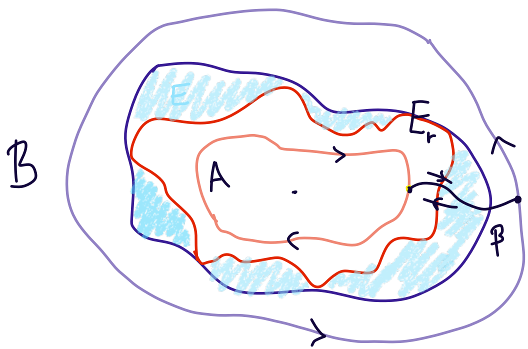
Proof, Step 1

Write $E = \mathbb{C} \setminus (f(\mathbb{D}) \cup g(\mathbb{D}^*))$ and for $r < 1$:

$A = f(z : |z| = r), B = g(w : |w| = r^{-1})$.

Then set $E_r = \mathbb{C} \setminus (f(z : |z| \leq r) \cup g(w : |w| \geq r^{-1}))$.

Choose β so that it is a smooth Jordan arc connecting the points $f(r)$ and $g(r^{-1})$ and otherwise not intersecting A, B.



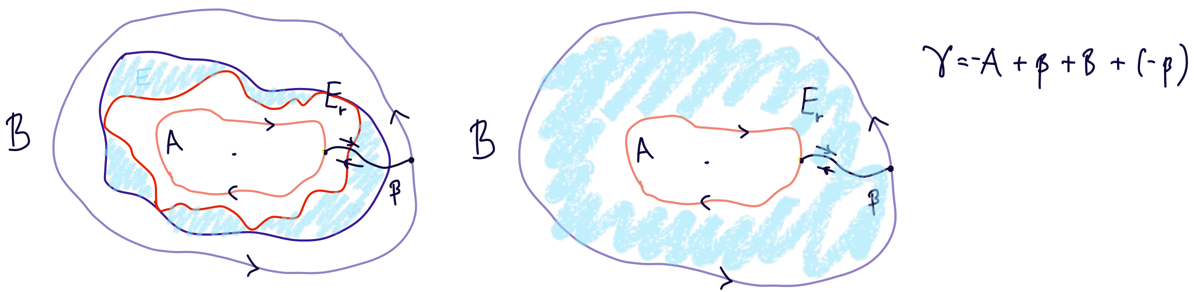
$\gamma = -A + \beta + B + (-\beta)$

Proof, Step 1

Apply **Green's theorem** with γ (note that the index of γ is $\equiv 1$ in $E_r \setminus \beta$)

$$\frac{1}{\pi}\int_{E_r\setminus\beta} |h'(z)|^2 \, dA(z) = -\frac{1}{2\pi i}\int_A \overline{h} h' dz + \frac{1}{2\pi i}\int_B \overline{h} h' dz - \overline{\lambda_0}\int_\beta h' dz\,.$$

The last term comes from the $-\lambda_0 \log z$ term in h. The RHS is non-negative.



Proof, Step 2

Change variables

$$\varphi(z) := h \circ f(z), |z| < 1, \quad \psi(w) := h \circ g(w), |w| > 1.$$

$$\frac{1}{\pi}\int_{E_r} |h'(z)|^2 \, dA(z) = -\frac{1}{2\pi i}\int_{|z|=r} \overline{\varphi}\varphi' dz + \frac{1}{2\pi i}\int_{|w|=r^{-1}} \overline{\psi}\psi' dw - \overline{\lambda_0}\left(\psi(r^{-1}) - \varphi(r)\right)$$

Use (1) and (2) for **Faber polynomials** (linking to Grunsky coefficients):

$$\varphi(z) = -\lambda_0 \log z + \lambda_0 \log \frac{z}{f(z)} + \sum_{k=1}^{m} \frac{\lambda_k}{k}\phi_k(f(z)) + \sum_{k=1}^{m} \frac{\lambda_{-k}}{k}\phi_{-k}\left(\frac{a}{f(z)}\right)$$

$$= \ldots = -\lambda_0 \log z + \sum_{k=1}^{m} \frac{\lambda_{-k}}{k} z^{-k} + \sum_{k=0}^{\infty} \alpha_k z^k.$$

$$\alpha_k = \sum_{l=-m}^{m} b_{-k,l}\lambda_l, \; \alpha_0 = \sum_{l=0}^{m} b_{0l}\lambda_l .$$

Recall

Faber polynomials

Key relations: (Compare expansions with definition of Grunsky coefficients.)

$$\Phi_k(f(z)) = k \sum_{j=0}^{\infty} b_{k,-j} z^j, \quad k \geq 1. \qquad (1)$$

$$\Phi_k(g(z)) = z^k + k \sum_{j=1}^{\infty} b_{kj} z^{-j}, \quad k \geq 1. \qquad (2)$$

Proof, Step 2

Change variables

$$\varphi(z) := h \circ f(z), |z| < 1, \quad \psi(w) := h \circ g(w), |w| > 1.$$

$$\frac{1}{\pi}\int_{E_r} |h'(z)|^2\, dA(z) = -\frac{1}{2\pi i}\int_{|z|=r} \overline{\varphi}\varphi'\, dz + \frac{1}{2\pi i}\int_{|w|=r^{-1}} \overline{\psi}\,\psi'\, dw - \overline{\lambda_0}\left(\psi(r^{-1}) - \varphi(r)\right)$$

Similar formula for ψ

$$\psi(z) = -\lambda_0 \log z + \lambda_0 \log \frac{z}{g(z)} + \sum_{k=1}^{m} \frac{\lambda_k}{k}\phi_k(g(z)) + \sum_{k=1}^{m} \frac{\lambda_{-k}}{k}\phi_{-k}\left(\frac{a}{g(z)}\right)$$

$$= \ldots = -\lambda_0 \log z + \sum_{k=1}^{m} \frac{\lambda_k}{k} z^k + \sum_{k=0}^{\infty} \delta_k z^{-k}.$$

$$\delta_k = \sum_{l=-m}^{m} b_{k,l}\lambda_l,\ \delta_0 = -\sum_{l=1}^{m} b_{0,-l}\lambda_{-l}\,.$$

Proof, Step 3

$$\frac{1}{\pi}\int_{E_r} |h'(z)|^2 \, dA(z) = -\frac{1}{2\pi i}\int_{|z|=r} \overline{\varphi}\varphi' dz + \frac{1}{2\pi i}\int_{|w|=r^{-1}} \overline{\psi}\psi' dw - \overline{\lambda_0}\left(\psi(r^{-1}) - \varphi(r)\right)$$

Integrate **term by term** (yes, **tedious**!) using formulas from previous slides...e.g., φ-term:

$$\varphi(z) = -\lambda_0 \log z + \sum_{k=1}^{m} \frac{\lambda_{-k}}{k} z^{-k} + \sum_{k=0}^{\infty} \alpha_k z^k \text{ with } \alpha_k = \sum_{l=-m}^{m} b_{-k,l}\lambda_l,\ \alpha_0 = \sum_{l=0}^{m} b_{0l}\lambda_l\,.$$

The φ-term becomes:

$$-\overline{\lambda_0}\varphi(r) + 2\Re \alpha_0 \overline{\lambda_0} + |\lambda_0|^2 (i\pi - 2\log r) + \sum_{k=1}^{m} k^{-1} |\lambda_{-k}|^2 r^{-2k} - \sum_{k=1}^{\infty} k\,|\alpha_k|^2 r^{2k}$$

Similar expression for ψ-term...

Proof, Step 3

The first term on the right-hand side is, by (11), equal to

$$\frac{1}{2\pi}\int_0^{2\pi}\left(i\bar{\lambda}_0 t - \bar{\lambda}_0 \log r + \sum_{k=1}^{m}\frac{\bar{\lambda}_{-k}}{k} r^{-k} e^{ikt} + \sum_{k=0}^{\infty}\bar{\alpha}_k r^k e^{-ikt}\right)$$

$$\left(\sum_{k=0}^{m}\lambda_{-k} r^{-k} e^{-ikt} - \sum_{k=1}^{\infty} k\alpha_k r^k e^{ikt}\right) dt$$

$$= -\bar{\lambda}_0\varphi(r) + \alpha_0\bar{\lambda}_0 + \bar{\alpha}_0\lambda_0 + |\lambda_0|^2(i\pi - 2\log r)$$

$$+ \sum_{k=1}^{m}\frac{|\lambda_{-k}|^2}{k} r^{-2k} - \sum_{k=1}^{\infty} k|\alpha_k|^2 r^{2k}$$

because

$$\frac{i}{2\pi}\int_0^{2\pi} t\, e^{ikt}\, dt = \frac{1}{k} \quad (k \neq 0), \quad = i\pi \quad (k = 0). \tag{18}$$

Similarly the second term is, by (13), equal to

$$\bar{\lambda}_0\psi(r^{-1}) - \beta_0\bar{\lambda}_0 - \bar{\beta}_0\lambda_0 + |\lambda_0|^2(-i\pi - 2\log r)$$

$$+ \sum_{k=1}^{m}\frac{|\lambda_k|^2}{k} r^{-2k} - \sum_{k=1}^{\infty} k|\beta_k|^2 r^{2k}.$$

Adding these contributions we obtain from (17) that

$$\frac{1}{\pi}\iint_{H(r)}|h'(w)|^2 d\Omega = \sum_{k=1}^{m}\frac{1}{k}(|\lambda_{-k}|^2 + |\lambda_k|^2) r^{-2k} - \sum_{k=1}^{\infty} k(|\alpha_k|^2 + |\beta_k|^2) r^{2k}$$

$$+ 2\,\mathrm{Re}[\bar{\lambda}_0(\alpha_0 - \beta_0)] - 4|\lambda_0|^2 \log r.$$

If we let $r \to 1 - 0$ we obtain the identity

$$\frac{1}{\pi}\iint_F |h'(w)|^2 d\Omega = \sum_{k=1}^{m}\frac{1}{k}(|\lambda_{-k}|^2 + |\lambda_k|^2) - \sum_{k=1}^{\infty} k(|\alpha_k|^2 + |\beta_k|^2) + 2\,\mathrm{Re}[\bar{\lambda}_0(\beta_0 - \alpha_0)] \tag{19}$$

Proof, conclusion

Area is non-negative

$$\varphi(z) := h \circ f(z), |z| < 1, \quad \psi(w) := h \circ g(w), |w| > 1.$$

$$0 \leq -\frac{1}{2\pi i}\int_{|z|=r} \overline{\varphi}\varphi' dz + \frac{1}{2\pi i}\int_{|w|=r^{-1}} \overline{\psi}\psi' dw - \overline{\lambda_0}\left(\psi(r^{-1}) - \varphi(r)\right)$$

The RHS is non-negative. Letting $r \to 1-$ **gives the result.**

That is, if (f, g) is a **disjoint** pair, then for $m \in \mathbb{N}_0$, and take $\lambda_k \in \mathbb{C}, k = -m, \ldots, m$, not all zero. Then:

$$\sum_{k=1}^{\infty} k \left| \sum_{l=-m}^{m} b_{kl}\lambda_l \right|^2 + \sum_{k=1}^{\infty} k \left| \sum_{l=-m}^{m} b_{-k,l}\lambda_l \right|^2$$

$$\leq \sum_{k=1}^{m} \frac{1}{k}(|\lambda_k|^2 + |\lambda_{-k}|^2) + 2\Re\left(\bar{\lambda}_0 \sum_{l=-m}^{m} b_{0,l}\lambda_l\right)$$

Some consequences

- Pick some l_0 and choose $\lambda_l = \delta_{l_0 l}$: the theorem gives for $l_0 \neq 0$ resp. $l_0 = 0$

$$\sum_{k=1}^{\infty} k\,|\,b_{kl_0}|^2 + \sum_{k=1}^{\infty} k\,|\,b_{-k,l_0}|^2 \leq \frac{1}{|\,l_0\,|} \quad (3)$$

$$\sum_{k=1}^{\infty} k\,|\,b_{k0}|^2 + \sum_{k=1}^{\infty} k\,|\,b_{-k,0}|^2 \leq 2\Re b_{00} = -\,2\log|\,a\,|\,. \quad (4)$$

(Special case $l_0 = 1$ follows from the usual area theorem since $b_{k1} = b_k, b_{-k,1} = a_k, k \geq 1$ and area$(f(\mathbb{D})) \leq$ area$(\mathbb{C} \setminus g(\mathbb{D}^*))$.)

- (4) implies (as pointed out in TT)

$$2\pi \log|\,a\,|^{-1} \geq \int_{\mathbb{D}} |\frac{f'}{f} - \frac{1}{z}|^2 dA + \int_{\mathbb{D}^*} |\frac{g'}{g} - \frac{1}{z}|^2 dA$$

- With work one can show, e.g., $\sum_{n=1}^{\infty} |\,a_n|^2 \leq e^{-|b_0|^2}$.

Consequences

- Take $\lambda_0 \in \mathbb{R}, \lambda_k \in \mathbb{C}, k \neq 0.$ Then (assuming the second series converges)

$$\Re \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} b_{kl}\lambda_k\lambda_l + \sum_{k=1}^{\infty} \frac{1}{k}(|\lambda_k|^2 + |\lambda_{-k}|^2) \geq 0$$

Pf. Assume $\lambda_k = 0$ for large k. The double sum is a **quadratic polynomial** in λ_0 :

$$q(\lambda_0) := \lambda_0^2 \Re b_{00} + 2\lambda_0 \Re \sum_{l\neq 0} b_{0l}\lambda_l + \Re \sum_{k\neq 0} \sum_{l\neq 0} b_{kl}\lambda_k\lambda_l.$$

If $\Re b_{00} > 0$ (cf (4)) there is a global minimum. Computing the value gives the statement.

Note: If $\lambda_0 = 0$ we get estimate on the modulus. (Cf. weak standard Grunsky.)

Goluzin via Loewner

If $f \in \Sigma$, $(\lambda_k) \in \mathbb{C}^N$:

$$\left| \sum_{k=1}^{N} \sum_{l=1}^{N} \lambda_k \lambda_l \log \frac{f(z_k) - f(z_l)}{z_k - z_l} \right| \leq - \sum_k \sum_l \lambda_k \bar{\lambda}_l \log\left(1 - \frac{1}{z_k \bar{z}_l}\right)$$

Theorem 1. *Let F denote a function belonging to Σ. Let $a_{\nu,\nu'}$, for $\nu, \nu' = 1, 2, \cdots, n$, where $n \geq 1$, denote real numbers such that $\sum_{\nu,\nu'=1}^{n} a_{\nu,\nu'} x_\nu x_{\nu'}$ is a positive quadratic form. Then, for arbitrary ζ_ν, $\nu = 1, 2, \cdots, n$, the inequalities*

$$\prod_{\nu,\nu'=1}^{n} \left| 1 - \frac{1}{\zeta_\nu \bar{\zeta}_{\nu'}} \right|^{a_{\nu,\nu'}} \leqslant \prod_{\nu,\nu'=1}^{n} \left| \frac{F(\zeta_\nu) - F(\zeta_{\nu'})}{\zeta_\nu - \zeta_{\nu'}} \right|^{a_{\nu,\nu'}} \leqslant \frac{1}{\prod_{\nu,\nu'=1}^{n} \left| 1 - \frac{1}{\zeta_\nu \bar{\zeta}_{\nu'}} \right|^{a_{\nu,\nu'}}} \tag{6}$$

hold throughout the domain $|\zeta| > 1$, where we should understand the factor $|F'(\zeta_\nu)|$ in the second product when $\zeta_\nu = \zeta_{\nu'}$.

Goluzin via Loewner

§2. Sharpening of the distortion theorems

As another application of the method of parametric representation, we now present sharpened forms of certain inequalities constituting distortion theorems for the classes S and Σ.[1] Let us first show a number of relationships that hold in the parametric representation of univalent functions and that play an important role in questions concerning bounds.

Consider an arbitrary function $f(z) \in C'$ defined in accordance with formula (1) of §1, where $f(z, t)$ satisfies in the interval $0 < t < \infty$ the differential equation (2) and initial condition (3) of §1. For the function $f(z, t)$, we have a number of interesting relationships. Specifically, let us take in $|z| < 1$ arbitrary points z_ν for $\nu = 1, \cdots, n$ and let us write for brevity $f_\nu = f(z_\nu, t)$. By using the equations

$$\frac{\partial f_\nu}{\partial t} = -f_\nu \frac{1 + k(t) f_\nu}{1 - k(t) f_\nu}, \qquad \nu = 1, \ldots, n,$$

we obtain by direct calculation the relations

$$\begin{aligned} &\frac{\partial}{\partial t} \log\left[\frac{e^{-it}}{f_\nu f_{\nu'}} \frac{f_\nu - f_{\nu'}}{z_\nu - z_{\nu'}}\right] = -2 \frac{kf_\nu}{1 - kf_\nu} \cdot \frac{kf_{\nu'}}{1 - kf_{\nu'}}, \\ &\frac{\partial}{\partial t} \log(1 - f_\nu \bar{f}_{\nu'}) = 2 \frac{kf_\nu}{1 - kf_\nu} \cdot \overline{\left(\frac{kf_{\nu'}}{1 - kf_{\nu'}}\right)} \end{aligned} \tag{1}$$

(where $k = k(t)$). If $z_\nu = z_{\nu'}$, the expression $(f_\nu - f_{\nu'})/(z_\nu - z_{\nu'})$ in (1) should be understood to mean $f'(z_\nu)$.

If we integrate equations (1) with respect to t from 0 to ∞ and keep (1) and (3) of §1 in mind, we obtain the following integral formulas:

$$\begin{aligned} &\log \frac{F(\zeta_\nu) - F(\zeta_{\nu'})}{\zeta_\nu - \zeta_{\nu'}} = -2 \int_0^\infty \frac{kf_\nu}{1 - kf_\nu} \cdot \frac{kf_{\nu'}}{1 - kf_{\nu'}}\, dt, \\ &\log\left(1 - \frac{1}{\zeta_\nu \bar{\zeta}_{\nu'}}\right) = -2 \int_0^\infty \frac{kf_\nu}{1 - kf_\nu} \cdot \overline{\left(\frac{kf_{\nu'}}{1 - kf_{\nu'}}\right)}\, dt, \end{aligned} \tag{2}$$

1) Goluzin [1948b, 1951a].

where we set

$$\zeta_\nu = \frac{1}{z_\nu}, \quad F(\zeta) = \frac{1}{f\left(\frac{1}{\zeta}\right)}; \quad F(\zeta) \in \Sigma.$$

We introduce the expansions

$$\begin{aligned} &\log \frac{F(\zeta) - F(\zeta')}{\zeta - \zeta'} = \sum_{k, l = 1}^\infty a_{k, l} \zeta^{-k} \zeta'^{-l}, \quad |\zeta| > 1, \ |\zeta'| > 1, \\ &\frac{k(t) f(z, t)}{1 - k(t) f(z, t)} = \sum_{k=1}^\infty b_k(t) z^k, \quad |z| < 1. \end{aligned} \tag{3}$$

When we substitute these expansions into (2) and identify the resulting equations with respect to ζ_ν and $\zeta_{\nu'}$, we arrive at the formulas

$$a_{k, l} = -2 \int_0^\infty b_k(t) b_l(t)\, dt \qquad (k, l = 1, 2, \ldots), \tag{4}$$

$$\int_0^\infty b_k(t) \overline{b_l(t)}\, dt = \begin{cases} 0 & \text{for } k \neq l, \\ \frac{1}{2k} & \text{for } k = l \end{cases} \qquad (k, l = 1, 2, \ldots). \tag{5}$$

Formulas (5) express the important property of orthogonality of the functions $b_k(t)$ for $k = 1, 2, \cdots$.

We shall now prove two theorems dealing with the applications of formulas (2) to inequalities.

Theorem 1. *Let F denote a function belonging to Σ. Let $\alpha_{\nu, \nu'}$, for $\nu, \nu' = 1, 2, \cdots, n$, where $n \geq 1$, denote real numbers such that $\sum_{\nu, \nu' = 1}^n \alpha_{\nu, \nu'} x_\nu x_{\nu'}$ is a positive quadratic form. Then, for arbitrary ζ_ν, $\nu = 1, 2, \cdots, n$, the inequalities*

$$\prod_{\nu, \nu' = 1}^n \left|1 - \frac{1}{\zeta_\nu \bar{\zeta}_{\nu'}}\right|^{\alpha_{\nu, \nu'}} \leqslant \prod_{\nu, \nu' = 1}^n \left|\frac{F(\zeta_\nu) - F(\zeta_{\nu'})}{\zeta_\nu - \zeta_{\nu'}}\right|^{\alpha_{\nu, \nu'}} \leqslant \frac{1}{\prod_{\nu, \nu' = 1}^n \left|1 - \frac{1}{\zeta_\nu \bar{\zeta}_{\nu'}}\right|^{\alpha_{\nu, \nu'}}} \tag{6}$$

hold throughout the domain $|\zeta| > 1$, where we should understand the factor $|F'(\zeta_\nu)|$ in the second product when $\zeta_\nu = \zeta_{\nu'}$.

Proof. Let us first prove inequalities (6) for the functions $F(\zeta) = 1/f(1/\zeta)$, where $f(z) \in S'$. In this case, if we set

Goluzin via Loewner

$$\frac{kf_\nu}{1-kf_\nu} = X_\nu + iY_\nu, \quad \nu = 1, \ldots, n,$$

and take out the real parts in (2), we obtain

$$\log\left|\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}}\right| = -2\int_0^\infty (X_\nu X_{\nu'} - Y_\nu Y_{\nu'})\,dt,$$

$$\log\left|1-\frac{1}{\zeta_\nu\bar{\zeta}_{\nu'}}\right| = -2\int_0^\infty (X_\nu X_{\nu'} + Y_\nu Y_{\nu'})\,dt.$$

Then, by adding and subtracting, we arrive at the following formulas

$$\log\left|\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}}\right| = -\log\left|1-\frac{1}{\zeta_\nu\bar{\zeta}_{\nu'}}\right| - 4\int_0^\infty X_\nu X_{\nu'}\,dt,$$

$$\log\left|\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}}\right| = \log\left|1-\frac{1}{\zeta_\nu\bar{\zeta}_{\nu'}}\right| + 4\int_0^\infty Y_\nu Y_{\nu'}\,dt.$$

We multiply these formulas by the numbers $\alpha_{\nu,\nu'}$ and sum each of them with with respect to ν, $\nu' = 1, \cdots, n$. Then, remembering that for numbers $\alpha_{\nu,\nu'}$ th satisfy the conditions of the theorem, we have

$$\sum_{\nu,\nu'=1}^n \alpha_{\nu,\nu'} X_\nu X_{\nu'} \geq 0, \quad \sum_{\nu,\nu'=1}^n \alpha_{\nu,\nu'} Y_\nu Y_{\nu'} \geq 0.$$

We then immediately obtain inequalities (6). But these inequalities, which were proved for the functions $F(\zeta) \in \Sigma$ represented in the form $F(\zeta) = 1/f(1/\zeta)$, where $f(z) \in S'$, obviously remain valid for arbitrary functions $F(\zeta) \in \Sigma$ that can be represented in the form $F(\zeta) = 1/f(1/\zeta) + \text{const}$, where $f(z) \in S$, and consequently they remain valid for the entire class Σ. This completes the proof of the theorem.

Theorem 2. *Let n denote a positive integer. For $\nu = 1, \cdots, n$, let γ_ν denote arbitrary complex numbers and let ζ_ν denote complex numbers in the domain $|\zeta| > 1$. Then*

$$\left|\sum_{\nu,\nu'=1}^n \gamma_\nu\gamma_{\nu'}\log\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}}\right| \leq -\sum_{\nu,\nu'=1}^n \gamma_\nu\bar{\gamma}_{\nu'}\log\left(1-\frac{1}{\zeta_\nu\bar{\zeta}_{\nu'}}\right) \qquad (7)$$

for $F(\zeta) \in \Sigma$.[1)]

Proof. Just as above, it will be sufficient to prove (7) for the functions $F(\zeta) = 1/f(1/\zeta)$, where $f(z) \in S'$. But, in the present case, we have from the first of formulas (2)

$$\sum_{\nu,\nu'=1}^n \gamma_\nu\gamma_{\nu'}\log\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}} = -2\int_0^\infty \sum_{\nu,\nu'=1}^n \gamma_\nu\gamma_{\nu'}\frac{kf_\nu}{1-kf_\nu}\,\frac{kf_{\nu'}}{1-kf_{\nu'}}\,dt = -2\int_0^\infty\left(\sum_{\nu=1}^n \gamma_\nu\frac{kf_\nu}{1-kf_\nu}\right)^2 dt$$

and, consequently,

$$\left|\sum_{\nu,\nu'=1}^n \gamma_\nu\gamma_{\nu'}\log\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}}\right| \leq 2\int_0^\infty\left|\sum_{\nu=1}^n \gamma_\nu\frac{kf_\nu}{1-kf_\nu}\right|^2 dt$$

$$= 2\int_0^\infty \sum_{\nu,\nu'=1}^n \gamma_\nu\bar{\gamma}_{\nu'}\frac{kf_\nu}{1-kf_\nu}\cdot\overline{\left(\frac{kf_{\nu'}}{1-kf_{\nu'}}\right)}\,dt.$$

From the second of formulas (2) applied to the last integral, this yields (7) and completes the proof of the theorem.

Further remarks

Grunsky proved his estimate and introduced the coefficients in 1939. (Area principle.)
Goluzin proved his ineq using Loewner equation (late 1940s), from which Grunsky follows. Goluzin's book (1950s).
Generalized and "exponentiated" by **Milin, Lebedev** (1965, 1977) [Ch 5 in Duren], used to prove special cases of Bieberbach. **Garabedian-Schiffer**, extension, also using Loewner. ... **de Branges**, 1985.

Faber, 1903. **Schiffer** studied Faber polynomials systematically in 1948, proof of Grunsky via variation of Faber polynomials. Survey by Curtiss, 1971.

Generalized Grunsky: **Hummel**, 1972 (area-principle).

Grunsky matrix as operator (**Milin, Pommerenke, Pederson** 1960s).
Quasicircle characterization (**Schiffer** 1960s, Pommerenke?), further classes (Pommerenke, Takhtajan-Teo, Shen... 1980s, 2000s).

Acting on Bergman spaces (**Bergman-Schiffer**, 1960s (?))

Further remarks

Links to CFT and physics ...

* Further implications of (generalized) Grunsky, e.g., links to composition operators, spectral characterization theorems, applications in integrable systems, CFT: e.g. arXiv:1607.08373 (Grunsky coeff), arXiv:hep-th/0005259 ("tau function", Schwarzian),