Quasidisk geometry and the Grunsky operator (continued)

Tim Mesikepp

Bozeman virtual summer school 2020

August 13, 2020

Philosophy/big picture

$f \in \Sigma$, $f(z) = z + a_0 + a_1 z^{-1} + \cdots$, $z \to \infty$

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{m,n=1}^{\infty} b_{mn} z^{-m} \zeta^{-n}$$

These Grunsky coefficients $(b_{mn})_{m,n=1}^{\infty}$ give rise to the natural operator

$$x = (x_m)_{m=1}^{\infty} \mapsto Gx := \left(\sum_{n=1}^{\infty} \sqrt{mn}\, b_{mn} x_n \right)_{m=1}^{\infty}$$

I.e. "matrix multiplication" of the infinite, symmetric matrix $(\sqrt{mn}\, b_{mn})_{m,n=1}^{\infty}$ by the infinite column vector $(x_m)_{m=1}^{\infty} \in \ell^2$.

Philosophy/big picture

We continue to explore the idea of how the geometry of $E := \mathbb{C}\backslash f(\mathbb{D}^*)$ translates into operator-theoretic properties of the Grunsky operator, working "in coordinates" (b_{mn}).

We've already seen:

- Peter: G is unitary iff $\text{Area}(E) = 0$ (both directions hold). G is a strict contraction $\|Gx\| < \|x\|$ iff $\text{Area}(E) > 0$.
- Steffen: $\|G\| < 1$ iff E is a quasidisk.

Philosophy/big picture

We continue to explore the idea of how the geometry of $E := \mathbb{C} \backslash f(\mathbb{D}^*)$ translates into operator-theoretic properties of the Grunsky operator, working "in coordinates" (b_{mn}).

We've already seen:

- Peter: G is unitary iff $\text{Area}(E) = 0$ (both directions hold). G is a strict contraction $\|Gx\| < \|x\|$ iff $\text{Area}(E) > 0$.
- Steffen: $\|G\| < 1$ iff E is a quasidisk.
- Our goal:

Theorem (Shen, TT)

G is a compact operator iff E is an asymptotically-conformal quasidisk.

A more abstract formulation:

Note we have a map

$$\mathscr{G} : \Sigma \to \overline{B_1}(\mathcal{B}(\ell^2)).$$

Now,

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \log \frac{g(z) - g(\zeta)}{z - \zeta} \quad \Leftrightarrow \quad f(z) = g(z) + c,$$

so let's mod out by translation at ∞ to make $\mathscr{G}$ injective:

$$\Sigma_0 = \{\, f \in \Sigma \;:\; f(z) = z + a_1/z + \cdots, z \to \infty\},$$

Then, $\mathscr{G} : \Sigma_0 \hookrightarrow \overline{B_1}(\mathcal{B}(\ell^2))$.

Restricting the domain to $\bigcup_{\kappa<1} \Sigma_0(\kappa) = T(1)$ puts us in the realm of Teichmüller theory. TT considered this "period mapping"

$$\hat{\mathscr{P}} : T(1) \hookrightarrow B_1(\mathcal{B}(\ell^2))$$

and showed

Theorem (TT (Appendix B), Shen)

$\hat{\mathscr{P}}$ is a holomorphic inclusion of Banach manifolds.

See Shen's paper for details on what this means and a fairly simple proof (which uses some Teichmüller machinery).

Restricting the domain to $\bigcup_{\kappa<1} \Sigma_0(\kappa) = T(1)$ puts us in the realm of Teichmüller theory. TT considered this "period mapping"

$$\hat{\mathscr{P}} : T(1) \hookrightarrow B_1(\mathcal{B}(\ell^2))$$

and showed

Theorem (TT (Appendix B), Shen)

$\hat{\mathscr{P}}$ is a holomorphic inclusion of Banach manifolds.

See Shen's paper for details on what this means and a fairly simple proof (which uses some Teichmüller machinery).

Stated in this framework, our desired theorem becomes

$$\hat{\mathscr{P}}^{-1}(\mathcal{C}(\ell^2)) = \text{Asymptotically-conformal quasidisks}/\sim$$

As an aside, an interesting (and open?) question: characterize the range of

$$\hat{\mathscr{P}} : T(1) \to B_1(\mathcal{B}(\ell^2))$$

(Perhaps answering our question if $\|G\| = \kappa \Leftrightarrow f$ is $\frac{1+\kappa}{1-\kappa}$-QC would help.)

Overview

1. “Crash course” review of compact operators

2. G compact iff f asymptotically conformal

Bounded operators

Let A, B be normed vector spaces, $T : A \to B$ a linear map. Recall:

- T is *bounded* if there exists $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in A$.
- The *operator norm* of T is $\|T\| := \sup\{\, \|Tx\| \; : \; \|x\| = 1\,\}$
- Note $\|Tx\| \leq \|T\|\|x\|$ for all $x \in A$.
- Boundedness of T is equivalent to continuity of T.
- $\mathcal{B}(A, B) := \{T : A \to B \; : \; T \text{ linear, bounded}\}$ is itself a normed vector space, and, if B is Banach, so is $\mathcal{B}(A, B)$. $\mathcal{B}(A) := \mathcal{B}(A, A)$.

Compact operators

$T : A \to B$ is *compact* if the image (Tx_n) of any bounded sequence (x_n) in A has a convergent subsequence.

- Equivalently, the image $T(B_1) \subset B$ of the unit ball $B_1 \subset A$ is pre-compact (we're in a metric space, so compactness $\Leftrightarrow$ sequential compactness).
- Note that if T is compact, then T is bounded.
  - Otherwise, find a sequence (x_n) with $\|x_n\| = 1$ such that $\|T(x_n)\| \to \infty$. Contradicts subsequential convergence.

Compact operators: examples

Let's build our intuition with several simple examples.

1. $T \in \mathcal{B}(A, B)$ with finite rank is compact.
   - Suppose $T(A) \subset F$, a finite-dimensional subspace.
   - Closed, bounded sets are compact in F, since they are in Euclidean spaces, and any two norms on a finite-dimensional normed space are equivalent.
   - Now, (x_n) bounded $\Rightarrow$ (Tx_n) bounded, and the closure of (Tx_n) is compact.
   - In particular, all linear operators on finite-dimensional vector spaces (matrices) are compact.

Compact operators: examples

2. The identity map $I : H \to H$ on a infinite-dimensional Hilbert space is not compact. (The unit ball in H is not pre-compact: the sequence of basis vectors (e_n) has no Cauchy subsequence.)

Compact operators: examples

2. The identity map $I : H \to H$ on a infinite-dimensional Hilbert space is not compact. (The unit ball in H is not pre-compact: the sequence of basis vectors (e_n) has no Cauchy subsequence.)

3. $T \in \mathcal{B}(H)$, $T = \mathrm{diag}(\lambda_1, \lambda_2, \dots)$, i.e.

$$T(e_n) = \lambda_n e_n$$

for a basis (e_n) for H. Then T is compact iff $\lambda_n \to 0$.

- Hint for $\Leftarrow$: Use truncation operators $T_N(\sum_{n=1}^{\infty} \alpha_n e_n) := \sum_{n=1}^{N} \alpha_n \lambda_n e_n$. T_N compact, argue $\|T_N - T\| \to 0$, and use the next theorem.

Compact operators

$$\mathcal{C}(A, B) := \{T \in \mathcal{B}(A, B) \ : \ T \text{ compact}\}.$$

Theorem

If A and B are Banach, $\mathcal{C}(A, B)$ is a closed subspace of $\mathcal{B}(A, B)$ with respect to the operator topology.

Proof sketch: Easy to see $\mathcal{C}(A, B)$ is a subspace. Closed? Diagonalization $+ \ \epsilon/3$ argument.

Diagonalization argument

We have (T_n) compact, $\|T_n - T\| \to 0$.

Let $(x_n) \subset A$ with $\|x_n\| \le M$. We want to find $(x_{n_k}) \subset (x_n)$ such that (Tx_{n_k}) converges.

- Extract a subsequence $(x_n^1) \subset (x_n)$ for which $(T_1 x_n^1)$ converges.
- Extract a further subsequence $(x_n^2) \subset (x_n^1)$ for which $(T_2 x_n^2)$ converges.
- Continue in this manner. Then the "diagonal" sequence (x_n^n) is such that $(T_m x_n^n)_{n=1}^{\infty}$ converges for every m.

Show (Tx_n^n) converges.

$\epsilon/3$ argument

$$\begin{aligned}\|Tx_n^n - Tx_m^m\| &\leq \|(T - T_p)x_n^n\| + \|T_px_n^n - T_px_m^m\| + \|(T - T_p)x_m^m\| \\ &\leq \|T - T_p\|\, M + \|T_px_n^n - T_px_m^m\| + \|T - T_p\|\, M \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\end{aligned}$$

for all $n, m > N = N(p)$. $\square$

A last observation: T^*T

- Note that $\mathcal{C}(A)$ is a two-sided ideal of $\mathcal{B}(A)$: For $T \in \mathcal{C}(A)$ and $S \in \mathcal{B}(A)$,

$$\begin{aligned} Tx_{n_k} \to y \quad &\Rightarrow \quad STx_{n_k} \to Sy, \\ \|x_n\| \leq M \quad &\Rightarrow \quad \|Sx_n\| \leq \|S\| \, \|x_n\| = M'. \end{aligned}$$

So both $ST, TS \in \mathcal{C}(A)$.

A last observation: T^*T

- Note that $\mathcal{C}(A)$ is a two-sided ideal of $\mathcal{B}(A)$: For $T \in \mathcal{C}(A)$ and $S \in \mathcal{B}(A)$,

$$\begin{aligned} Tx_{n_k} \to y \quad &\Rightarrow \quad STx_{n_k} \to Sy, \\ \|x_n\| \leq M \quad &\Rightarrow \quad \|Sx_n\| \leq \|S\| \, \|x_n\| = M'. \end{aligned}$$

  So both $ST, TS \in \mathcal{C}(A)$.

- In particular, $T \in \mathcal{C}(A) \Rightarrow T^*T \in \mathcal{C}(A)$. Thus T^*T is a compact, self-adjoint operator. For $T \in \mathcal{C}(H)$, a separable Hilbert space, the *spectral theorem* for such operators states that there exists a complete orthonormal sequence (e_n) such that $T^*Te_n = \lambda_n e_n$.
  - Note $\lambda_n \to 0$.

2. G compact iff f asymptotically conformal

2. G compact iff f asymptotically conformal

Asymptotic conformality is any of the following equivalent properties:

Theorem (Becker-Pommerenke '78, Gardiner-Sullivan '92)

Let $f \in \Sigma$ such that $E = \mathbb{C} \backslash f(\mathbb{D}^*)$ is a Jordan domain. TFAE:

1. *$f(\mathbb{S}^1)$ is a Jordan curve $J \subset \mathbb{C}$ satisfying*

$$\max_{w \in J(a,b)} \frac{|a-w| + |w-b|}{|a-b|} \to 1 \quad \text{as } a, b \in J, \ |a-b| \to 0,$$

   where $J(a,b)$ is the arc of J of smaller diameter between a and b.

2. G compact iff f asymptotically conformal

Asymptotic conformality is any of the following equivalent properties:

Theorem (Becker-Pommerenke '78, Gardiner-Sullivan '92)

Let $f \in \Sigma$ such that $E = \mathbb{C} \backslash f(\mathbb{D}^*)$ is a Jordan domain. TFAE:

1. *$f(\mathbb{S}^1)$ is a Jordan curve $J \subset \mathbb{C}$ satisfying*

$$\max_{w \in J(a,b)} \frac{|a-w|+|w-b|}{|a-b|} \to 1 \quad \text{as } a, b \in J,\ |a-b| \to 0,$$

   where $J(a,b)$ is the arc of J of smaller diameter between a and b.

2. *The welding homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ of $f(\mathbb{S}^1)$ is "symmetric," i.e. its lift $\hat{\varphi} : \mathbb{R} \to \mathbb{R}$ defined via $\varphi(e^{2\pi i x}) = e^{2\pi i \hat{\varphi}(x)}$ satisfies*

$$\lim_{t \to 0} \frac{\hat{\varphi}(x+t) - \hat{\varphi}(x)}{\hat{\varphi}(x) - \hat{\varphi}(x-t)} = 1 \qquad \text{uniformly in } x \in \mathbb{R}$$

Theorem (asymptotic conformality equivalences, cont)

3. $(1-|z|^2)\frac{f''(z)}{f'(z)} \to 0$ *as* $|z| \to 1^+$.

4. $(1-|z|^2)^2 S(f)(z) \to 0$ *as* $|z| \to 1^+$.

5. $f \in \bigcup_{\kappa<1} \Sigma(\kappa)$ *where* $\mu := \overline{\partial} f / \partial f$ *satisfies* $\mu(z) \to 0$ *as* $|z| \to 1^-$.

Lemma: Grunsky coefficients and the Schwarzian

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{m,n=1}^{\infty} b_{mn} z^{-m} \zeta^{-n},$$

and so

$$\begin{aligned} \frac{\partial^2}{\partial z \partial \zeta} \log \frac{f(z) - f(\zeta)}{z - \zeta} &= \frac{f'(z) f'(\zeta)}{(f(z) - f(\zeta))^2} - \frac{1}{(z - \zeta)^2} \\ &= -\sum_{m,n=1}^{\infty} mn b_{mn} z^{-m-1} \zeta^{-n-1}. \end{aligned}$$

Now let $\zeta \to z$. Taylor expand $f(\zeta)$ and $f'(\zeta)$ at $\zeta = z$, and be careful. You get:

Lemma: Grunsky coefficients and the Schwarzian

Lemma

$$-\frac{1}{6}\mathcal{S}(f)(z) = \sum_{m,n=1}^{\infty} mnb_{mn}z^{-(m+n+2)}$$

Proof $\Rightarrow$

We present Shen's argument.

Suppose first that G is compact. We show

$$(1-|z|^2)^2 S(f)(z) \to 0 \quad \text{as} \quad |z| \to 1^+.$$

Idea: build a cleverly-chosen sequence $x(z) = (x_m(z)) \in B_1(\ell^2)$ for each $z \in \mathbb{D}^*$ such that $\langle Gx(z), \overline{x(z)}\rangle$ gives a multiple of $(1-|z|^2)^2 S(f)(z)$, but where we know, via compactness, that $\langle Gx(z), \overline{x(z)}\rangle \to 0$ as $|z| \to 1^+$.

Proof $\Rightarrow$

For $z \in \mathbb{D}^*$, define

$$x(z) = (x_m(z))_{m=1}^{\infty} = \left(\frac{1-|z|^2}{|z|} \cdot \frac{\sqrt{m}}{z^m} \right)_{m=1}^{\infty}. \tag{1}$$

Claim 1: $\|x(z)\| = 1$.

Proof $\Rightarrow$

For $z \in \mathbb{D}^*$, define

$$x(z) = (x_m(z))_{m=1}^{\infty} = \left(\frac{1-|z|^2}{|z|} \cdot \frac{\sqrt{m}}{z^m} \right)_{m=1}^{\infty}. \tag{1}$$

Claim 1: $\|x(z)\| = 1$.
Indeed, $\sum_{m=1}^{\infty} m x^m = \frac{x}{(1-x)^2}$ for $|x| < 1$. Therefore,

$$\begin{aligned} \|x(z)\|^2 &= \frac{(1-|z|^2)^2}{|z|^2} \sum_{m=1}^{\infty} m \left(\frac{1}{|z|^2} \right)^m \\ &= \frac{(1-|z|^2)^2}{|z|^2} \cdot \frac{1/|z|^2}{(1-1/|z|^2)^2} = 1, \end{aligned}$$

as claimed.

Proof $\Rightarrow$

Claim 2: $x(z) \rightharpoonup 0$ as $|z| \to 1^+$. I.e. for any fixed $y \in \ell^2$,

$$\langle x(z), y\rangle \to 0 \quad \text{as} \quad |z| \to 1^+$$

Proof $\Rightarrow$

Claim 2: $x(z) \rightharpoonup 0$ as $|z| \to 1^+$. I.e. for any fixed $y \in \ell^2$,

$$\langle x(z), y\rangle \to 0 \quad \text{as} \quad |z| \to 1^+$$

Indeed, choose M such that $\left(\sum_{m=M+1}^{\infty} |y_m|^2\right)^{1/2} < \epsilon/2$. Since for each fixed m we have $x_m(z) \to 0$ as $|z| \to 1^+$,

$$\begin{aligned}\langle x(z), y\rangle &= \sum_{m=1}^{M} x_m(z)\overline{y}_m + \sum_{m=M+1}^{\infty} x_m(z)\overline{y}_m \\ &\leq \frac{\epsilon}{2} + 1 \cdot \Big(\sum_{m=M+1}^{\infty} |y_m|^2\Big)^{1/2} < \epsilon\end{aligned}$$

for all $|z|$ close to 1^+, as claimed.

Proof $\Rightarrow$

Claim 3: $Gx(z) \to 0$ as $|z| \to 1^+$.

Proof $\Rightarrow$

Claim 3: $Gx(z) \to 0$ as $|z| \to 1^+$.

Indeed, along any sequence $(z_n) \subset \mathbb{D}^*$ with $|z_n| \to 1^+$, we have that $(Gx(z_n))$ has a convergent subsequence $Gx(z_{n_k}) \to y \in \ell^2$ as $k \to \infty$. Therefore

$$\langle Gx(z_{n_k}), y \rangle \to \langle y, y \rangle = \|y\|^2 ,$$

while simultaneously

$$\langle Gx(z_{n_k}), y \rangle = \langle x(z_{n_k}), G^* y \rangle \to 0,$$

showing $y = 0$.

Proof $\Rightarrow$

Follows that any subsequence of $(Gx(z_n))$ has a further subsequence which converges to 0, and hence $Gx(z_n)$ itself converges to 0.

Since this holds for *any* sequence $(x(z_n))$ where $|z_n| \to 1^+$, we conclude $Gx(z) \to 0$ as $|z| \to 1^+$, as claimed.

In particular, $\langle Gx(z), \overline{x(z)} \rangle \to 0$ as $|z| \to 1^+$.

Proof $\Rightarrow$

Recall $Gx := \left(\sum_{n=1}^{\infty} \sqrt{mn} b_{mn} x_n\right)_{m=1}^{\infty}$, and so

$$\langle Gx, \overline{x} \rangle = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{mn} b_{mn} x_n x_m.$$

In particular, for $x(z) = \left(\frac{1-|z|^2}{|z|} \cdot \frac{\sqrt{m}}{z^m}\right)_m$,

$$\begin{aligned}
\langle Gx(z), \overline{x(z)} \rangle &= \frac{(1-|z|^2)^2}{|z|^2} \sum_{m,n=1}^{\infty} mn b_{mn} z^{-(n+m)} \\
&= \left(\frac{z}{|z|}\right)^2 (1-|z|^2)^2 \sum_{m,n=1}^{\infty} mn b_{mn} z^{-(m+n+2)} \\
&= -\frac{1}{6}\left(\frac{z}{|z|}\right)^2 (1-|z|^2)^2 S(f)(z) \to 0.
\end{aligned}$$

Proof $\Rightarrow$

So

$$|(1-|z|^2)^2 S(f)(z)| \to 0$$

as $|z| \to 1^+$, yielding asymptotic conformality.

Proof $\Rightarrow$

So

$$|(1-|z|^2)^2 S(f)(z)| \to 0$$

as $|z| \to 1^+$, yielding asymptotic conformality.

Q: If we merely assume that $\|G\| \leq \kappa$, can we use an argument like this with the pre-schwarzian $\mathcal{P}(f)$ or $\mathcal{S}(f)$ for $f \in \bigcup_{\kappa<1} \Sigma(\kappa)$?

Proof $\Leftarrow$

Suppose f is asymptotically conformal. Idea: Work with continuity of μ at $\partial\mathbb{D}$ to build a sequence of "truncated operators", each of which is clearly compact, which converge to G (in operator norm). Will need to borrow from:

- Yilin's talk tomorrow.
- Teichmüller theory

Proof $\Leftarrow$

Truncations: For $0 < r < 1$, set

$$\mu_r(z) := \chi_{r\mathbb{D}}(z)\mu(z).$$

Then the associated conformal maps f_{μ_r} are conformal on the larger region $\{|z| > 1 - r\} \supset \mathbb{D}^*$, and now the quasicircles $f_{\mu_r}(\mathbb{S}^1)$ are actually analytic Jordan curves.

Proof $\Leftarrow$

Truncations: For $0 < r < 1$, set

$$\mu_r(z) := \chi_{r\mathbb{D}}(z)\mu(z).$$

Then the associated conformal maps f_{μ_r} are conformal on the larger region $\{|z| > 1 - r\} \supset \mathbb{D}^*$, and now the quasicircles $f_{\mu_r}(\mathbb{S}^1)$ are actually analytic Jordan curves.

From Yilin's talk tomorrow, the operators $G_r := G(f_{\mu_r})$ are Hilbert-Schmidt, and hence compact.

Suffices to show $\|G_r - G\| \to 0$ as $r \to 1^-$.

Aside on Teichmüller theory

Recall that one model of the universal Teichmüller space $T(1)$ is

$$\begin{aligned} T(1) :=& B_1(L^\infty(\mathbb{D}))/\sim \\ &= \{\, \mu \in L^\infty(\mathbb{D}) \;:\; \text{ess sup}_{z\in\mathbb{D}}|\mu(z)| < 1 \,\}/\sim, \end{aligned}$$

where $\sim$ is as follows:

For $\mu \in B_1(L^\infty(\mathbb{D}))$, extend μ to zero in $\mathbb{D}^*$, and solve the Beltrami equation to obtain f_μ conformal on $\mathbb{D}^*$ with hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$. Then $\mu \sim \nu$ iff $f_\mu|_{\mathbb{D}^*} \equiv f_\nu|_{\mathbb{D}^*}$.

Aside on Teichmüller theory

The *Teichmüller distance* between points $[\mu], [\nu] \in T(1)$ is

$$\tau([\mu],[\nu]) := \inf \left\{ \frac{1}{2} \log \frac{1 + \left\| \frac{\mu_1 - \nu_1}{1 - \overline{\mu}_1 \nu_1} \right\|_{L^\infty(\mathbb{D})}}{1 - \left\| \frac{\mu_1 - \nu_1}{1 - \overline{\mu}_1 \nu_1} \right\|_{L^\infty(\mathbb{D})}} \ : \ \mu \sim \mu_1, \ \nu \sim \nu_1 \right\}.$$

Aside on Teichmüller theory

The *Teichmüller distance* between points $[\mu], [\nu] \in T(1)$ is

$$\tau([\mu],[\nu]) := \inf\left\{ \frac{1}{2}\log\frac{1+\left\|\frac{\mu_1-\nu_1}{1-\overline{\mu}_1\nu_1}\right\|_{L^\infty(\mathbb{D})}}{1-\left\|\frac{\mu_1-\nu_1}{1-\overline{\mu}_1\nu_1}\right\|_{L^\infty(\mathbb{D})}} \ : \ \mu\sim\mu_1,\ \nu\sim\nu_1 \right\}.$$

Claim: $\tau([\mu],[\mu_r]) \to 0$. Easy to see:

$$\left\|\frac{\mu-\mu_r}{1-\overline{\mu}\mu_r}\right\|_{L^\infty(\mathbb{D})} = \text{ess sup}_{|z|\geq r}|\mu(z)| \to 0$$

as $r \to 1^-$, by asymptotic conformality.

Note that the Grunsky operator is well-defined on universal Teichmüller space via $G([\mu]) = G(f_\mu|_{\mathbb{D}^*})$.

Recall that this period map $\hat{\mathscr{P}} : T(1) \to \mathcal{B}(\ell^2)$ is actually *holomorphic*. In particular, it is continuous, and so

$$\mu_r \to \mu \qquad \Rightarrow \qquad G_r \to G$$

in $\mathcal{B}(\ell^2)$. Thus G is compact, as claimed. $\square$

Summary of equivalences

Geometry	Area$(E) = 0$	E quasidisk	E asymptotically-conformal quasidisk
Conformal map		QC extension to $\hat{\mathbb{C}}$	$(1-\|z\|^2)^2 S(f)(z) \to 0,\ \|z\| \to 1^+$
Welding		Quasisymmetric	Symmetric
Dilitation		Exists, $\\|\mu\\|_{L^\infty(\mathbb{C})} < 1$	$\|\mu(z)\| \to 0$ as $\|z\| \to 1^-$
Operator	G unitary	$\\|G\\| < 1$	G compact

Some questions:

- What would the operator property be for asymptotically-smooth quasidisks (can we fill in that column)?
- What can we say about the spectrums of the G's in the various columns?

References

Gardiner, Sullivan (1992)
Symmetric Structures on a Closed Curve
American Journal of Mathematics

Pommerenke (1992)
Boundary Behaviour of Conformal Maps (§11.2)

Yuliang Shen (2007)
On Grunsky Operator
Science in China Series A: Mathematics

Takhtajan, Teo (2006)
Weil-Petersson Metric on the Universal Teichmüller Space
Memoirs of the AMS