Grunsky is Hilbert-Schmidt $\Leftrightarrow$ Weil-Petersson q.circle

I) Hilbert-Schmidt operators $\mathcal{B}(H,H)$

Let $(H, \langle \cdot \rangle)$ be a separable Hilbert space
$(e_n)_{n\geq 1}$ an orthonormal basis of H
$\langle x, x \rangle := \|x\|^2$

Definition 2.8. *An bounded operator $T : H \to H$ is Hilbert-Schmidt if it has finite Hilbert-Schmidt norm defined as*

$$\|T\|_{HS}^2 = \sum_{n\geq 1} \|Te_n\|_H^2 = \sum_{n\geq 1} \langle e_n, T^*Te_n \rangle = \mathrm{Tr}(T^*T). < \infty$$

In particular, T is Hilbert-Schmidt if and only if T^*T is of trace class.

Lemma 2.9. *If T is Hilbert-Schmidt, then T is compact.*

Proof. We show that T is limit (for the operator norm) of finite rank operators.

$\forall x \in H, \quad x = \sum_{n\geq 1} \langle x, e_n \rangle e_n \qquad Tx = \sum_{n\geq 1} \langle x, e_n \rangle Te_n$

with $\|x\|^2 = \sum_{n\geq 1} \langle x, e_n \rangle^2$

Consider the operator T_m :

$$T_m x = \sum_{n=1}^{m} \langle x, e_n \rangle Te_n$$

$$\Rightarrow \|Tx - T_m x\|^2 = \Big\| \sum_{n\geq m+1} \langle x, e_n \rangle Te_n \Big\|^2$$

$$\leq \Big(\sum_{n\geq m+1} |\langle x, e_n \rangle| \, \|Te_n\| \Big)^2$$

$$\leq \underbrace{\left(\sum_{n\geq m+1} |\langle x, e_n\rangle|^2\right)}_{\substack{\leq \\ \|x\|^2}} \underbrace{\left(\sum_{n\geq m+1} \|Te_n\|^2\right)}_{\substack{\downarrow\, m\to\infty \\ 0}}$$

$$\Rightarrow \quad \|T - T_m\| \xrightarrow[m\to\infty]{} 0 \qquad \blacksquare$$

T Hilbert-Schmidt $\Rightarrow$ T is compact $\Rightarrow$ T^*T compact and self adjoint

$\Rightarrow$ Diagonalizable w/ $\underbrace{\lambda_1^2 \geq \lambda_2^2 \geq \cdots}_{\text{spectrum of } T^*T} \geq 0$

Spectral theorem

($\lambda_i = \sqrt{\lambda_i^2} \geq 0$ is called the singular value of T)

$$\Rightarrow \|T\|_{HS}^2 = \sum_{n\geq 0} \lambda_n^2 < \infty$$

<u>Remark.</u> T is compact $\Leftrightarrow$ $\lambda_i \xrightarrow[i\to\infty]{} 0$.

<u>Remark</u> HS-norm is the p-Schatten norm with $p=2$ $\quad p\in[1,\infty)$

$$\|T\|_p := \left(\sum_{n\geq 1} \lambda_n^p\right)^{\frac{1}{p}}$$

II) Grunsky ⇔ Hilbert-Schmidt

$g'(\infty) = 1$ $\quad g : \mathbb{D}^* \longrightarrow \mathbb{C}$, $\quad g \in \bigcup_{k<1} \Sigma(k)$

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} b_{mn}(g) z^{-m} \zeta^{-n}. \qquad (*)$$

Our Grunsky operator G

$G_{mn} := \sqrt{mn}\, b_{mn}(g)$

(B_4 in Fredrik's talk. [TT])

$g \in \Sigma(k) \Leftrightarrow \|G\| \leq k$

Steffen's lecture $\Rightarrow$ $g(\mathbb{D}^*)$ is quasidisk

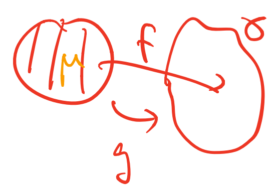
$g(S^1) := \gamma$

Tim: G compact $\Leftrightarrow$ γ is asymptotically conformal.

[TT] [Shen]

$\boxed{\text{Thm: } G \text{ is Hilbert-Schmidt} \Leftrightarrow \gamma \text{ is Weil-Petersson}}$

Theorem 3.8 (Equivalent characterization of Weil-Petersson quasicircles). *The curve* γ *is a* Weil-Petersson quasicircle *if and only if one of the following equivalent conditions holds:*



- $\iint_{\mathbb{D}} |\nabla \operatorname{Re}(\log f'(z))|^2 \,|\mathrm{d}z|^2 = \iint_{\mathbb{D}} |f''(z)/f'(z)|^2 \,|\mathrm{d}z|^2 < \infty$;
- $\iint_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \,|\mathrm{d}z|^2 < \infty$;
- $\iint_{\mathbb{D}} |\mathcal{S}(f)|^2 \rho^{-2}(z) |\mathrm{d}z|^2 < \infty$; $\quad$ $S(g)$ is Schwarzian of g
- ① $\iint_{\mathbb{D}^*} |\mathcal{S}(g)|^2 \rho^{-2}(z) |\mathrm{d}z|^2 < \infty$;
- ② *g has quasiconformal extension to $\mathbb{C}$, whose complex dilation $\mu = \bar\partial g / \partial g$, supported in $\mathbb{D}$, satisfies*
$$\iint_{\mathbb{D}} |\mu(z)|^2 \rho^2(z) |\mathrm{d}z|^2 < \infty;$$
- *$\varphi = g^{-1} \circ f|_{S^1}$ is absolutely continuous with respect to the arc-length measure, and $\log \varphi'$ belongs to the Sobolev space $H^{1/2}(S^1)$.*

In the above, $\rho(z) = 1/(1 - |z|^2)$.

We follow [Shen]

Proof : G is HS $\Rightarrow$ 1

Differentiating (*) in both z and ζ

$$\mathcal{S}(g,z,\zeta) := \frac{g'(z)g'(\zeta)}{(g(z)-g(\zeta))^2} - \frac{1}{(z-\zeta)^2} = -\sum_{m,n=1}^{\infty} mnb_{mn}(g)z^{-m-1}\zeta^{-n-1}$$

Let $\zeta \to z$

$$\mathcal{S}(g)(z) = 6\lim_{\zeta\to z}\mathcal{S}(g,z,\zeta) = -6\sum_{m,n=1}^{\infty} mnb_{mn}(g)z^{-m-n-2}$$

$$= -6\sum_{k\geq 2}\left(\sum_{m+n=k} mnb_{mn}(g)\right)z^{-k-2}.$$

Lemma 3.10. *If* $\phi(z) = \sum_{n=2}^{\infty} a_n z^{-n-2}$, $\psi(z) = \sum_{n=2}^{\infty} b_n z^{-n-2}$, *then*

$$\iint_{\mathbb{D}^*} \phi(z)\overline{\psi}(z)\rho^{-2}(z)|dz|^2 = 2\pi\sum_{n=2}^{\infty}(n^3-n)^{-1}a_n\bar{b}_n.$$

$\mathbb{D}^*$

$$\Rightarrow \int_{\mathbb{D}^*} |S(g)|^2(z)\,\rho^{-2}(z)|dz|^2 = 72\pi\sum_{k=2}^{\infty}(k^3-k)^{-1}\left|\sum_{m+n=k} mn\, b_{mn}\right|^2$$

$$\leq 72\pi\sum_{k=2}^{\infty}\cancel{(k^3-k)^{-1}}\underbrace{\cancel{\sum_{m+n=k} mn}}_{\frac{\cancel{k^3-k}}{6}}\sum_{m+n=k} mn\, b_{mn}^2$$

$$= 12\pi\sum_{m,n\geq 1} G_{mn}^2$$

$$= 12\pi\, \mathrm{Tr}(G^*G)$$

$$= 12\pi\, \|G\|_{HS}^2 < \infty \qquad \Rightarrow 1$$

2 $\Rightarrow$ G is HS.

$$\mathcal{S}(g,z,\zeta) := \frac{g'(z)g'(\zeta)}{(g(z)-g(\zeta))^2} - \frac{1}{(z-\zeta)^2} = -\sum_{m,n=1}^{\infty} mnb_{mn}(g)z^{-m-1}\zeta^{-n-1}$$

$$= -\pi\sum_{m,n\geq 1} G_{mn}\, e_m(z)\, e_n(\zeta)$$

where $e_m := e_m(z) = \sqrt{\frac{m}{\pi}}\, z^{-m-1}$

is an ONB of $A^2(\mathbb{D}^*) = \{ L^2\text{-integrable holomorphic function in } \mathbb{D}^* \}$

Bergman

We use this ONB to simplify the computation.

For instance:

$$\iint_{\mathbb{D}^*} \mathcal{S}(g,z,\zeta)\overline{\zeta}^{-n-1}|\mathrm{d}\zeta|^2 = \pi^{1/2}n^{-1/2}\iint_{\mathbb{D}^*} \mathcal{S}(g,z,\zeta)\overline{e_n(\zeta)}|\mathrm{d}\zeta|^2$$
$$= -\pi^{3/2}n^{-1/2}\sum_{m=1}^{\infty} G_{mn}e_m(z).$$

Therefore,

$$\underbrace{\mathrm{Tr}(G^*G)}_{\text{Goal: show } <\infty} = \sum_{n\geq 1}\|\sum_{m=1}^{\infty} G_{mn}e_m\|^2 = \pi^{-3}\sum_{n\geq 1} n\underbrace{\iint_{\mathbb{D}^*}\left(\iint_{\mathbb{D}^*}\mathcal{S}(g,z,\zeta)\overline{\zeta}^{-n-1}|\mathrm{d}\zeta|^2\right)^2|\mathrm{d}z|^2}$$

$$\iint_{\mathbb{D}^*} \mathcal{S}(g,z,\zeta)\overline{\zeta}^{-n-1}|\mathrm{d}\zeta|^2 \xrightarrow[\mathbb{D}^* \rightsquigarrow \mathbb{D}]{\text{Stokes' formula}} = -\iint_{\mathbb{D}}\frac{g'(z)\bar{\partial}g(\zeta)}{(g(z)-g(\zeta))^2}\zeta^{n-1}|\mathrm{d}\zeta|^2.$$

(g extended to $\mathbb{D}$ with $\int_{\mathbb{D}} \mu^2\rho^2 < \infty$)

$$\underbrace{\iint_{\mathbb{D}^*}\left|\iint_{\mathbb{D}^*}\mathcal{S}(g,z,\zeta)\overline{\zeta}^{-n-1}|\mathrm{d}\zeta|^2\right|^2|\mathrm{d}z|^2} = \iint_{\mathbb{D}^*}\left|\iint_{\mathbb{D}}\frac{g'(z)\bar{\partial}g(\zeta)}{(g(z)-g(\zeta))^2}\zeta^{n-1}|\mathrm{d}\zeta|^2\right|^2|\mathrm{d}z|^2$$

$w = g(z)$, $\eta = g(\zeta)$

change of variable

$$= \iint_{\Omega^*}\left|\iint_{\Omega}\frac{\bar{\partial}g^{-1}(\eta)}{(w-\eta)^2}[g^{-1}(\eta)]^{n-1}|\mathrm{d}\eta|^2\right|^2|\mathrm{d}w|^2$$

$$\leq \iint_{\mathbb{C}}\left|\iint_{\mathbb{C}}\frac{\overbrace{\bar{\partial}g^{-1}(\eta)}^{=0 \text{ outside of } \Omega}}{(w-\eta)^2}[g^{-1}(\eta)]^{n-1}|\mathrm{d}\eta|^2\right|^2|\mathrm{d}w|^2$$

extend using principle value

(is the Beurling transform of $\bar\partial g^{-1}(g^{-1})^{n-1}$

$\mathcal{B}(\mu)(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\mu(\zeta)}{(\zeta - z)^2} d\xi d\eta$, isometry of $L^2(\mathbb{C})$.)

using $\bar\partial g = 0$ in $\mathbb{C}\setminus\Omega$

$$= \pi^2 \iint_{\Omega} \left|\bar\partial g^{-1}(w)(g^{-1}(w))^{n-1}\right|^2 |\mathrm{d}w|^2$$

change of variable

$$= \pi^2 \iint_{\mathbb{D}} \frac{\left|\bar\partial g(\zeta)\zeta^{n-1}\right|^2}{|\partial g|^2 - |\bar\partial g|^2} |\mathrm{d}\zeta|^2$$

$$= \pi^2 \iint_{\mathbb{D}} \frac{|\mu(\zeta)|^2}{1 - |\mu(\zeta)|^2} |\zeta|^{2n-2} |\mathrm{d}\zeta|^2.$$

$$\begin{aligned}
\mathrm{Tr}(G^*G) &\le \pi^{-1} \sum_{n\ge1} n \iint_{\mathbb{D}} \frac{|\mu(\zeta)|^2}{1 - |\mu(\zeta)|^2} |\zeta|^{2n-2} |\mathrm{d}\zeta|^2 \\
&= \pi^{-1} \iint_{\mathbb{D}} \frac{|\mu(\zeta)|^2}{1 - |\mu(\zeta)|^2} \sum_{n\ge1} n |\zeta|^{2n-2} |\mathrm{d}\zeta|^2 \\
&= \pi^{-1} \iint_{\mathbb{D}} \frac{|\mu(\zeta)|^2}{1 - |\mu(\zeta)|^2} \sum_{n\ge1} n |\zeta|^{2n-2} |\mathrm{d}\zeta|^2 \\
&\le \pi^{-1} \iint_{\mathbb{D}} \frac{|\mu(\zeta)|^2}{1 - \|\mu\|_\infty^2} \frac{1}{(1 - |\zeta|^2)^2} |d\zeta|^2 \\
&< \infty. \Rightarrow 2
\end{aligned}$$

$(\|\mu\|_\infty < 1)$

$\boxed{}$

same for the Grunsky associated to f.

Finally, [TT06, Ch.II.2] shows that the singular values $(\lambda_n)_{n\ge1}$ of the Grunsky operator G of g coincides with the *Fredholm eigenvalues* associated to a quasicircle, defined as the eigenvalues of classical Poincaré-Fredholm integral operator. This result was first proved by Schiffer [Sch81] for C^3 curves.

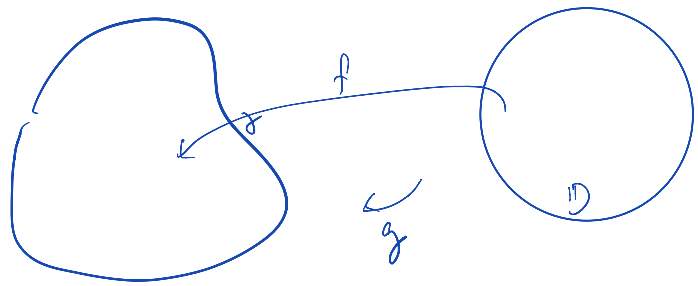
Theorem 5.11 (Takhtajan-Teo [TT06]). *Let* $\mathrm{Det}_F(\gamma)$ *denote the determinant of* $I - G^*G$, *namely,* $\prod_{n\ge1}(1 - \lambda_n^2)$. *We have*

$$I^L(\gamma) = -12 \log \mathrm{Det}_F(\gamma),$$

where

$$I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} |f''/f'|^2 + \frac{1}{\pi}\int_{\mathbb{D}^*} |g''/g'|^2 + 4\log|f'(0)/g'(\infty)|$$

is the universal Liouville action (and the Loewner energy) of γ.



Summary :

From C.Bishop's paper

CURVES OF FINITE TOTAL CURVATURE AND THE WEIL-PETERSSON CLASS

CHRISTOPHER J. BISHOP

$\log f'$	$\mathcal{B}_0$ $\supset$	BMOA $\supset$	VMOA $\supset$	Dirichlet
$(\log f')'(1-\lvert z\rvert^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S_f(z)(1-\lvert z\rvert^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$\varphi = g^{-1}\circ f$	symmetric	strongly quasisymmetric	$\log\varphi' \in \mathrm{VMO}$	$\log\varphi' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	$\sum \beta^2 < \infty$
$\bigcup_{k<1}\Sigma(k) \supset$ Grunsky	Compact	??	??	H-S

$\infty > p > 2$

$\uparrow$ $2 > p \geq 1$

Other characterizations

Gavin Jones : G has finite p-Schatten norm. $p \in [1,\infty)$

$\iff \mu \in L^p(\mathbb{D};\ \rho^2(z)\,|dz|^2)$

$\iff S(g) \in L^p(\mathbb{D}^*,\ \rho^{2-2p}(z)\,|dz|^2)$