Beurling and Grunsky.

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Bergman Space

Let Ω be a domain in $\mathbb{C}$. The *Bergman space* $\mathcal{A}^p(\Omega)$ is the subspace of analytic functions in $L^p(\Omega)$

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We will mostly talk about $p = 2$.

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is an orthonormal basis of $\mathcal{A}^2(\mathbb{D})$, and for $f(z) = \sum a_n z^n \in \mathcal{A}^2(\mathbb{D})$:

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$$\mathcal{P}g(z) := \sum_{n=0}^{\infty} < g, \mathfrak{e}_n > \mathfrak{e}_n(z)$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \int_{\mathbb{D}} g(w) \overline{w}^n \, dA(w) \frac{z^n}{n+1} = \frac{1}{\pi} \int_{\mathbb{D}} \frac{g(w)}{(1 - z\overline{w})^2} \, dA(w)$$

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The interchange of the integration and summation is justified because the sum $\sum_{n=0}^{\infty} \frac{\overline{w}^n z^n}{n+1}$ converges uniformly on $\mathbb{D}$ for each fixed z.

Grunsky kernel

Let $\psi \in \Sigma$. Its Grunsky decomposition:

$$\log\left(\frac{\psi(z)-\psi(w)}{z-w}\right) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_{nk} z^{-n} w^{-k}.$$

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We can assume $\psi \in \Sigma_0$, so consider

$$\phi(z) := \frac{1}{\psi(1/z)} \in S$$

Then

$$\log\left(\frac{\phi(z)-\phi(w)}{z-w}\right) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_{nk} z^{n} w^{k} + \log\frac{\phi(z)}{z} + \log\frac{\phi(w)}{w}.$$

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Now differentiate to get the *Grunsky kernel*:

$$\Phi(z,w) := \tfrac{\partial^2}{\partial z \partial w}\log\left(\tfrac{\phi(z)-\phi(w)}{z-w}\right) =$$

$$\frac{\phi'(z)\phi'(w)}{(\phi(z)-\phi(w))^2} - \frac{1}{(z-w)^2} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} nk\gamma_{nk} z^{n-1} w^{k-1}$$

Grunsky operator

Grunsky operator is an anti-linear operator on $L^2(\Omega)$ with kernel $\Phi(z,w)$:

$$\overline{\Gamma}_\phi f(z) := \frac{1}{\pi}\int_{\mathbb{D}} \left(\frac{\phi'(z)\phi'(w)}{(\phi(z)-\phi(w))^2} - \frac{1}{(z-w)^2} \right) \overline{f}(w)\, dA(w).$$

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Observe

$$\overline{\Gamma}_\phi \mathfrak{e}_{m-1} = -\frac{1}{\pi\sqrt{\pi m}} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \int_{\mathbb{D}} nk\gamma_{nk} z^{n-1} w^{k-1} \overline{w}^{m-1}\, dA(w) =$$
$$- \frac{1}{\pi\sqrt{\pi m}} \sum_{n=1}^{\infty} m\pi n \gamma_{nm} z^{n-1} = -\sum_{n=1}^{\infty} \sqrt{nm}\gamma_{nm}\mathfrak{e}_{n-1}$$

So $\overline{\Gamma}_\phi$ maps $\mathcal{A}^2(\mathbb{D})$ to $\mathcal{A}^2(\mathbb{D})$, with the Grunsky matrix.

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Grunsky inequalities are equivalent to the fact that $\overline{\Gamma}_\phi$ *is a contraction for all* $\phi \in S$, *i.e.* $\|\overline{\Gamma}_\phi f\|_2 \leq \|f\|_2$.
Equivalently, a linear version $\Gamma_\phi f := \overline{\Gamma}_\phi \overline{f}$ is a contraction from $\overline{\mathcal{A}}^2(\mathbb{D})$ to $\mathcal{A}^2(\mathbb{D})$.

Cauchy and Beurling transforms

Let $f \in C_0^\infty(\mathbb{C})$. Its *Cauchy transform:*

$$(\mathfrak{C}f)(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{z-w}\, dA(w).$$

By Green, $(\mathfrak{C}f_{\overline{z}}) = (\mathfrak{C}f)_{\overline{z}} = f$. Holds in weak sense for any distribution.

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Beurling transform of f:

$$(\mathfrak{B} f)(z) := \frac{\partial}{\partial z}(\mathfrak{C} f)(z) = (\mathfrak{C} f_z)(z) = \lim_{r \to 0} \frac{1}{\pi} \int_{|w-z|>r} \frac{f_z(w)}{z - w}\, dA(w) \overset{Green}{=}$$

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The fundamental property: $\mathfrak{B}f_{\overline{z}} = f_z$, since

$$\mathfrak{B}\circ\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z}\circ\mathfrak{C}\circ\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z}.$$

Properties of Beurling transform

$\mathfrak{B}$ is symmetric:

$$\int_{\mathbb{C}} f(z)(\mathfrak{B}g)(z)\,dA(z) = \lim_{r\to 0} \frac{-1}{\pi} \int_{\mathbb{C}} \int_{|w-z|>r} \frac{f(z)g(w)}{(z-w)^2}\,dA(w)\,dA(z) =$$

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It is unitary for g of the form $g = f_{\bar{z}}$, $f \in C_0^\infty$:

$$\int_{\mathbb{C}} |g(z)|^2\,dA(z) = \int_{\mathbb{C}} f_{\bar{z}}(z)\overline{f_{\bar{z}}}(z)\,dA(z) =$$
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Since functions of the form f_z, $f \in C_0^\infty$ and of the form $f_{\bar{z}}$, $f \in C_0^\infty$ are dense in $L^2(\mathbb{C})$ (the functions orthogonal to all of them must be anti-entire or entire correspondingly), $\mathfrak{B}$ extends to a unitary operator on $L^2(\mathbb{C})$.

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In terms of Fourier transform: $\widehat{(\mathfrak{B}f)}(\zeta) = \frac{\zeta}{\bar{\zeta}}\hat{f}(\zeta)$.

By the standard Calderon-Zygmund theory, $\mathfrak{B} : L^p \mapsto L^p$, $1 < p < \infty$.

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By scaling and translation, if $f(z) := \mathbb{1}_{\mathbb{D}(a,r)}$:

$$\mathfrak{B} f(z) = -\frac{r^2}{(z-a)^2} \mathbb{1}_{\mathbb{C} \setminus \mathbb{D}(a,r)}.$$

Pointwise existence of Beurling transform: following Mateu and Verdera.

For $f \in L^p(\mathbb{C})$ $\infty > p > 1$, $\mathfrak{B}f \in L^p(\mathbb{C})$ is defined a.e. But is it equal to the principal value of the integral?

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By the previous observation:

$$\frac{-1}{\pi}\int_{|w-z|>r} \frac{f(w)}{(z-w)^2}\, dA(w) = \frac{1}{\pi r^2}\int_{\mathbb{C}} f(w)(\mathfrak{B}\mathbb{1}_{\mathbb{D}(z,r)})(w)\, dA(w) \overset{\text{by symmetry}}{=} \frac{1}{\pi r^2}\int_{\mathbb{C}}(\mathfrak{B}f)(w)\mathbb{1}_{\mathbb{D}(z,r)}(w)\, dA(w) = \frac{1}{\text{Area}(\mathbb{D}(z,r))}\int_{\mathbb{D}(z,r)} \mathfrak{B}f(w)\, dA(w)$$

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So

$$\lim_{r\to 0} \frac{-1}{\pi}\int_{|w-z|>r} \frac{f(w)}{(z-w)^2}\, dA(w) = \mathfrak{B}f(z)$$

for all Lebesgue points of $\mathfrak{B}f$.

The transferred Beurling transform

Let $\Omega \subset \mathbb{C}$ be a domain. For $f \in L^p(\Omega)$, the *Restricted Beurling transform* of f is:

$$\mathfrak{B}_\Omega f(z) := \frac{-1}{\pi} \,\mathrm{p.v.} \int_\Omega \frac{f(w)}{(z-w)^2}\, dA(w) = \mathbb{1}_\Omega(z) \mathfrak{B} f(z).$$

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If Ω is a simply-connected domain, let $\phi : \mathbb{D} \mapsto \Omega$ be its Riemann map. Let the isometry $T^p : L^p(\Omega) \mapsto L^p(\mathbb{D})$ (and $\mathcal{A}^p(\Omega) \mapsto \mathcal{A}^p(\mathbb{D})$) be defined by

$$T^p_\phi f(z) := \phi'(z)^{2/p} f(\phi(z)).$$

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If Ω is a simply-connected domain, let $\phi : \mathbb{D} \mapsto \Omega$ be its Riemann map. Let the isometry $T^p : L^p(\Omega) \mapsto L^p(\mathbb{D})$ (and $\mathcal{A}^p(\Omega) \mapsto \mathcal{A}^p(\mathbb{D})$) be defined by

$$T^p_\phi f(z) := \phi'(z)^{2/p} f(\phi(z)).$$

For $g \in L^p(\mathbb{D})$, define the *transferred Beurling transform*:

$$\mathfrak{B}^p_\phi g := T^p_\phi \circ \mathfrak{B}_\Omega \circ \left(T^p_\phi\right)^{-1} \circ \mathfrak{M}_h g,$$

The transferred Beurling transform

Let $\Omega \subset \mathbb{C}$ be a domain. For $f \in L^p(\Omega)$, the *Restricted Beurling transform* of f is:

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The integral form of transferred Beurling transform and Grunsky inequality.

In the integral form

$$\left(\mathfrak{B}_\phi^p g\right)(z) = \frac{1}{\pi}\,\text{p. v.}\int_{\mathbb{D}} \frac{\phi'(z)^{2/p}\phi'(w)^{2-2/p}}{(\phi(z)-\phi(w))^2} g(w)\,dA(w).$$

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Collect what we know about $p = 2$:

$$(\Gamma_{\phi} g)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \left(\frac{\phi'(z)\phi'(w)}{(\phi(z) - \phi(w))^2} - \frac{1}{(z - w)^2} \right) g(w) \, dA(w)$$

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But on $\overline{\mathcal{A}}^2(\mathbb{D})$, $\mathfrak{B}^2_{\mathbb{D}} \equiv 0$, so for $g \in \overline{\mathcal{A}}^2(\mathbb{D})$:

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Since $\mathfrak{B}_\phi^2$ is a contraction, it proves the Grunsky inequality.

Generalized Grunsky inequality: transferred Beurling.

Essentially the same, but we need two univalent maps $\psi : \mathbb{C} \setminus \mathbb{D} \mapsto \Omega_-$, $\phi : \mathbb{D} \mapsto \Omega_+$, with $\Omega_- \cap \Omega_+ = \emptyset$. $\Omega := \Omega_- \cup \Omega_+$. Then define

$$T^p_{\phi,\psi} f(z) := \begin{cases} \phi'(z)^{2/p} f(\phi(z)) & |z| < 1 \\ \psi'(z)^{2/p} f(\psi(z)) & |z| > 1 \end{cases}.$$

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And, in integral form

$$\left(\mathfrak{B}^p_{\phi,\psi} g\right)(z) = \frac{1}{\pi} \, \mathrm{p.\,v.} \int_{\mathbb{D}} \frac{\phi'(z)^{2/p} \phi'(w)^{2-2/p}}{(\phi(z) - \phi(w))^2} g(w) \, dA(w) + \frac{1}{\pi} \int_{\mathbb{C}\setminus\mathbb{D}} \frac{\phi'(z)^{2/p} \psi'(w)^{2-2/p}}{(\phi(z) - \psi(w))^2} g(w) \, dA(w); \quad |z| < 1.$$

and similar expression for $|z| > 1$.

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and similar expression for $|z| > 1$. A compression for $p = 2$.

Generalized Grunsky inequality.

Generalized Grunsky:

$$\log\left(\frac{\psi(z)-\psi(w)}{z-w}\right) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_{-n,-k}z^{-n}w^{-k}.$$

$$\log\left(\frac{\phi(z)-\phi(w)}{z-w}\right) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_{n,k}z^{n}w^{k} + \log\frac{\phi(z)}{z} + \log\frac{\phi(w)}{w}.$$

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By differentiating $\partial_z\partial_w$ and comparing we see that the operator

$$\overline{\Gamma}_{\phi,\psi} := \mathfrak{B}^2_{\phi,\psi} - \mathfrak{B}^2_{\mathbb{D}} - \mathfrak{B}^2_{\mathbb{C}\setminus\mathbb{D}}$$

has the generalized Grunsky matrix on $\overline{\mathcal{A}}(\mathbb{D}) \oplus \overline{\mathcal{A}}(\mathbb{C}\setminus\mathbb{D})$.

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Grunsky identity

Let $f \in \mathcal{A}^1(\mathbb{D})$. Then

$$(\Im f)(w) := \frac{1}{\pi} \int_{\mathbb{D}} \frac{w}{1-\bar{\zeta} w} f(\zeta)\, dA(\zeta) = \int_{[0,w]} f(\zeta)\, d\zeta$$

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Indeed,

$$(\mathfrak{I}f)'(w) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{1}{((1-\bar{\zeta}w))^2} f(\zeta) dA(\zeta) = (\mathcal{P}f)(w) = f(w)$$

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For a conformal ϕ, apply it to $f(w) := \frac{\phi'(z)\phi'(w)}{\phi(z)-\phi(w)} - \frac{1}{z-w}$:

$$\log\left(\frac{z(\phi(z)-\phi(w))}{(z-w)\phi(z)}\right) = (\mathfrak{I}f)(w) =$$

$$\frac{1}{\pi}\int_{\mathbb{D}} \left(\frac{\phi'(z)\phi'(\zeta)}{\phi(z)-\phi(\zeta)} - \frac{1}{z-\zeta}\right) \frac{w}{1-\bar{\zeta}w}\, dA(\zeta)$$

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For $f(\zeta) := \log(1-\overline{\zeta}w)$, $f(0)=0$, $f_\zeta = 0$, $f_{\overline{\zeta}} = -\frac{w}{1-\overline{\zeta}w}$, so, by generalized Cauchy:

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Apply $\partial_w \partial_z$ and apply to a test function η to get

$$\text{p. v.} \int_{\mathbb{D}} \left(\frac{\phi'(z)\phi'(w)}{(\phi(z) - \phi(w))^2} - \frac{1}{(z-w)^2} \right) \eta(w) \, dA(w) =$$
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or $\Gamma_\phi = \mathfrak{B}^2_\phi - \mathfrak{B}^2_{\mathbb{D}} = \mathfrak{B}^2_\phi\overline{\mathcal{P}}$ – a contraction on $L^2(\mathbb{D})$, maps it to $\mathcal{A}^2(\mathbb{D})$!

Skewed Grunsky identities

Expand p-Beurling kernel near diagonal:

$$\frac{\phi'(z)^{2/p}\phi'(w)^{2-2/p}}{(\phi(w)-\phi(z))^2} - \frac{1}{(z-w)^2} + \left(\frac{2}{p}-1\right)\frac{\phi''(w)}{(w-z)\phi'(w)} = O(1)$$

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So it is an analytic kernel, and for any $f \in L^2(\mathbb{D})$,

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So it does not change under projection to analytic functions:

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In the case $p=2$, it gives a version of Grusky identity:

$$\mathfrak{B}^2_\phi - \mathfrak{B}_\mathbb{D} = \mathcal{P}\mathfrak{B}^2_\phi$$

Grunsky operator in the domain Ω.

$$\overline{\Gamma}_\Omega f := \mathfrak{B}_\Omega \overline{f} = \frac{-1}{\pi}\,\mathrm{p.\,v.}\int_\Omega \frac{\overline{f}(w)}{(z-w)^2}\,dA(w)$$

is an anti-linear contraction of $\mathcal{A}^2(\Omega)$.

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Since $\mathfrak{B}_\mathbb{D} \overline{T_\phi^2 f} = 0$ for $f \in \mathcal{A}^2(\Omega)$, we have

$$\int_\Omega L_\Omega(z,w)\overline{f}(w)\, dA(w) \equiv 0$$

Grunsky operator in the domain Ω.

$$\overline{\Gamma}_\Omega f := \mathfrak{B}_\Omega \overline{f} = \frac{-1}{\pi}\,\mathrm{p.\,v.}\int_\Omega \frac{\overline{f}(w)}{(z-w)^2}\,dA(w)$$

is an anti-linear contraction of $\mathcal{A}^2(\Omega)$.
Let

$$L_\Omega(z,w) := \frac{-2}{\pi}\frac{\partial^2 G_\Omega(z,w)}{\partial z \partial w} = \frac{1}{\pi(z-w)^2} - l_\Omega(z,w)$$

Since $\mathfrak{B}_\mathbb{D}\overline{T_\phi^2 f} = 0$ for $f \in \mathcal{A}^2(\Omega)$, we have

$$\int_\Omega L_\Omega(z,w)\overline{f}(w)\,dA(w) \equiv 0$$

which means that on $\mathcal{A}^2(\Omega)$,

$$\overline{\Gamma}_\Omega f = \int_\Omega l_\Omega(z,w)\overline{f}(w)\,dA(w)$$